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APPLIED MECHANICS FOR ENGINEERS

BY

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PREFACE

THE author's object in writing this book has been to provide a practical statement of the principles of Mechanics. The arrangement adopted is similar to that of his *Applied Mechanics for Beginners*. Great pains have been taken to make the treatment adequate; principles have been illustrated by numerous fully worked-out examples, and exercises for home or class work have been provided at the ends of the chapters. The working out of typical exercises must be done by every student of Mechanics, but the mere ability to solve examination questions is not the only service the study of Applied Mechanics can render the Engineer. The problems met with in actual engineering practice often differ greatly from the text-book form of exercise, and the student of Mechanics, in addition to a sound knowledge of principles, must learn to appreciate the assumptions involved and the consequent limitations which arise in their practical applications

Consequently, the student must be provided with frequent opportunities for performing suitable experiments under workshop conditions. In the mechanical laboratory he must come into touch with practical problems, and there learn to test and apply his knowledge of principles, and in this work he should have the assistance of a teacher and the criticism of fellow-students. But if the whole value of such laboratory work is to be secured, no slip-shod working out of results must be tolerated. In recognition of the supreme importance of the experience gained in the laboratory, many suitable experiments have been described, and these have been arranged on p. xi to provide a connected course of practical work. The nature and scope of the apparatus available in different laboratories vary greatly, and some of the experiments included are given as suggestions only, so as to be applicable to any form of machine or instrument.

Students using the book must have a knowledge of Algebra up to quadratic equations, and of Trigonometry to the simple properties of triangles. They should be acquainted also with about half-a dozen rules of the Calculus, and these are given in Chapter I. Students able to integrate $x^n dx$, and to differentiate x^n, $\sin x$, and $\cos x$, will be able to understand practically the whole volume.

Though no particular examination syllabus has been followed, the book should be of service to students preparing for University degrees in Engineering, for the examinations of the Institutions of Civil Engineers and of Mechanical Engineers, and for the higher examinations of the Board of Education and the City and Guilds of London Institute.

Exercises marked B.E. are from recent examination papers of the Board of Education, and are reprinted by permission of the Controller of H.M. Stationery Office; those marked I.C.E. are taken from recent examination papers of the Institution of Civil Engineers, and are reprinted by permission of the publishers, Messrs. W. Clowes & Sons. Exercises marked L.U. are reprinted, with permission, from recent examination papers for B.Sc. (Eng.) of London University.

It is impossible to give in a book of moderate size a complete statement of all subjects of Applied Mechanics. For fuller information on special matters the student is referred to separate treatises; the names of some of these are noted in the text, and the author takes the opportunity of acknowledging his own indebtedness to them, especially to *Strength of Materials*, by Sir J. A. Ewing (Cambridge University Press), and to *Machine Design*, by Prof. W. C. Unwin (Longmans).

Sir Richard Gregory and Mr. A. T. Simmons have read the proofs, and to their expert knowledge of books and book production the author owes a heavy debt of gratitude. Thanks are also due to Mr. L. Wyld, B.Sc., Assistant Lecturer at West Ham Institute, who has read the proofs and checked the whole of the mathematical work and the answers to the exercises; it is hoped that his care has had the effect of reducing the number of errors to a minimum.

The apparatus represented in Figs. 706, 707 and 715 is made by Mr. A. Macklow-Smith, Queen Anne's Chambers, Westminster, and the illustrations have been reproduced from working drawings kindly supplied by him. The illustration of a chain (Fig. 585) is inserted by permission of Messrs. Hans Renold, Ltd. The Tables of Logarithms and Trigonometrical Ratios are reprinted from Mr. F. Castle's *Machine Construction and Drawing* (Macmillan).

J. DUNCAN.

WEST HAM, September, 1913.

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COURSE OF LABORATORY EXPERIMENTS

INSTRUCTIONS FOR CARRYING OUT LABORATORY WORK

General Instructions.—Two Laboratory Note books are required, in one rough notes of the experiments should be made, and in the other a fair copy of them in ink should be entered.

Before commencing any experiment, make sure that you understand what its object is, and also the construction of the apparatus and instruments employed.

Reasonable care should be exercised in order to avoid damage to apparatus, and to secure fairly accurate results

In writing up the results, enter the notes in the following order:

(1) The title of the experiment and the date on which it was performed.

(2) Sketches and descriptions of any special apparatus or instruments used

(3) The object of the experiment.

(4) Dimensions, weights, etc, required for working out the results; from these values calculate any constants required

(5) Log of the experiment, entered in tabular form where possible, together with any remarks necessary.

(6) Work out the results of the experiment and tabulate them where possible.

(7) Plot any curves required.

(8) Work out any general equations required.

(9) Where possible, state any general conclusions which may be deduced from the results, and compare the results obtained with those which may be derived from theory. Account for any discrepancies

Notes should not be left in the rough form for several days; it is much better to work out the results and enter them directly after the experiments have been performed.

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PART I.

MATERIALS AND STRUCTURES.

CHAPTER I.

INTRODUCTORY PRINCIPLES.

Definition of terms. Applied mechanics treats of those laws of force and the effects of force upon matter which apply to works of human art. It will suffice to define **matter** as anything which occupies space. Matter exists in many different forms, and can often be changed from one form to another, but man cannot create it, nor can he annihilate it. Any given piece of matter, occupying a definite space, is called a **body**. **Force** may exert push or pull on a body; force may change or tend to change a body's state of rest or of motion.

Statics is that part of the subject embracing all questions in which the forces applied to a body do not produce a disturbance in its state of rest or motion. When we speak of a body's motion we mean its motion relative to other bodies. Rest is merely a relative term; no body, so far as we are aware, is actually at rest; but if its position is not changing in relation to other neighbouring bodies, we say it is at rest. In the same way, when we speak of a body's motion we mean the change of position which is being effected relative to neighbouring bodies. Change of the state of rest or of motion may be secured by the application of a force or forces, but if the forces applied are self-equilibrating, *i.e.* balance among themselves, no change of motion will occur. **Kinetics** includes all problems in which change of motion occurs as a consequence of the application of forces.

There is another division of the subject called **kinematics**. This division may be defined as the geometry of motion, and has no reference to the forces which may be required for the production of the motion. Problems arise in kinematics such as the curves described by moving points in a mechanism, and the velocities of these points at any instant.

Measurement of matter. Matter is measured by the **mass**, or quantity of matter, it contains. The standard unit of mass for this country is the **pound** mass, which may be defined as the quantity of matter contained in a certain piece of platinum preserved in the Exchequer Office. A gallon of water at 62° F. has a mass of 10 pounds. In cases where a larger unit is desirable, the **ton**, containing 2240 pounds, or the **hundredweight**, containing 112 pounds, may be used. Generally speaking, it is best to state results in tons and decimals of a ton, or in pounds and decimals of a pound.

In countries using the metric system, the unit of mass employed is the **gram**. This may be defined as the quantity of matter contained in a cubic centimetre of pure water at the temperature of 4° C. Where a larger unit is required, the **kilogram** may be used, being a mass of 1000 grams.

The term **density** refers to the mass of unit volume of a substance. Thus, in the British system, the density of water is about 62·5, there being 62·5 pounds mass in one cubic foot of water. The density of cast iron in the same system is about 450 pounds per cubic foot. The density of water in the metric system is 1, and of cast iron 7·2, these numbers giving the mass in grams in one cubic centimetre of water and cast iron respectively.

Measurement of force. Forces may be measured by comparison with the weight of the unit of mass. Thus, the weight of the one pound mass, or that of the gram, may be taken as units of force, and as these depend on gravitational effort they are referred to as **gravitational units of force**. The attraction exerted by the earth in producing the effect known as the weight of a body varies in different latitudes, hence gravitational units of force have the disadvantage of possessing variable magnitudes. The variation can be disregarded in many engineering calculations, as it affects the result to a very small extent only. Other practical gravitational units of force are the weight of one ton (2240 lb.) and the weight of a kilogram (1000 grams or 2·2 lb. nearly).

An **absolute unit of force** does not vary, as it is defined in relation to the invariable units of mass, length and time belonging to the system. In the British system, the absolute unit of force is called the **poundal**, and has such a magnitude that, if it acts on one pound mass, assumed to be perfectly free to move, for one second, it will produce a velocity of one foot per second. The metric absolute unit of force is the **dyne**, and will produce a velocity of one centimetre per second if it acts for one second on a gram mass which s perfectly free to move. The poundal is equal roughly to the weight of half-an-ounce, or, accurately, it is equal to $\frac{1}{g}$ lb. weight, g being the rate at which a body falling freely increases its speed. For all parts of Britain g may be taken as 32·2 in feet and second units, or 981 in centimetre and second units. On this basis, the dyne will be $\frac{1}{g}$ gram weight, or 981 dynes equal one gram weight nearly.

Newton's laws of motion. In connection with the above definitions, it is useful to study the laws of motion laid down by Newton. These laws form the basis of all principles in mechanics, and are three in number.

First law. Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by forces o change that state.

Second law. Change of momentum is proportional to the applied orce, and takes place in the direction in which the force acts.

Third law. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

The first law expresses what is called the **inertia** of a body, *i.e.* that property whereby it resists any effort made to change either the magnitude of its velocity or the direction of its motion. In the second law, the term **momentum** may be here understood to mean quantity of motion, measured by the product of the body's mass and velocity. The law expresses the observed facts that change in the magnitude of the velocity of a given body is proportional to the force applied, and change in the direction of motion takes place in the line of the force. The third law also expresses observed facts. It is impossible to apply a single force; there must always be an equal opposite force. One end of a string cannot be pulled unless an equal opposite pull be applied to the other end. If the body used be free to move and an effort be applied, the velocity will change continuously and the inertia of the body provides the resistance equal and opposite to the force applied.

Experimental measurement of mass and force. Masses may be compared by means of a common balance (Fig. 1). In this appliance, a beam AB, pivoted at its centre, will become horizontal, or will describe small equal angles on each side of the horizontal when equal forces are applied at A and B. Such equal forces will arise when bodies C and D, having equal masses, are placed in the pans. This follows as a consequence of the fact that equal masses have equal weights at the same part of the earth's surface. Further, no matter at what part of the earth the balance is used, it will always indicate equal masses. It therefore follows that such a balance could not be used to indicate the variation of a body's weight in different places.

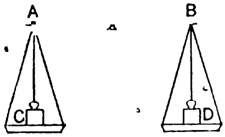


FIG. 1.—Common balance.

Spring balances (Fig. 2) may be used to measure forces by observation of the extensions produced in a spring. As equal masses have equal weights, such balances will indicate the same scale reading for equal masses, but as it is the weight of the body which produces the extension of the spring, and as it is known that the extension is proportional to the force applied, it follows that change of weight, such as would be produced by taking the balance to another part of the earth's surface, will be evidenced by a different scale reading. As has been already mentioned, such difference is very small. Spring balances are generally calibrated in a vertical position, as shown in Fig. 2, and will not indicate quite the same force when the balance is used in an inclined or inverted position. This is owing to zero on the scale being marked for the spring extension corresponding to the weights of the parts of the balance suspended from the spring, but no load on the hook or scale pan. Consequently the zero will change if the balance is used in any position other than that shown.

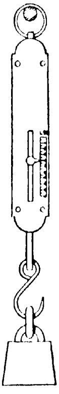


FIG. 2.—Spring balance.

Specific gravity. The specific gravity of a substance is the weight of a given volume of the substance as compared with the weight of an equal volume of pure water. Specific gravities are usually measured at a temperature of 60° Fahrenheit.

Let V = volume of a given body in cubic feet,

ρ = specific gravity of material,

W = weight of body in lb.

Then $W = 62{\cdot}5V$ lb. weight if the material is water,
$W = 62{\cdot}5V\rho$ lb. weight for the given substance.

Hence $\rho = \dfrac{W}{62{\cdot}5V}$.

This expression enables the specific gravity of a given body to be found roughly by first weighing it, then calculating its volume from the measured dimensions.

The following table gives the weights and specific gravities of some common substances:

WEIGHTS AND SPECIFIC GRAVITIES.

Material	Weight of One cub. foot	Weight of One cub. inch	Weight of a sheet 1″ thick, 1 sq. foot area.	Specific Gravity
	lb.	lb.	lb.	
Wrought iron	480	0·28	40	7·7
Steel - -	490	0·28	41	7·8
Cast iron -	450	0·26	37½	7·2
Copper - -	550	0·32	46	8·8
Brass - -	525	0·30	44	8·4
Gun metal -	540	0·31	45	8·6
Aluminium -	165	0·095	14	2·6
Zinc - -	450	0·26	37½	7·2
Tin - -	465	0·27	39	7·4
Lead - -	710	0·41	59	11·4
Fresh water -	62·5	0·036	-	1·0
Sea water -	64	0·037		1·024

Mathematical formulae. The following mathematical notes are given for reference. It is assumed that the reader has studied the principles involved, or that he is doing so conjointly with his course in mechanics. It may be noted here that a knowledge of the elementary rules of the calculus given below is not required in reading the first five chapters of this book.

MENSURATION.

Determination of areas.

Square, side s; area $= s^2$.
Rectangle, adjacent sides a and b; area $= ab$.
Triangle, base b, perpendicular height h; area $= \frac{1}{2}b \times h$.
Triangle, sides a, b and c. $2s = a + b + c$.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Parallelogram; area = one side × perpendicular distance from that side to the opposite one.

Any irregular figure bounded by straight lines; split it up into triangles, find the area of each separately and take the sum.

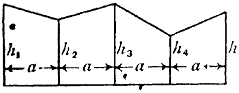


FIG. 3—Trapezoidal figure.

Trapezoid; area = half the sum of the end ordinates × the base.

A trapezoidal figure having equal intervals (Fig. 3);

$$\text{area} = a\left(\frac{h_1 + h_5}{2} + h_2 + h_3 + h_4\right).$$

Simpson's rule for the area bounded by a curve (Fig. 4); take an odd number (say 7) of equidistant ordinates; then

$$\text{area} = \frac{a}{3}(h_1 + 4h_2 + 2h_3 + 4h_4 + 2h_5 + 4h_6 + h_7).$$

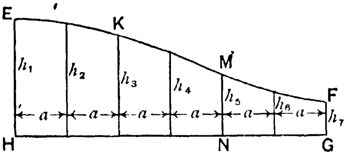


FIG. 4—Illustration of Simpson's rule.

Circle, radius r, diameter d; area $= \pi r^2 = \dfrac{\pi d^2}{4}$.

(Circumference $= 2\pi r = \pi d$.)

Parabola, vertex at O (Fig. 5); area OBC $= \frac{2}{3}ab$.

Cylinder, diameter d, length l, area of curved surface $= \pi dl$.

Sphere, diameter d, radius r; area of curved surface $= \pi d^2 = 4\pi r^2$.

Cone; area of curved surface = circumference of base × $\frac{1}{2}$ slant height.

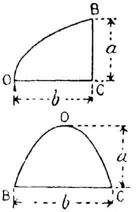


FIG. 5—Area of a parabola.

Determination of volumes.

Cube, edge s; volume $= s^3$.

Cylinder or prism, having its ends perpendicular to its axis; volume = area of one end × length of cylinder or prism.

Sphere, radius r; volume $= \frac{4}{3}\pi r^3$.

Cone or pyramid; volume = area of base × $\frac{1}{3}$ perpendicular height.

TRIGONOMETRY.

A **degree** is the angle subtended at the centre of a circle by an arc of $\frac{1}{360}$th of the circumference.

A **radian** is the angle subtended at the centre of a circle by an arc equal to the radius of the circle.

There are 2π radians in a complete circle, hence

$$\begin{aligned} 2\pi \text{ radians} &= 360 \text{ degrees.} \\ \tfrac{3}{2}\pi \quad ,, \quad &= 270 \quad ,, \\ \pi \quad ,, \quad &= 180 \quad ,, \\ \tfrac{1}{2}\pi \quad ,, \quad &= 90 \quad ,, \end{aligned}$$

Let l be the length of arc subtended by an angle, and let r be the radius of the circle, both in the same units; then angle $= \frac{l}{r}$ radians.

Trigonometrical ratios. In Fig. 6 let OB revolve anti-clockwise about O, and let it stop successively in positions OP_1, OP_2, OP_3, OP_4; the angles described by OB are said to be as follows:

P_1OB, in the first quadrant COB.

P_2OB, in the second quadrant COA.

P_3OB (greater than 180°), in the third quadrant AOD.

P_4OB (greater than 270°), in the fourth quadrant BOD.

Drop perpendiculars such as P_1M_1 from each position of P on to AB. OP is always regarded as positive, OM is positive if on the right and negative if on the left of O, PM is positive if above and negative if below AB.

Name of ratio	Ratio as written	Value of ratio	Algebraic sign of ratio			
			1st quad.	2nd quad.	3rd quad.	4th quad.
sine POM -	sin POM	$\frac{PM}{OP}$	+	+		
cosine POM -	cos POM	$\frac{OM}{OP}$	+	-	-	+
tangent POM -	tan POM	$\frac{PM}{OM}$	+	-	+	
cosecant POM	cosec POM	$\frac{OP}{PM}$	+	+		
secant POM -	sec POM	$\frac{OP}{OM}$	+	-	-	+
cotangent POM	cot POM	$\frac{OM}{PM}$	+	−	+	-

The values of the ratios are not affected by the length of the radius OP; taking OP to be unity, we have

sin POM = PM (Fig. 6),

cos POM = OM (Fig. 6),

tan POM = P′B or P′A, depending on the quadrant (Fig. 7).

Figs. 6 and 7 show clearly both the sign and the varying values these ratios, and enable the following table to be deduced:

Name of ratio.	Values of the ratios for angles of				
	0°	90°	180°	270°	360°
sin POM - -	0	1	0	-1	0
cos POM - -	1	0	-1	0	1
an POM - -	0	∞	0	$-\infty$	0

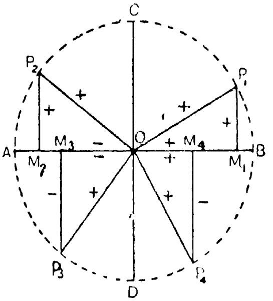


FIG. 6.—Trigonometrical ratios.

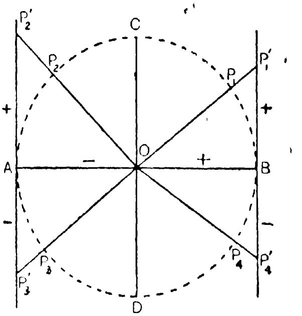


FIG. 7.—Tangents of angles.

The following formulae are given for reference:

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}; \quad \cot A = \frac{1}{\tan A}.$$

$$\tan A = \frac{\sin A}{\cos A}; \quad \cot A = \frac{\cos A}{\sin A}, \quad \cos^2 A + \sin^2 A = 1$$

$$\tan^2 A + 1 = \sec^2 A; \qquad \cot^2 A + 1 = \operatorname{cosec}^2 A.$$

$$\sin A = \cos(90^\circ - A); \quad \sin A = \sin(180^\circ - A).$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

If the angles of a triangle are A, B and C, and the sides opposite these angles are a, b and c respectively, the following relations hold:

$$a = b \cos C + c \cos B.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

ALGEBRA.

Solution of simple simultaneous equations. If the given equations are

$$a_1x + b_1y = c_1, \quad \ldots \quad (1)$$

$$a_2x + b_2y = c_2, \quad \ldots \quad (2)$$

then

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Solution of a quadratic equation. If

$$ax^2 + bx + c = 0,$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

CALCULUS.

Differential calculus Let AB (Fig. 8) represent the relation of two quantities x and y which are connected in some definite manner. Consider two points P_1 and P_2 on AB separated by a short distance P_1P_2; then

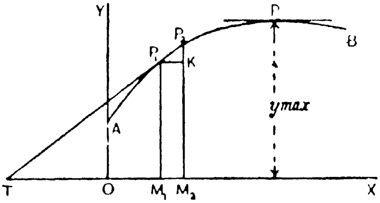


FIG. 8.—Graphic illustration of a differential coefficient

$$OM_1 = x_1; \quad P_1M_1 = y_1.$$

$$OM_2 = x_2; \quad P_2M_2 = y_2.$$

The difference between the abscissae OM_1 and OM_2 will be

$x_2 - x_1$), and may be written δx, the symbol δ signifying "the difference in"; similarly with the ordinates P_1M_1 and P_2M_2. Hence

$$\delta x = x_2 - x_1 = M_1M_2 = P_1K.$$

$$\delta y = y_2 - y_1 = P_2K.$$

The ratio of these will be

$$\frac{\delta y}{\delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{P_2K}{P_1K}.$$

The value of this ratio depends on the proximity of P_1 and P_2. If these points are taken indefinitely close together, the ratio tends o take a definite value which depends on the given relationship of x and y. This value is called a **differential coefficient**, and serves o measure the rate of growth of y with x.

If P_1 and P_2 are very close together, P_1P_2 is practically a straight line, and we have

$$\frac{P_2K}{P_1K} = \tan P_2P_1K.$$

If P_1 and P_2 are indefinitely close together, P_1P_2 is in the direction of the tangent P_1T drawn to touch the curve at P_1; in this case δy and δx are written dy and dx, and the final value of the ratio is

$$\frac{dy}{dx} = \tan P_1TM_1 = \frac{P_1M_1}{TM_1}.$$

For example, suppose a graph such as AB in Fig. 8 to have been plotted from the equation,

$$y = x^2 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Then

$$y + \delta y = (x + \delta x)^2$$
$$= x^2 + 2x \,.\, \delta x + (\delta x)^2 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Taking the difference between (2) and (1) gives

$$\delta y = 2x \,.\, \delta x + (\delta x)^2.$$

Now $(\delta x)^2$ is the square of a quantity which ultimately becomes very small, and therefore becomes negligible. Hence we may write

$$dy = 2x \,.\, dx,$$

or

$$\frac{dy}{dx} = 2x \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Suppose, as another example, we take

$$y = ax^2, \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

when a is a constant. It will be evident, on repeating the above process, that

$$\frac{dy}{dx} = 2ax, \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

thus giving the rule that any constant factor appears unaltered in the value of the differential coefficient.

Take now the following equation :

$$y = x^2 + a \ldots\ldots\ldots\ldots \quad (6)$$

The effect of the addition of a constant a to the right hand side of (1) is simply to raise the graph to a higher level above OX in Fig. 8 ; its shape will be exactly as before, and hence the tangent at any point will make the same angle with OX. Therefore the differential coefficient will have the same value as (3), viz.

$$\frac{dy}{dx} = 2x \ldots\ldots\ldots\ldots \quad (7)$$

It will also be clear that, if the equation is

$$y = ax^2 + b, \ldots\ldots\ldots\ldots \quad (8),$$

then

$$\frac{dy}{dx} = 2ax \ldots\ldots\ldots\ldots \quad (9)$$

The rule may be expressed that a constant quantity added to the right-hand side disappears from the differential coefficient.

The following differential coefficients are useful ; the methods of obtaining them may be studied in any book dealing with the calculus. The symbol e represents the base of the Naperian or hyperbolic system of logarithms, viz. 2·71828.

DIFFERENTIAL COEFFICIENTS.

$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$y = ax^n$	$\frac{dy}{dx} = anx^{n-1}$
$y = e^x$	$\frac{dy}{dx} = e^x$	$y = ae^{bx}$	$\frac{dy}{dx} = abe^{bx}$
$y = \log_e x$	$\frac{dy}{dx} = \frac{1}{x}$	$y = a \log_e x$	$\frac{dy}{dx} = \frac{a}{x}$
$y = \sin x$	$\frac{dy}{dx} = \cos x$	$y = a \sin bx$	$\frac{dy}{dx} = ab \cos bx$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$	$y = a \cos bx$	$\frac{dy}{dx} = -ab \sin bx$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$y = a \tan bx$	$\frac{dy}{dx} = ab \sec^2 bx$
$y = a$	$\frac{dy}{dx} = 0$		

Differentiation rules. The following rules may also be stated here.

If the right-hand side takes the form of the sum of a number of terms each depending on x, then the differential coefficient is the sum of the differential coefficients of the terms taken separately. Thus:

$$y = ax^3 + bx^2 + cx + d,$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c.$$

To differentiate the product of a number of factors, each of which depends on x, multiply the differential coefficient of each factor by all the other factors and take the sum. Thus:

$$y = x^2 \sin x,$$

$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x.$$

To differentiate a fraction in which both numerator and denominator depend on x, proceed thus:

$$\frac{dy}{dx} = \frac{\text{diff. coeff. of numerator} \times \text{denominator} - \text{diff. coeff. of denominator} \times \text{numerator}}{\text{square of denominator}}.$$

EXAMPLE. Let $y = \dfrac{x^2}{\sin x}$.

The differential coefficient of the numerator is $2x$ and that of the denominator is $\cos x$, hence, by the above rule:

$$\frac{dy}{dx} = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}.$$

Supposing we have to find the differential coefficient of

$$y = \sin^3 x,$$

it should be noticed that the given expression, viz. the cube of $\sin x$, depends on another function of x. The rule to be followed is to differentiate the expression as given, viz. $(\sin x)^3$, the result being $3(\sin x)^2$; then multiply this result by the differential coefficient of the function on which the given expression depends, viz. $\sin x$, for which the differential coefficient is $\cos x$.

Hence,
$$\frac{dy}{dx} = 3(\sin x)^2 \cos x$$
$$= 3 \sin^2 x \cos x.$$

In **successive differentiation**, the differential coefficient of the given function is taken as a new function of x and its differential coefficient is found; the latter is called the second differential coefficient, and is written $\dfrac{d^2y}{dx^2}$. The operation may be repeated as many times as may be necessary.

EXAMPLE. Let

$$y = ax^5,$$

$$\frac{dy}{dx} = 5ax^4,$$

$$\frac{d^2y}{dx^2} = 20ax^3,$$

$$\frac{d^3y}{dx^3} = 60ax^2.$$

The **maximum value** of a given function of x may often be found by application of the following simple method. It will be noted that, in Fig. 8, at the point in AB for which y has its maximum value, the tangent to the curve is parallel to OX, and hence $\frac{dy}{dx}$ for this point will be zero. The rule therefore is, take the differential coefficient and equate to zero; this will give the value of x corresponding to the maximum value of y. By inserting this value of x in the given equation connecting x and y, the maximum value of y may be found. Thus:

Let

$$y = \sin x,$$

$$\frac{dy}{dx} = \cos x = 0 \text{ for the maximum value of } y.$$

Now when $\cos x = 0$, x is either 90° or 270°, *i.e.* $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ radians, hence

$$\text{Maximum value of } y = \sin\frac{\pi}{2} \text{ or } \sin\frac{3\pi}{2}.$$

As the numerical value of $\sin\frac{\pi}{2}$ is unity, it follows that the maximum value of y is also unity.

As another example, take

$$y = ax - x^2,$$

then,

$$\frac{dy}{dx} = a - 2x = 0;$$

$$\therefore x = \frac{a}{2} \text{ for the maximum value of } y.$$

$$\text{Maximum value of } y = \frac{a^2}{2} - \frac{a^2}{4} = \frac{a^2}{4}.$$

Integral calculus. In this branch of mathematics, rules are formed for the addition of the indefinitely small portions into which a quantity may be imagined to be divided. In Fig. 9, OA and OB are two distances measured along the same straight line from O. Let these be a and b respectively, then the length of AB will be

$$AB = b - a. \qquad \text{(1)}$$

The line AB might be measured also by the process of dividing it up into a large number of small portions δx_1, δx_2, δx_3, etc. The total length of AB will then be

$$AB = \delta x_1 + \delta x_2 + \delta x_3 + \text{etc.}$$
$$= b - a, \text{ from } (1). \quad \ldots\ldots \quad (2)$$

The symbol Σ or $\int$ (sigma) is used to denote the phrase "the algebraic sum of," and if any expression follows the symbol Σ, it is understood to be one only of a number of terms which are all of the same type. Thus, $\Sigma\,\delta x$ means "the algebraic sum of all terms of which δx is given as a type." If we write Σ_a^b, it is to be understood that we are to begin taking small portions, such as δx_1, at a distance a from the origin, and to finish at a distance b. Hence we may write (2),

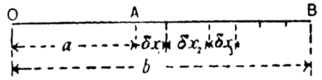


FIG. 9

$$\Sigma_a^b dx = b - a. \quad \ldots\ldots \quad (3)$$

In Fig. 10 is shown another example. As before, OA = a and OB = b, and the figure ABCD is constructed by making AD = a and BC = b, both being perpendicular to OB. The area of the figure ABCD may be calculated by deducting the area of the triangle OAD from that of the triangle OBC. Thus:

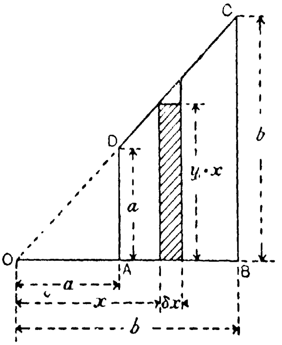


FIG. 10.

Area of ABCD

$$= (b \times \tfrac{1}{2}b) - (a \times \tfrac{1}{2}a)$$
$$= \frac{b^2}{2} - \frac{a^2}{2} \quad \ldots\ldots \quad (4)$$

Alternatively, the area may be estimated by cutting the figure into strips, such as the one shown shaded. It is evident from the construction that its height y is equal to x; let δx be its breadth, then

$$\text{Area of the strip} = x \, . \, \delta x. \quad \ldots\ldots \quad (5)$$

Any similar strip will have a similar expression for its area, hence

$$\text{Total area of the strips} = \Sigma_a^b x \, \delta x. \quad \ldots\ldots \quad (6)$$

The area stated in (5) is taken as that of a rectangle, and hence omits a small triangle at the top of the strip. If, however, the strips be taken indefinitely narrow, these triangles will practically vanish, and the area expressed in (6) will be the area of ABCD. Hence from (6) and (4),

$$\Sigma_a^b x \, . \, dx = \frac{b^2}{2} - \frac{a^2}{2}. \qquad \ldots(7)$$

In mathematical books, it is shown that if x is raised to a power n in equation (7), n having any value except -1, then the result is as follows:

$$\Sigma_a^b x^n . dx = \frac{b^{n+1} - a^{n+1}}{n+1}. \qquad \ldots(8)$$

If n is -1, then the result may be shown to be

$$\Sigma_a^b x^{-1} . dx = \Sigma_a^b \frac{dx}{x} = \log_e \frac{b}{a}. \qquad .(9)$$

If n is zero, then $x^0 = 1$, and we have

$$\Sigma_a^b x^0 dx = \Sigma_a^b dx = \frac{b-a}{1} = b - a \qquad \ldots (10)$$

The above are examples of definite integrals, taken between given limits a and b; the sum may be stated in an indefinite manner, leaving the limits to be inserted afterwards. Thus:

$$\Sigma x^n dx = \frac{x^{n+1}}{n+1}. \qquad . (11)$$

It is also shown in mathematics that a constant term c should be added to the result. The value of c depends on the conditions of the problem, and can be found usually from the data. The complete solution of (11) would thus be

$$\Sigma x^n dx = \frac{x^{n+1}}{n+1} + c. \qquad . (12)$$

Similarly,

$$\Sigma \frac{dx}{x} = \log_e x + c. \qquad (13)$$

If a constant factor is given on the left-hand side, it will appear unaltered on the right-hand side. Thus:

$$\Sigma a x^2 dx = a \frac{x^3}{3} + c.$$

If a number of terms be given, the result will be obtained by applying the rules to each term separately and then summing for the total. Thus:

$$\Sigma (x^3 + x^2 + a) dx = \frac{x^4}{4} + \frac{x^3}{3} + ax + c.$$

The rules (8), (9), (12) and (13) should be learned thoroughly. Some examples are given.

EXAMPLE 1. Find the area of the triangle given in Fig. 11.

Taking a narrow strip parallel to the base and at a distance y from O, let the breadth of the strip be δy and its length b.

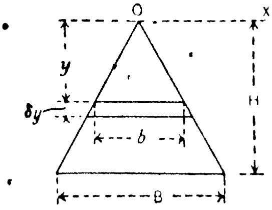


FIG. 11.—Area of a triangle.

$$\text{Area of the strip} = b \,.\, \delta y.$$

$$\text{Now} \qquad \frac{b}{B} = \frac{y}{H};$$

$$\therefore\ b = \frac{B}{H}y;$$

$$\therefore\ \text{area of the strip} = \frac{B}{H}\,.\,y\,\delta y.$$

Any other similar strip will have a similar expression for its area, hence

$$\text{Total area} = \sum_0^H \frac{B}{H}y\,dy,$$

$$= \frac{B}{H}\left(\frac{H^2}{2} - \frac{0^2}{2}\right)$$

$$= \frac{BH}{2}.$$

EXAMPLE 2. Find the volume of a cone of height H and radius of base R (Fig. 12).

In this case take a thin slice parallel to the base; let the radius of the slice be r and its thickness δh. Then

$$\text{Volume of the slice} = \pi r^2 \,.\, \delta h.$$

$$\text{Now} \qquad \frac{r}{R} = \frac{h}{H},$$

$$\therefore\ r = \frac{R}{H}h;$$

$$\therefore\ \text{volume of the slice} = \pi\frac{R^2}{H^2}h^2\delta h.$$

Any other similar slice will have a similar expression for its volume, hence

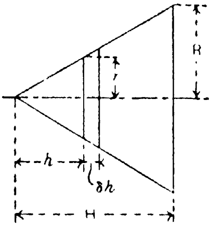


FIG. 12.—Volume of a cone.

$$\text{Total volume} = \pi\frac{R^2}{H^2}\sum_0^H h^2\,dh$$

$$= \pi\frac{R^2}{H^2}\cdot\frac{H^3}{3}$$

$$= \pi R^2 \times \tfrac{1}{3}H.$$

No constant of integration need be added in either of these examples. Instances where a constant is necessary will occur later.

The following table of indefinite integrals is given here for reference.

INTEGRALS.

$\int x^n . dx = \frac{x^{n+1}}{n+1}$	$\int \sec^2 x . dx = \tan x$
$\int \frac{1}{x} . dx = \log_e x$	$\int \operatorname{cosec}^2 x . dx = -\cot x$
$\int \frac{a}{x} . dx = a \log_e x$	$\int a \cos bx . dx = \frac{a}{b} \sin bx$
$\int e^x . dx = e^x$	$\int a \sin bx . dx = -\frac{a}{b} \cos bx$
$\int ae^{bx} . dx = \frac{a}{b} e^{bx}$	$\int a \sec^2 bx . dx = \frac{a}{b} \tan bx$
$\int \cos x . dx = \sin x$	$\int a \operatorname{cosec}^2 bx . dx = -\frac{a}{b} \cot bx$
$\int \sin x . dx = -\cos x$	$\int \frac{\sin x}{\cos^2 x} dx = \sec x$
$\int \tan x . dx = \log \sec x$	$\int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x$

EXERCISES ON CHAPTER I.

1. A masonry wall is trapezoidal in section, one face of the wall being vertical. Height of wall, 20 feet; thickness at top, 4 feet; thickness at base, 9 feet. The masonry weighs 150 lb per cubic foot. Find the weight of a portion of the wall 1 foot in length.

2. A trapezoidal figure, having equal intervals of 10 feet each, has ordinates in feet as follows: 0, 100, 140, 120, 80, 0. Find the total area in square feet.

3. Draw a parabolic curve on a base $a = 60$ feet; the height y feet of the curve at any distance x from one end of the base is given by

$$y = 2x - \frac{x^2}{30}.$$

Find the area by application of Simpson's rule; check the result by use of the rule: area $= \frac{2}{3}ab$, where b is the maximum height of the curve.

4. Write down the differential coefficients of the following:

(a) $y = 5x^3$.

(b) $y = 3x^2 - 7x^5$.

(c) $y = 2 \sin x - 3 \cos x$.

(d) $y = \sin^2 x + \cos^2 x$.

(e) $y = \sin^3 x + \cos^3 x$.

(f) $y = 3 \tan x - \cos x$.

5. In Question 3, from a point M on the base, distant 15 feet from one end, draw a perpendicular to cut the curve at a point P. At P draw a tangent to the curve cutting the base produced in a point T. Measure PM and MT and evaluate the ratio $\frac{PM}{MT}$. The result gives the differential coefficient for the curve at P; compare this result with that obtained by differentiation of $y = 2x - \frac{x^2}{30}$ and putting $x = 15$ in the expression for $\frac{dy}{dx}$. How do you account for the discrepancy, if any?

6. Take the equation $y = (4 - x)x$. Find the value of x for which y attains its maximum value, and find also the maximum value of y. Check your result by plotting a graph from the equation.

7. Write down the indefinite integrals of the following:

(*a*) $3x^2 dx$.

(*b*) $(4x^3 - 2x^2)dx$.

(*c*) $(2 - x)^2 dx$.

(*d*) $(2x^2 + \cos x)dx$.

(*e*) $0{\cdot}4\frac{d\theta}{\theta}$.

8. Find the value of the following expression when $R_1 = 12$ inches and $R_2 = 6$ inches. No constant of integration is required.

$$I = 2\pi \int_{R_2}^{R_1} r^3 . dr.$$

9. Find the value of the following expression when $B = 4$ inches and $H = 8$ inches. No constant of integration is required.

$$I = B \int_{-\frac{1}{2}H}^{+\frac{1}{2}H} y^2 . dy.$$

CHAPTER II.

FORCES ACTING AT A POINT.

Representation of a force. Any force is specified completely when we are given the following particulars: (*a*) its magnitude, (*b*) its point of application, (*c*) its line of direction, (*d*) its sense, *i.e.* to state whether the force is pushing or pulling at the point of application.

A straight line may be employed to represent a given force, for it may be drawn of any length, and so represent to a given scale the magnitude of the force. The end of the line shows the point of application, the direction of the line gives the direction, and an arrow point on the line will indicate the sense of the force. Thus a pull of 5 lb. acting at a point O in a body (Fig. 13) at 45° to the horizontal would be completely represented by a line OA, of length $2\frac{1}{2}''$ to a scale of $\frac{1}{2}''$ to a lb., and an arrow point as shown. OA is called a **vector**; any physical quantity for which a line of direction must be stated in order to have a complete specification is called a **vector quantity**. Other quantities, such as mass and volume, into which the idea of direction does not enter, are called **scalar quantities**.

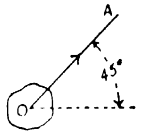


FIG. 13.—Representation of a force.

The expression "force acting at a point" must not be taken literally. No material is so hard that it would not be penetrated by even a very small force applied to it at a mathematical point. What is meant is that the force may be imagined to be concentrated at the point in question without thereby affecting the condition of the body as a whole.

Forces acting in the same straight line. A body is said to be in **equilibrium** if the forces applied to it balance one another. Thus, if two equal and opposite pulls P, P (Fig. 14) be applied at a point O in a body, both in the same straight line, they will evidently balance one another, and the body will be in equilibrium.

Examples of this principle occur in ties, and in struts and columns.

Ties are those parts of a structure intended to be under pull (Fig. 15), **struts** and **columns** are those parts intended to be under push (Fig. 16). These parts remain at rest under the action of the equal and opposite forces applied in the same straight line.

It is impossible for a single force to act alone. To every force there must be an equal and opposite force, or what is exactly equivalent to an equal and opposite force. The term **reaction** is often used to distinguish the resistance offered by bodies to which a given body is connected when forces are applied to the latter body. An example of the use of the term will be found in the reactions of the piers supporting a bridge girder. Loads applied to the girder are balanced by the reactions of the piers.

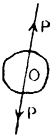


FIG. 14.—Two equal opposite forces

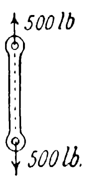


FIG. 15.—Equilibrium of a tie.

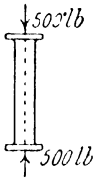


FIG. 16.—Equilibrium of a column

If several forces in the same straight line act at a point, the point will be in equilibrium if the sum of the forces of one sense is equal to the sum of those of opposite sense. Calling those forces of one sense positive and those of opposite sense negative, the condition may be expressed by stating that the algebraic sum of the given forces must be zero. Thus, the forces P_1, P_2, P_3, etc. (Fig. 17), will balance, provided

$$P_1 + P_2 - P_3 - P_4 - P_5 = 0.$$

or,

$$\Sigma P = 0,$$

the interpretation being that the algebraic sum of all the forces of which one only is given as a type immediately after the symbol Σ must be equal to zero.

Suppose in a given case it is found that the algebraic sum of the given forces is not zero. We may infer from this that a single force may be substituted for the given forces without altering the effect. Thus, in Fig. 18, calling forces of sense from A towards B positive, we have

$$2 + 3 + 5 - 8 - 1 = +1.$$

The given forces can be replaced by a single force of 1 lb. weight of sense from A towards B. The single force which may be substituted for a given system of forces without altering the effect on the body is called the **Resultant** of the system. To find the resultant R of the system we have been considering above, we have

$$\Sigma P = R.$$

The resultant R may be balanced by applying an equal opposite force in the same straight line, and, since R is equivalent to the given system of forces, the same force would also balance the given system.

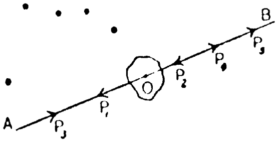


FIG. 17.—Forces in the same straight line

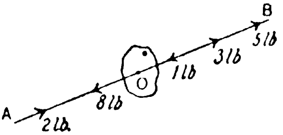


FIG. 18.

Any force which balances a given system of forces is called the **equilibrant** of the system. Thus, the equilibrant E of the system shown in Fig. 18 is a force of 1 lb. weight of sense from B towards A.

Two intersecting forces. To find the resultant of two intersecting forces, the following construction may be employed. Let P and Q be two pulls applied to a nail at O (Fig. 19(*a*)), their joint tendency will be to carry the nail upwards to the right, and the resultant must produce exactly the same tendency. Set off, in the direction in which P acts, OA, to some suitable scale, equal to P, and OB, to the same scale, equal to Q and in the direction in which Q acts. Complete the parallelogram OACB, and draw its diagonal OC. This diagonal will represent R completely, the magnitude being measured by the length of OC to the same scale. The method is called the **parallelogram of forces.** P and Q are called **components** of R.

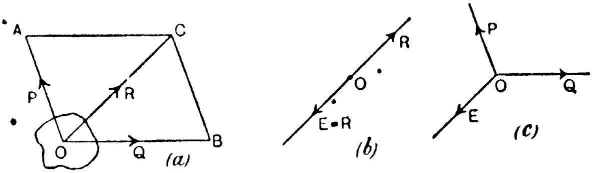


FIG. 19.—Resultant and equilibrant of two intersecting forces

As R is equivalent in its effects to P and Q jointly, we may apply either P and Q together, or R alone, without altering the effect on the nail. This may be expressed by stating that the resultant may be substituted for the components, or *vice versa*.

Substituting R for P and Q (Fig. 19 (*b*)), we may balance R by applying an equilibrant E = R as shown. Again, replacing R by P and Q (Fig. 19 (*c*)), it will be evident that P, Q and E are in equilibrium.

Experimental verification. The most satisfactory proof that the engineering student can have of the truth of the parallelogram of forces is experimental.

EXPT. 1.—**Parallelogram of forces.** In Fig. 20 is shown a board attached to a wall and having three pulleys A, B and C capable of being clamped to any part of the edge of the board. These pulleys should run very easily. Pin a sheet of drawing paper to the board. Clamp the pulleys A and B in any given positions. Tie two silk cords to a split key ring, pass a bradawl through the ring into the board at O, and lead the cords over the pulleys at A and B. The ends of the cords should have scale pans attached, in which weights may be placed. Thus, known forces P and Q are applied to the ring at O. Take care in noting these forces that the weight of the scale pan is added to the weight you have placed in it. Mark carefully the directions of P and Q on the paper, and find their resultant R by means of the parallelogram O*abc*. Produce the line of R, and by means of a third cord tied to the ring apply a force E equal to R, bringing the cord exactly into the line of R by using the pulley C clamped to the proper position on the board. Note that the proper weight to place in the scale pan is E less the weight of the scale pan, so that weight and scale pan together equal E. If the method of construction is correct, the bradawl may be withdrawn without the ring altering its position.

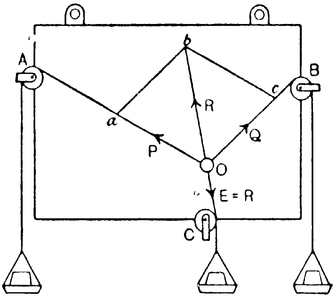


FIG. 20.—Apparatus for demonstrating the parallelogram of forces

In general it will be found that, after the bradawl is removed, the ring may be made to take up positions some little distance from O. This is due to the friction of the pulleys and to the stiffness of the cords bending round the pulleys, giving forces which cannot easily be taken into account in the above construction.

Notice that, before attempting to apply the parallelogram of forces, **both given forces must be made to act either towards or from the point of application.** Thus, given P′ pushing and Q pulling at O (Fig. 21), the tendency will be to carry O downwards to the right. Substitute P = P′, pulling at O for P′; complete the parallelogram OACB, when OC will give the resultant R.

It will also be noticed that any one of the forces P, Q and E (Fig. 19 (*c*)) will be equal and opposite to the resultant of the other two if the three forces are in equilibrium.

Rectangular components of a force. Very frequently it becomes useful in a given problem to deal with the components of a given force instead of using the force itself. These components are generally taken along two lines at 90° intersecting on the line of the given force. Thus, given P acting at O (Fig. 22), and two lines OA and OB at 90° intersecting at O, and in the same plane as P. The components will be found by making OC equal to P, and completing the parallelogram of forces OBCA, which in this case is a rectangle. S equal to OB and T equal to OA will be the rectangular components of P.

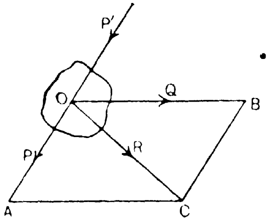


Fig. 21.—Parallelogram of forces applied to a push and a pull.

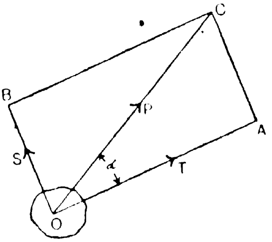


Fig. 22.—Rectangular components of a force.

The following will be seen easily from the geometry of the figure:

$$OC^2 = OA^2 + AC^2$$
$$= OA^2 + OB^2;$$
$$\therefore\ P^2 = S^2 + T^2.$$

Also, let the angle COA = α, then

$$\frac{OA}{OC} = \cos\alpha,$$

$$OA = OC.\cos\alpha;$$

$$\therefore\ T = P.\cos\alpha.$$

Again,

$$\frac{AC}{OC} = \sin\alpha,$$

$$AC = OB = OC.\sin\alpha,$$

$$S = P.\sin\alpha.$$

Triangle of forces. It will now be understood that the conditions which must be fulfilled in order that three forces whose lines intersect may be in equilibrium are: (*a*) the forces must all be in the same plane, *i.e.* **uniplanar**; (*b*) their lines must intersect in the same point; (*c*) any one of them must be equal and opposite to the resultant of the other two forces.

Condition (*c*) may be stated in another manner. In Fig 23, P and Q have a resultant R, found by the parallelogram of forces OACB. A force E has been applied equal and opposite to R as shown; hence the forces E, P and Q are in equilibrium. The following relation evidently holds:

$$R : Q : P = OC : OB : OA.$$

Note the order in which the letters of the lines have been written, thus, R is represented by OC, not by CO, the order being so chosen as to show the sense of the force. Now E is equal to R, and OA is equal to BC; hence we may write

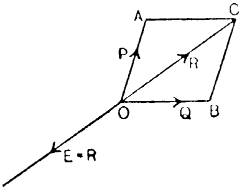


Fig. 23.

$$E : Q : P = CO : OB : BC,$$

OC having been altered to CO so as to give the proper sense to E.

Expressed in words, the proportion states that **the three forces in equilibrium are proportional respectively to the sides of a triangle taken in order.** The triangle OBC in Fig. 23 may be drawn anywhere on the paper, and is called the **triangle of forces** for the forces E, Q, P.

EXAMPLE 1 Given three uniplanar forces P, Q, S′ (Fig. 24) acting at O; test for their equilibrium.

Using a convenient scale of force, draw *ab*, *bc* and *ca′* parallel and proportional respectively to the forces P, Q and S′. If the given forces are in equilibrium, the lines so drawn will form a closed triangle. In Fig. 24, It will be noticed that there is a gap *aa'*. S' will therefore not equilibrate P and Q, but may be made to do so if it is redrawn as S, parallel and proportional to *ca*, the closing line of the triangle *abc*.

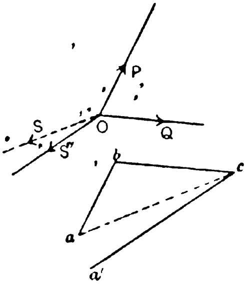


Fig. 24 — Triangle of forces

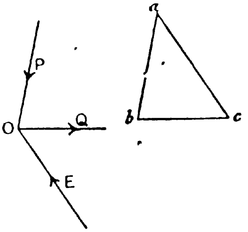


Fig. 25 — Triangle of forces applied to a push and a pull

EXAMPLE 2. Given two forces P and Q (Fig. 25) acting at O; find their equilibrant.

It will be observed that, in applying the triangle of forces, there is no necessity for first making both the given forces pushes or pulls, provided attention is paid to drawing the sides of the triangle in proper order. Thus, draw *ab* to represent P and *bc* to represent Q, then *ca* will represent the equilibrant, which should now be drawn as E acting at O, parallel and proportional to *ca* and of sense shown by the order of the letters *ca*. Note carefully that the problem is not finished until E has been applied on the drawing acting at the proper place O.

EXAMPLE 3. Three given forces are known to be in equilibrium (Fig. 26(*a*)); draw the triangle of forces.

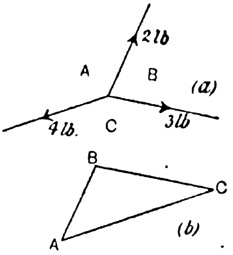


Fig. 26 — Application of Bow's Notation

This example is given to illustrate a convenient method of lettering the forces called **Bow's Notation.** This method will be found to simplify many of the problems which have to be discussed, and consists in giving letters to the spaces instead of to the forces. In Fig. 26(*a*) this plan has been carried out by calling the space between the 4 lb and the 2 lb A, that between the 2 lb and the 3 lb B, and the remaining space C. Starting, say, in space A and crossing over into space B, a line AB (Fig. 26(*b*)) is drawn parallel and proportional to the force crossed, and the letters are so placed that their order A to B represents the sense of that force. Now cross from space B into space C and draw BC to represent completely the force crossed. Finish the construction by crossing from space C into space A, when CA in Fig. 26(*b*) will represent the third force completely.

Examining these diagrams, it will be observed that a complete rotation round the point of application has been performed in Fig. 26(*a*), and that there has been no reversal of the direction of rotation. Also that, in Fig. 26(*b*), if the same order of rotation be followed out, the sides correctly represent the senses of the various forces. Either sense of rotation may be used in proceeding round the point of application, clockwise or anti-clockwise, but once started there must be no reversal.

Relation of forces and angles. In Fig. 27 (*a*) there are three given forces in equilibrium, viz., P, Q and S, and in Fig. 27 (*b*) is shown the triangle of forces for them. From what has been said above, we may write

$$P : Q : S = AB : BC : CA.$$

It is shown in trigonometry that the sides of any triangle are proportional to the sines of the opposite angles. Hence, in Fig. 27 (*b*),

$$AB : BC : CA = \sin\gamma : \sin\alpha : \sin\beta,$$

or,

$$P : Q : S = \sin\gamma : \sin\alpha : \sin\beta.$$

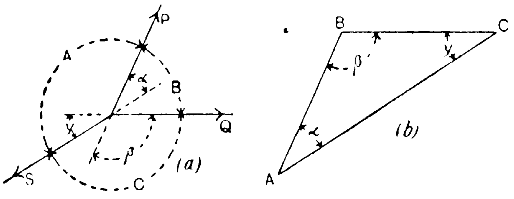


Fig. 27.—Relation of forces and angles

It will be noticed in Fig. 27 (*a*), as shown by dotted lines, that α, β, γ are respectively the angles between the produced directions of S and P, P and Q, and Q and S, also that the angles or spaces denoted by A, B and C in the same figure are the supplements of these angles. As the sine of any angle is equal to the sine of its supplement, we have, in Fig. 27 (*a*),

$$P : Q : S = \sin C : \sin A : \sin B.$$

We infer from this that **each force is proportional to the sine of the angle between the other two forces.**

Any number of uniplanar forces acting at a point. The net effect of such a system of forces may be found by taking components of each force along two rectangular axes which meet in the point of intersection and are in the same plane as the given forces. It is best, in order to comply with the usual trigonometrical conventions regarding the algebraic signs of sines and cosines, to arrange the forces to be either all pulls or all pushes.

In Fig. 28, P_1, P_2, P_3 and P_4 are the given forces acting at O, and OX and OY are two rectangular axes. The angles of direction of the forces are stated with reference to OX as α_1, α_2, α_3 and α_4. Taking components along OX and OY, we have:

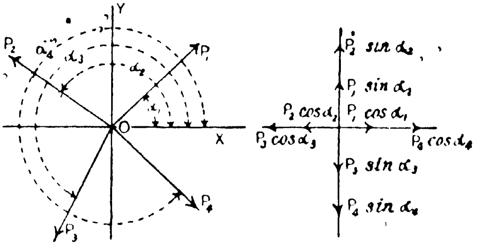


FIG. 28.—System of uniplanar forces acting at a point

Components along OX, $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, $P_3 \cos \alpha_3$, $P_4 \cos \alpha_4$.

Components along OY, $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, $P_3 \sin \alpha_3$, $P_4 \sin \alpha_4$.

Paying attention to the algebraic signs of these, it will be observed that components acting along OX towards the right are positive, and those acting towards the left are negative; also, of the components acting along OY, those acting upwards are positive, while those acting downwards are negative. Each of these sets of components may have a resultant, or they may be in equilibrium. Suppose each to have a resultant, and denote that along OX by R_X, also that along OY by R_Y; then

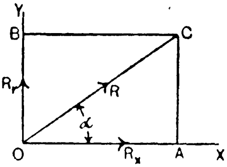


FIG. 29.—Resultant of the system shown in Fig. 28

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4 = R_X,$$
$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + P_4 \sin \alpha_4 = R_Y.$$

Using the abbreviated system of writing these, we have

$$\Sigma P \cos \alpha = R_X, \qquad \ldots \ldots (1)$$
$$\Sigma P \sin \alpha = R_Y. \qquad \ldots \ldots (2)$$

The system being now reduced to two forces R_X and R_Y acting in lines at 90° to each other, we have for the resultant (Fig. 29),

$$R = \sqrt{OA^2 + OB^2}$$
$$= \sqrt{R_X^2 + R_Y^2}. \qquad \ldots \ldots (3)$$

Also,
$$\tan\alpha = \frac{CA}{OA} = \frac{OB}{OA}$$
$$= \frac{R_Y}{R_X} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

It may so happen that either R_X or R_Y may be zero, in which case the resultant of the system is a force acting along either OX or OY, depending upon which of the forces is zero. For equilibrium of the given system both R_X and R_Y must be zero. This condition may be written
$$\Sigma P \cos\alpha = 0, \quad \ldots\ldots\ldots\ldots\ldots (5)$$
$$\Sigma P \sin\alpha = 0; \quad \ldots\ldots\ldots\ldots\ldots (6)$$
a pair of simultaneous equations which will serve for the solution of any problem connected with the equilibrium of any system of uniplanar forces acting at a point.

Graphical solution. A graphical solution of the same problem may be obtained by repeated application of the parallelogram of forces. Thus, given P, Q, S and T acting at O (Fig. 30). First find R_1 of P and S, then R_2 of Q and T by applications of the parallelogram of forces. The resultant R is found by a third application of the parallelogram, as shown. A better solution is obtained by repeated application of the triangle of forces.

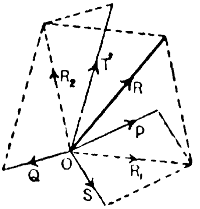


FIG. 30 – Resultant by application of the parallelogram of forces

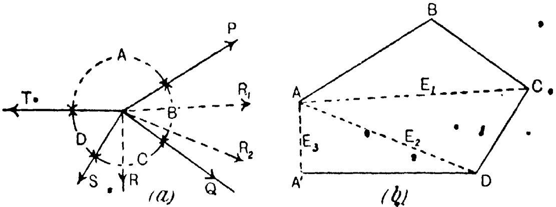


FIG. 31.—Resultant by application of the triangle of forces

In Fig. 31 (*a*), four forces P, Q, S and T are given To ascertain the net effect of the system, first find the equilibrant E_1 of P and Q by the triangle of forces ABC (Fig. 31 (*b*)). E_1 reversed in sense will give R_1, the resultant of P and Q, and is so shown in Fig. 31 (*a*), and is represented by AC in Fig. 31 (*b*) Now find the equilibrant E_2 of R_1 and S by means of the triangle of forces ACD (Fig. 31 (*b*)). E_2 reversed gives R_2, the resultant of R_1 and S, and hence the resultant of P, Q and S. R_2 will be represented in Fig. 31 (*b*) by

A′D). R_2 and T being the only forces remaining in Fig. 31(*a*), their resultant R will be found from the triangle of forces ADA′ (Fig. 31(*b*)), which gives their equilibrant E_3, represented by A′A, and on reversal gives R.

It will be noticed that, had the given forces been in equilibrium, E_3 would have been zero, and A′ would have coincided with A. This case is shown in Fig. 32, giving a closed polygon ABCD, the sides of which, taken in order, represent respectively the given forces. We therefore infer that **a given system of uniplanar forces acting at a point will be in equilibrium, provided a closed polygon can be drawn which shall have its sides respectively parallel and proportional to the given forces taken in order.** Should the polygon not close, then the line required in order to close it will represent the equilibrant of the given forces, and, the sense being reversed, the same line will give the resultant of the given system. The figure ABCD (Fig. 32(*b*)) is called the **polygon of forces** for the given forces. Note, as before, that no problem can be regarded as completed until R or E, as the case may require, is actually shown on the drawing acting at its proper place O.

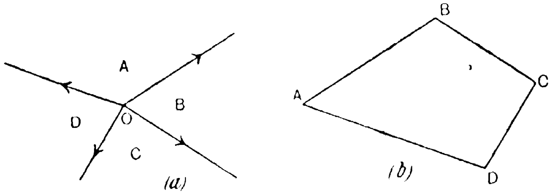


FIG. 32.—Polygon of forces.

EXPT. 2.—**Pendulum.** Fig. 33(*a*) shows a pendulum consisting of a heavy bob at A suspended by a cord attached at B and having a spring balance at F. Another cord is attached to A and is led horizontally to E, where it is fastened. A spring balance at D enables the pull to be read. Find the pulls T and P of the spring balances F and D respectively when A is at gradually increased distances x from the vertical. Check these by calculation as shown below, and plot P and x.

Since P, W and T are respectively horizontal, vertical and along AB, it follows that ABC is the triangle of forces for them. Hence

$$\frac{P}{W} = \frac{CA}{BC} = \frac{x}{h} \text{ (Fig. 33}b\text{)},$$

$$P = \frac{x}{h} W$$

$$= W \tan \alpha \quad \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Also,

$$\frac{T}{W} = \frac{AB}{BC} = \frac{l}{h},$$

$$T = \frac{l}{h} W$$

$$= W \sec \alpha. \quad \ldots \ldots \ldots \ldots (2)$$

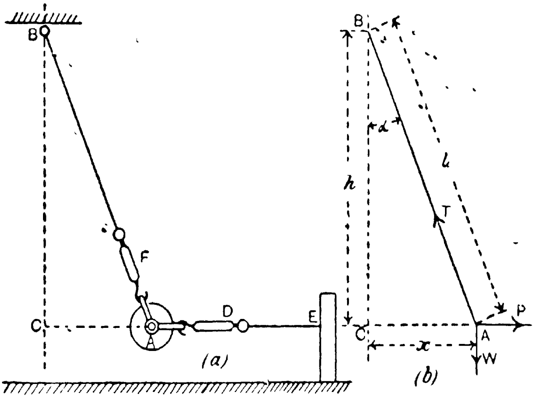


FIG. 33.—Experiment on a pendulum.

Measure l, also x and h, for each position of the bob, and calculate P and T by inserting the required quantities in (1) and (2). Tabulate thus: Weight of bob in lb. = W =

Length of AB in inches = l =

x inches.	h inches.	Calculated values		Observed values from spring balances	
		$P = \frac{x}{h} W$ lb	$T = \frac{l}{h} W$ lb.	P lb.	T lb

The curve will resemble that shown in Fig. 34. Note how nearly straight it is for comparatively small values of x.

EXPT. 3.—**Roof truss.** In Fig. 35 is shown a simple model of a roof truss consisting of two rafters made of wooden bars AB and BC hinged by means of a bolt at B and connected at the bottom by a cord AC, which takes the place of the tie-bar in the actual truss.

Compression spring balances D and E and an ordinary spring balance F enable the forces in the various parts to be measured. C is pivoted by two pointed set screws, as shown in the end elevation, and a roller at A, also shown in end elevation, permits the span of the truss to be altered by adjusting the length of the cord AC. A weight W is hung from B.

Set up the apparatus, and observe the push in each rafter AB and CB, and also the pull in the tie AC. Measure and note the lengths AB, BC and AC when the load is on. Repeat the experiment, using different weights and spans, being careful in each case to note the altered dimensions of the parts. Compare each set of readings with those found by application of the triangle of forces, as shown below.

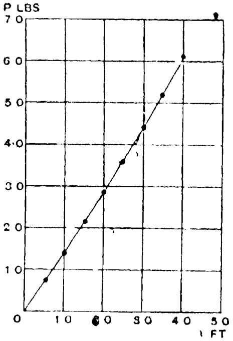


Fig. 34.—Graph of P and l for a pendulum

Make an outline drawing of the truss to scale (Fig. 36(*a*)). If the truss is symmetrical, each rafter will give equal pushes, say P lb., to the joints at B, A and C. The tie will apply equal forces T, T at A and C. The reactions of the supports, R_1 and R_2, may be assumed to be vertical. Considering the forces acting at the point B, which is in equilibrium, and setting off *ab* to represent W (Fig. 36(*b*)), and *ac* and *bc* parallel respectively to AB and BC, we have the triangle of forces *abc* for P, W and P acting at B. Now *ca* represents P acting at B, and *ac* may be taken to represent P of opposite sense acting at A. Draw *cd* parallel to AC. Then the triangle *acd* is the triangle of forces for P, R_1 and T acting at A. In the same way *bcd* is the triangle of forces for P, R_2 and T acting at C. Therefore,

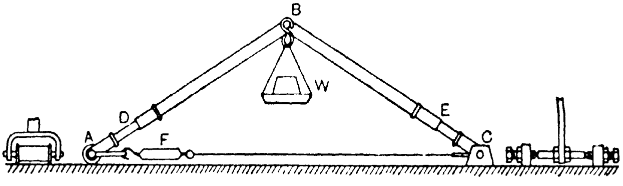


Fig. 35.—Experimental roof truss

$$P = ac,$$
$$T = cd.$$

The results for P and T as obtained from the diagram will agree fairly well with those obtained from the spring balances, provided due allowance be made for the effects of the weights of the various parts before the application of W. To do this, remove W and note the readings of the balances. These readings should be deducted from those taken after W is applied, when the corrected results will show the forces in the parts due to the application of W alone. The results should be tabulated thus:

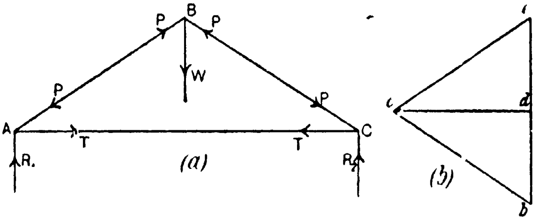


Fig. 36.—Forces in a simple roof truss.

Lengths in inches			Forces in lb from diagram		Corrected forces in lb from spring balances.	
AB	BC	AC	P, push	T, pull.	P, push.	T, pull

From your experiments, give a general statement of how P and T vary for the same value of W, but with increasing lengths of span.

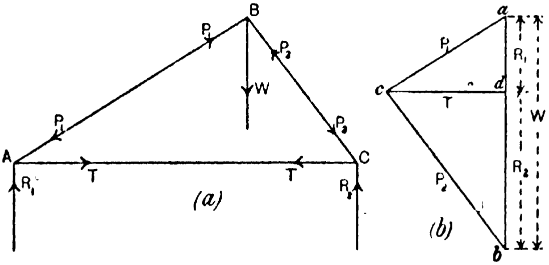


Fig. 37.—Unsymmetrical roof truss.

The case of an unsymmetrical roof truss may be worked out in a similar manner, and is shown in Fig. 37, the lettering of which corresponds with that of Fig. 36.

EXPT. 4.—**Derrick crane.** A derrick crane model is shown in Fig. 38, consisting of a post AB firmly fixed to a base board which

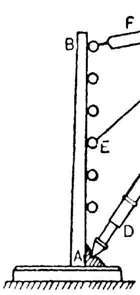


FIG. 38.—Model derrick crane.

is screwed to a table, a jib AC has a pointed end at A bearing in a cup recess, a pulley at C and a compression spring balance at D. A tie BC supports the jib and is of adjustable length; a spring balance for measuring the pull is inserted at F. The weight is supported by a cord led over the pulley at C and attached to one of the screw-eyes on the post. The inclination of the jib may be altered by adjusting the length of BC, and the inclinations of EC and BC may be changed by making use of different screw eyes.

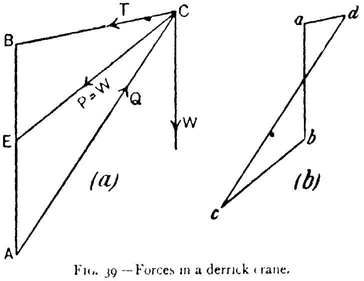


FIG. 39.—Forces in a derrick crane.

Find the push in the jib and the pull in the tie for different values of W and different dimensions of the apparatus by observing the spring balances. Check the results by means of the polygon of forces.

The methods are similar to those adopted for the roof truss (p. 30). It may be assumed that the pulley at C merely changes the direction of the cord without altering the force in it. Hence

P = W (Fig. 39(*a*)). The polygon of forces is shown in Fig. 39(*b*), in which

$$W = ab, \qquad Q = cd,$$
$$P = bc, \qquad T = da.$$

The observed and graphical results should be compared in tabular form as before:

Lengths in inches				Forces in lb from diagram		Corrected forces in lb from spring balances.	
AB	BC	AC	AE	Q, push	T, pull	Q, push	T, pull.

EXPT. 5.—**Wall crane.** A model wall crane is shown in Fig. 40(*a*). Its construction is similar to that of the derrick crane, and the method of experimenting is the same. The outline diagram is given in Fig. 40(*b*), and the polygon of forces in Fig. 40(*c*). These are lettered to correspond with those for the derrick crane, and will be followed readily.

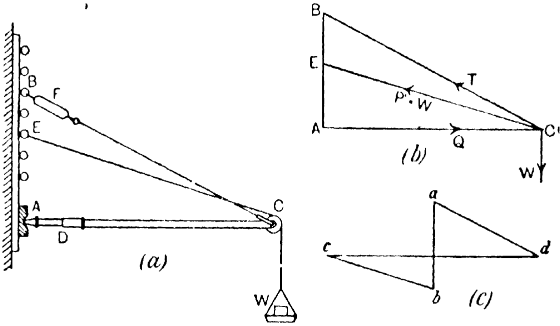
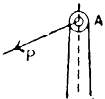


FIG. 40.—Experimental wall crane.

Forces acting at a point but not in the same plane. In Fig. 41 is shown in outline a pair of **sheer legs** such as is used for moving heavy loads. Two legs AB and BC are jointed together at the top B, and are hinged at the ground at A and C so as to be capable of rotating as a whole about the line AC in the plan. The legs are supported in any given position by means of a back leg DE, which is

jointed to ·ll P of 5 lb. and another force Q of unknown magnitude act
moved horizontally balanced by a force of 7 lb. Find the magnitude of Q.
load W is hung fro 3, supposing P and Q
legs; these are shown.
the same vertical plan ig 43) has its arms at
AC = 15 inches, BC = 6
inclined plane which f 35 lb. is applied at A
legs are symmetrical, the another Q is applied equal, and will have a
resultant S, which will fall in the same vertical plane as T and W.



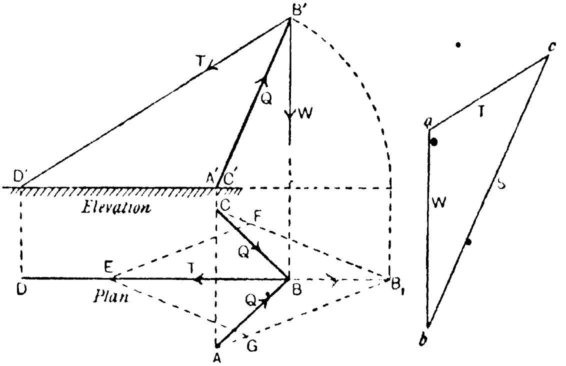


Fig. 41 Forces in a pair of sheer legs

Draw the triangle of forces *abc* by making *ab* represent W, and *bc* and *ca* parallel to A′B′ and B′D′ respectively. Then *ca* gives the pull T in the back leg, and *cb* gives the force S. To obtain the forces Q, Q, rotate the plane of ABC about the line AC, as shown, until it lies on the ground, when the true shape of the triangle ABC will be seen in the plan as AB_1C. Mark off B_1E to represent S, and draw the parallelogram of forces B_1GEF, when the equal lines GB_1 and FB_1 will give the values of the equal forces Q, Q.

A **tripod** is worked out in Fig. 42. Three poles AD, BD and CD are lashed together at their tops, and have their lower ends resting on the ground. Often the poles are equal in length, but for greater generality they have been taken unequal in the example chosen. To draw the plan (Fig. 42), first construct a plan of the triangular base ABC from the given distances between the feet of the poles. Construct the triangles AFC and BEC by making AF equal to the length of the pole AD, BE equal to that of BD, and CF and CE each equal to that of CD. It is clear that AFC and BEC are respectively the

P = W (Fig. 39(*a*)). The polygon of forces is shown in F[illegible]AC and in which

$W = ab$, $Q = cd$, [illegible] the position of

$P = bc$, $T = da$. [illegible] D and perpendicular

The observed and graphical results should be [illegible] Q and S be the forces form as before: [illegible] same vertical plane,

and may be balanced [illegible] Forces in [illegible] in the same vertical plane and also contained by the [illegible] of ADC. The line of Z in the plan will be DG, obtained by producing BD. To obtain a true

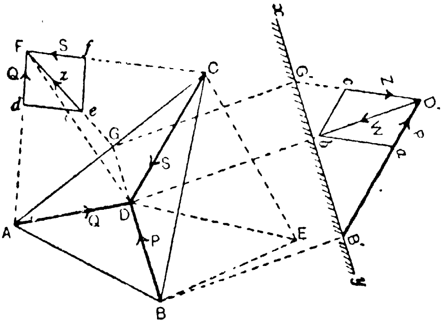


Fig. 42.—Forces in a tripod.

view of the forces P, W and Z, take an elevation on the ground line xy, which is parallel to BD; in this elevation, B′D′ is the true length of the pole BD. The lines of P and Z are shown by B′D′ and G′D′ in this view (Fig. 42). W will be perpendicular to xy, and by making D′b equal to W and drawing the parallelogram D′abc, the values of P and Z will be given by aD′ and cD′ respectively. To obtain Q and S, we have in the plan their lines lying on the ground at AF and CF, and GF will be the line of Z. Make Fe equal to Z and construct the parallelogram Fdef, when Q and S will be given by dF and fF respectively.

EXERCISES ON CHAPTER II.

1. Two pulls are applied to a point, one of 4 lb. and the other of 9 lb. Find graphically the magnitude and direction of the resultant when the forces are inclined to each other at angles of (*a*) 30°, (*b*) 45°, (*c*) 120°. Check your results by calculation.

2. Answer Question 1, supposing the 4 lb. force to be a push.

3. A pull P of 5 lb. and another force Q of unknown magnitude act at 90°. They are balanced by a force of 7 lb. Find the magnitude of Q.

4. Answer Question 3, supposing P and Q intersect at 45°.

5. A bent lever (Fig. 43) has its arms at 90° and is pivoted at C. AC = 15 inches, BC = 6 inches. A force P of 35 lb. is applied at A at 15° to the horizontal, and another Q is applied at B at 20° to the vertical. Find the magnitude of Q and the magnitude and direction of the reaction at C required to balance P and Q.

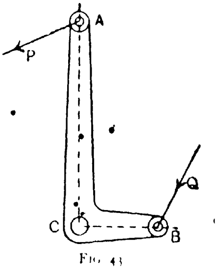


Fig. 43

6. A body weighing 24 lb. is kept at rest on an incline which makes 40° with the horizontal by a force P which is parallel to the plane (Fig. 44). Assume that the reaction R of the plane is at 90° to its surface, and find P.

7. Answer Question 6, supposing P to be horizontal.

8. Four loaded bars meet at a joint as shown (Fig. 45). P and Q are in the same horizontal line; T and W are in the same vertical; S makes 45° with P. Given that P = 15 tons, W = 12 tons, S = 6 tons, find Q and T.

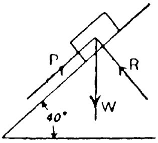


Fig. 44

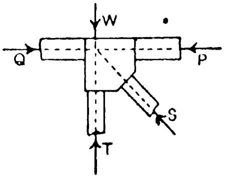


Fig. 45

9. Lines are drawn from the centre O of a hexagon to each of the corners A, B, C, D, E, F. Forces are applied in these lines as follows: From O to A, 6 lb.; from B to O, 2 lb.; from C to O, 8 lb.; from O to D, 12 lb.; from E to O, 7 lb.; from F to O, 3 lb. Find the resultant.

10. Two equal bars AC and BC are hinged at C (Fig. 46). A and B are capable of moving in guides in the straight line AB. A constant force P of 40 lb. is applied at C in a direction at 90° to AB, and is balanced by equal forces Q, Q applied at A and B in the line AB. Calculate the values of Q when the angle ACB has values as follows: 170°, 172°, 174°, 176°, 178°, 179°, 180°. Plot Q and the angle ACB from your results. (The arrangement is called a toggle joint.)

11. Five forces meet at a point O as shown (Fig. 47), and are in equilibrium. In the front elevation, P, Q and S are in the plane of the paper and T is at 45° to the plane of the paper; Q makes 135° with S. In the side elevation T and V are in the plane of the paper. V is perpendicular to the plane containing P, Q and S, and T makes 45° with V. Given Q = 40 tons, T = 25 tons, find P, S and V.

12. In a hinged structure, pieces BO and CO meet at the hinge O, and a force of 2 tons acts upon O in the direction AO. The angle AOB is 115°, BOC is 15° and the angle AOC is 130°; find the forces in the two pieces and say whether they are struts or ties. (B.E.)

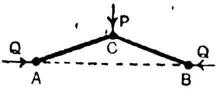


Fig. 46.

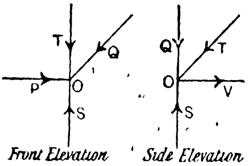


Fig. 47

13. There is a triangular roof truss ABC; AC is horizontal, the angle BCA is 25° and BAC is 55°; there is a vertical load of 5 tons at B. What are the compressive forces in BA and BC? What are the vertical supporting forces at A and C? What is the tensile force in AC? Find these answers in any way you please. (B.E.)

14. Each of the legs of a pair of sheer legs is 45 feet long; they are spread out 23 feet at their base. The length of the back stay is 60 feet. If a load of 40 tons is being lifted at a distance of 15 feet, measured in a perpendicular line from the line joining the feet of the two legs, find the forces in the legs and in the backstay due to this load. (It may be assumed that the load is simply hung from the top of the legs.) (B.E.)

15. A tripod has the following dimensions: The apex point is O, and the lengths of the three legs AO, BO and CO are respectively 18·0 feet, 17·5 feet and 16 feet. The lengths of the sides of the triangle formed by the feet AB, BC and CA are 9·0 feet, 9·5 feet and 10 feet respectively. Find graphically, or in any other way, the forces which act down each leg of the tripod when a load of 10 tons is suspended from it. (B.E.)

16. If a rigid body be acted on by two non-parallel forces whose points of application are different and be kept at rest by a third force, how must this third force act, and what must be its magnitude? A straight light rod xyz is pivoted freely at x, and the point y is attached to a pin w, vertically above x, by a light cord; xy is 3 feet, xw is 4 feet, yw is 2 feet, yz is 2 feet; from z is hung a weight of 30 lb. Find graphically the tension in the cord. (I.C.E.)

17. If three non-parallel forces are in equilibrium, prove that their lines of action must be concurrent. A uniform plank AB has length 6 feet and weight 80 pounds and is inclined at 40° to the vertical. Its lower end A is hinged to a support, while a light chain is fastened to a ring four feet vertically above A and to a point on the plank five feet from A. Find graphically, or otherwise, the tension in the chain and the magnitude and direction of the action of the hinge at A. (The weight of AB may be concentrated at the centre of the plank.) (L.U.)

18. Three cylinders, A, B and C, alike in all respects, are arranged as follows: A and B rest on a horizontal table and their curved surfaces touch one another. C rests on the top, its curved surface being in contact with both A and B. Each cylinder weighs 6 lb. Find, by calculation, the mutual pressure between C and A, also what minimum horizontal forces must be applied to A and B, passing through their axes, in order to preserve equilibrium. Frictional effects are to be disregarded.

19. Three similar spheres rest on a horizontal table and are in contact with each other. A fourth sphere, similar to the others, rests on the top of the three spheres. Each sphere weighs 10 lb. Find the pressure communicated by the top sphere to each of the other three spheres. Neglect frictional effects.

CHAPTER III.

PARALLEL FORCES

Parallel forces. Confining ourselves for the present to two forces only, there are two cases to be considered, viz. forces of like sense and forces of unlike sense. To find the resultant of two parallel forces P and Q (Fig. 48 (*a*)) of like sense, the following method may be employed. Let the given forces act at 90° to a rod, at the points A and B respectively. The equilibrium of the rod will not be disturbed by the application of equal opposite forces S, S, applied in the line of the rod at A and B. By means of the parallelogram of forces *Aha*, find R_1 of P and S acting at A; and by means of the parallelogram of forces *Befd*, find R_2 of Q and S acting at B. Produce the lines of R_1 and R_2 until they intersect at O, and let R_1 and R_2 act at O. Apply the parallelogram of forces *Ohkg* to find R of R_1 and R_2. R will clearly be the resultant of P and Q, and will balance P and Q if its sense be reversed. By measurement it will be found that R is equal to the sum of P and Q.

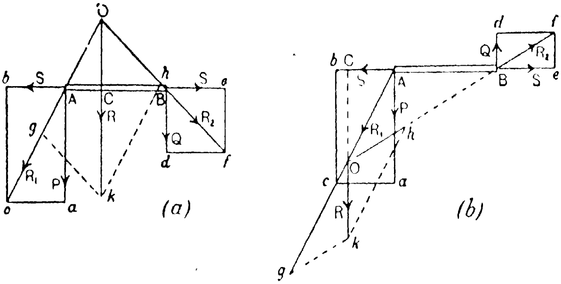


FIG. 48.—Resultant of two parallel forces.

The resultant of two parallel forces of unlike sense may be found by the same process. The construction is shown for two such forces, P and Q, in Fig. 48(*b*), the lettering of this diagram corresponds with that of Fig 48(*a*), and may be followed without further explanation. If the diagram be measured, it will be found that R is equal to the difference of P and Q.

Moment of a force. The **moment of a force means the tendency of a force to turn the body on which it acts about a given axis.** The moment of a given force depends upon (*a*) the magnitude of the force, and (*b*) the length of a perpendicular dropped from the axis of rotation on to the line of action of the force, and is therefore measured by taking the product of these quantities. Thus, in Fig. 49, the body is free to rotate about O, and a force P is acting on it. Draw OM at 90° to P, then

Moment of P = P × OM.

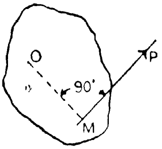


FIG. 49 – Moment of a force

To state the units in which a given moment is measured, both the unit of force employed and the unit of length must be mentioned. Thus, in the above case, if P is in lb. weight and OM in feet, the units will be lb.-feet. Other units which may be used are ton-foot, ton-inch, gram-centimetre, etc.

The sense of the moment of a force is best stated by reference to the direction of rotation of the hands of a clock. Thus the moment of a given force will be **clockwise** or **anti-clockwise** according as it tends to produce the same or opposite sense of rotation as that of the hands of a clock.

Principle of moments. The resultant moment of two or more forces, all of which tend to rotate the body on which they act in the same sense, will be found by first calculating the moment of each force, and then taking the sum. If some of the forces have moments of opposite sense, these may be designated negative, and the resultant moment will be found by taking the algebraic sum. Should the resultant moment be zero, the body will be in equilibrium so far as rotation is concerned. This leads to the statement that **a body will be in equilibrium as regards rotation provided the sum of the clockwise moments applied to it is equal to the sum of the anti-clockwise moments.** This is called the **principle of moments.**

EXAMPLE 1. A horizontal rod AB, the weight of which may be neglected, has a pivot at C (Fig. 50), and has two vertical forces P and Q applied at A and B respectively. Find the relation of P and Q if the rod is in equilibrium.

Let AC = a,
BC = b.

Taking moments about C,

$$\text{clockwise moment} = \text{anti-clockwise moment},$$

$$Q \times b = P \times a,$$

$$\frac{P}{Q} = \frac{b}{a}.$$

It will be seen from this result that the forces are inversely proportional to the segments into which the rod is divided by the pivot. It will also be evident that the equilibrant of P and Q acts through C.

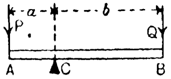


FIG. 50.

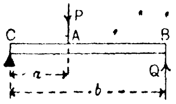


FIG. 51.

EXAMPLE 2. A horizontal rod BC, the weight of which may be neglected, has a pivot at C, and has two vertical forces P and Q of unlike sense applied at A and B respectively (Fig 51). Find the relation of P and Q if the rod is balanced.

Let $AC = a,$

$BC = b.$

Taking moments about C,

$$P \times a = Q \times b,$$

$$\frac{P}{Q} = \frac{b}{a}.$$

Again we may say that each force is proportional to the distance of the other force from the pivot, and that the equilibrant of P and Q acts through C.

EXAMPLE 3. A horizontal rod AB, the weight of which may be neglected, has a weight W applied at C and is supported at A and B, the reactions P and Q being vertical (Fig. 52). Find P and Q.

Let $AB = l,$

$AC = a,$

then $BC = l - a.$

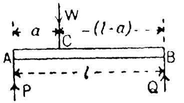


FIG. 52.

Taking moments about B,

$$P \times l = W(l - a) + (Q \times 0),$$

$$P = \left(\frac{l - a}{l}\right) W. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Taking moments about A,

$$W \times a = (Q \times l) + (P \times 0),$$

$$Q = \frac{a}{l} . W. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It is of interest to find the sum of P and Q, using their values as found above; thus

$$\begin{aligned} P+Q &= \left(\frac{l-a}{l}\right)W+\frac{a}{l}W \\ &= W\left\{\left(\frac{l-a}{l}\right)+\frac{a}{l}\right\} \\ &= W\left\{1-\frac{a}{l}+\frac{a}{l}\right\} \\ &= W. \end{aligned}$$

Resultant of two parallel forces. Examining the results of these examples together with what has been said regarding two parallel forces on p. 40, we may state that the resultant of two parallel forces has the following properties:

(1) **The resultant is equal to the sum or difference of the given forces according as they are of like or unlike sense.**

(2) **The resultant is parallel to the given forces and acts nearer to the larger; it falls between the given forces if these are of like sense and outside the larger force if of unlike sense.**

(3) **The perpendicular distances from the line of the resultant to the given forces are inversely proportional to the given forces.**

We may state properties (1) and (3) algebraically:

$$R = P \pm Q, \qquad\qquad (1)$$

$$\frac{P}{Q} = \frac{b}{a} \qquad\qquad (2)$$

A special case. The resultant of two **equal** parallel forces of opposite sense (Fig. 53) cannot be determined from these equations. Here Q is equal to P, hence

$$R = P - P = 0,$$

$$\frac{P}{P} = \frac{b}{a};$$

$$\therefore a = b$$

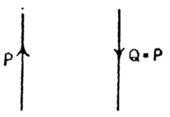


FIG. 53.—A couple.

These results show that no single force can form the resultant of such given forces, and we may infer from this that the resultant effect is to produce rotation solely. The name **couple** is given to this system.

Resultant of a number of parallel forces. In Fig. 54 is shown a horizontal rod AB acted on by a number of parallel vertical forces W_1, W_2, W_3, P and Q. For the rod to be in equilibrium under the action of these forces, the following conditions must be complied with: (1) **the forces must not produce any vertical movement, either upwards**

or downwards: (2) they must not produce any rotational movement, either clockwise or anti-clockwise.

The first condition will be satisfied provided the sum of the upward forces is equal to that of the downward forces; hence

$$W_1 + W_2 + W_3 = P + Q.$$

The second condition will be satisfied if, on taking moments about any point such as A, the sum of the clockwise moments is equal to that of the anti-clockwise moments, hence

$$W_1x_1 + W_2x_2 + W_3x_3 = Pa + Qb.$$

Supposing that it is found that the sum of the downward forces is not equal to that of the upward forces, then the rod may be equilibrated by application of a force E equal and opposite to the difference of these sums; thus $E = (W_1 + W_2 + W_3) - (P + Q)$.

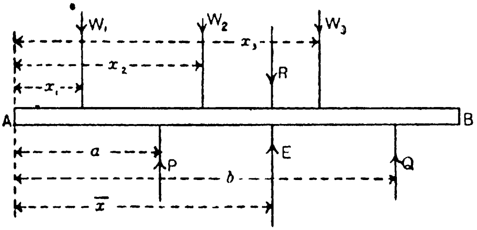


FIG. 54.—Resultant of parallel forces.

The distance $\bar{x}$ from A at which E must be applied (Fig. 54) may be found by taking moments about A; thus

$$E\bar{x} = (W_1x_1 + W_2x_2 + W_3x_3) - (Pa + Qb).$$

Having thus found the magnitude, position and sense of E, the resultant of the given forces may be found by reversing the sense of E.

We have therefore the following rules for finding the resultant of a number of parallel forces P_1, P_2, P_3, etc.

$$R = \Sigma P, \quad \text{.......................} (1)$$

$$R\bar{x} = \Sigma Px, \quad \text{.......................} (2)$$

or,

$$\bar{x} = \frac{\Sigma Px}{R}$$

$$= \frac{\Sigma Px}{\Sigma P}. \quad \text{.......................} (3)$$

Equation (1) will give the magnitude of R, and its position will be given by calculating $\bar{x}$, obtained by taking moments about any convenient point as indicated by equation (2).

EXAMPLE 1. Four parallel forces act on a rod AB as shown (Fig. 55(*a*)). Find their resultant.

$$R = \Sigma P$$
$$= 2 + 5 + 7 + 3$$
$$= \underline{17} \text{ lb., of sense downward.}$$

Taking moments about A, we have

$$R.\bar{x} = \Sigma P x,$$
$$17\bar{x} = (2 \times 2) + (5 \times 5) + (7 \times 6) + (3 \times 7)$$
$$= 92,$$
$$\bar{x} = \frac{92}{17} = \underline{5{\cdot}41} \text{ feet.}$$

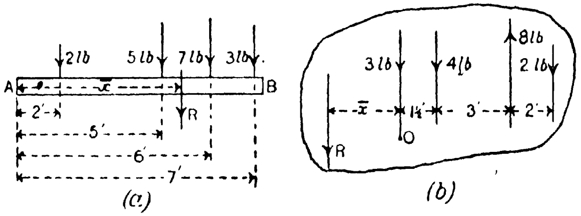


FIG. 55.

EXAMPLE 2. Parallel forces act on a body as shown in Fig. 55(*b*). Find their resultant.

$$R = \Sigma P$$
$$= (3 + 4 + 2) - 8$$
$$= 9 - 8 = \underline{1} \text{ lb., of sense downward.}$$

It is convenient to take moments about a point O on the line of the 3 lb. force.

$$R.x = \Sigma P.x,$$
$$1 \times \bar{x} = (3 \times 0) + (4 \times 1\tfrac{1}{2}) - (8 \times 4\tfrac{1}{2}) + (2 \times 6\tfrac{1}{2}),$$
$$\bar{x} = -\underline{17} \text{ feet.}$$

The negative sign indicates that R falls on the left side of O.

EXAMPLE 3. A beam of 16 feet span rests on supports at A and B and is loaded as shown (Fig. 56(*a*)). Find the reactions of its supports.

Taking moments about B we have

$$P \times 16 = (2 \times 14) + (1 \times 11) + (\tfrac{3}{4} \times 5) + (\tfrac{1}{2} \times 3),$$
$$P = \underline{2{\cdot}765} \text{ tons.}$$

Taking moments about A we have

$$Q \times 16 = (2 \times 2) + (1 \times 5) + (\tfrac{3}{4} \times 11) + (\tfrac{1}{2} \times 13),$$
$$Q = \underline{1{\cdot}484} \text{ tons.}$$

To check the work we have

$$P+Q=\Sigma W,$$
$$2{\cdot}765+1{\cdot}484=2+1+\tfrac{3}{4}+\tfrac{1}{2},$$
$$4{\cdot}249=4{\cdot}25,$$

results which agree within the limits of accuracy of the answers found for P and Q.

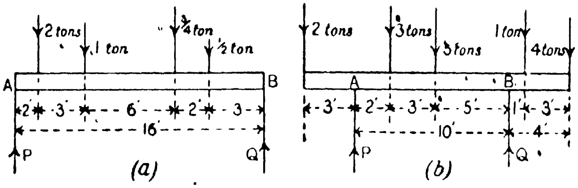


FIG. 56.—Reactions of the supports of beams.

EXAMPLE 4. A beam rests on supports at A and B (Fig. 56(*b*)), its ends overhanging the supports, and the beam is loaded as shown. Find the reactions P and Q.

Taking moments about B, we have

$$(P\times 10)+(1\times 1)+(4\times 4)=(2\times 13)+(3\times 8)+(5\times 5),$$
$$10\,P=75-17,$$
$$P=\underline{5{\cdot}8}\text{ tons.}$$

Taking moments about A, we have

$$(Q\times 10)+(2\times 3)=(4\times 14)+(1\times 11)+(5\times 5)+(3\times 2),$$
$$10Q=98-6,$$
$$Q=\underline{9{\cdot}2}\text{ tons.}$$

To check the work, we have

$$P+Q=2+3+5+1+4,$$
$$15=15$$

Graphical method of finding the reactions of a beam. The method will be illustrated by reference to Fig. 57, which shows a beam simply supported at A and B and carrying a single load W. Taking a base line CD projected from the drawing of the beam, set off CE at right angles to CD, and of length to scale to represent W. Join DE and project W downwards so as to intersect CD and DE in F and G. Taking moments about B, we have

$$Pl=Wb,$$
$$P=\frac{Wb}{l}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the similar triangles ECD and GFD, we have

$$\frac{CE}{CD}=\frac{FG}{FD},$$

or

$$\frac{W}{l}=\frac{FG}{b};$$

$$\therefore\ FG=\frac{Wb}{l}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

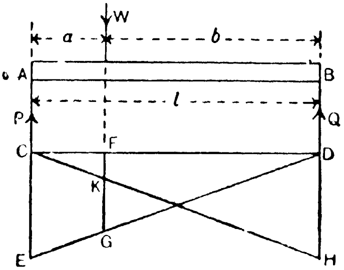


Fig. 57.—Beam carrying one load; reactions found graphically.

Hence FG represents P to the same scale that CE represents W. The value of Q may be found from

$$P+Q=W,$$

$$Q=W-P.$$

Or, by using the same construction, Q may be found by making DH equal to W, joining CH cutting FG in K, when FK gives the value of Q (Fig. 57).

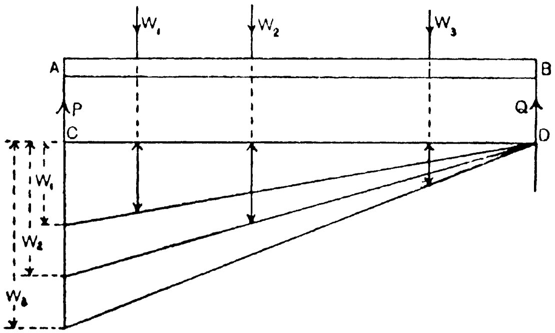


Fig. 58.—Graphical solution of P for a beam carrying several loads.

If the beam carries several loads (Fig. 58), the construction for P should be carried out for each load as indicated; the total sum of the intercepts will give the value of P. Q may be found by means of a similar construction carried out for the other end of the beam, and the result may be checked by comparing the sum of P and Q with the sum of the given loads.

Centre of parallel forces. Let two parallel forces P and Q act on a rod AB (Fig 59). Their resultant R will divide AB in the proportion

$$P : Q = BC : AC. \qquad \ldots (1)$$

Let the lines of P and Q be rotated to new positions P′, Q′, without altering the magnitudes. Through C draw DCE perpendicular to P′ and Q′. Then R′, the resultant of P′ and Q′, will divide DE in inverse proportion to P′ and Q′. Inspection of Fig. 59 will show that the triangles ACD and BCE are similar, hence

$$EC : DC = BC : AC$$
$$= P : Q$$

from (1). It therefore follows that R′ passes through the same point C. This point is called the **centre** of the parallel forces P and Q.

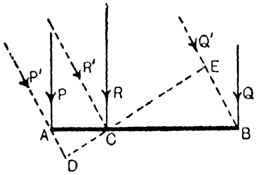


FIG. 59. Centre of parallel forces.

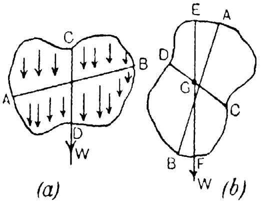


FIG. 60.—Centre of gravity.

If there are a number of parallel forces, it will be seen easily that their resultant always passes through the same point, whatever may be the inclination of the forces. A common example of this occurs in the case of the weight of a body. Each particle in the body possesses weight, hence gravitational effort on the body is really a large number of forces directed towards the earth's centre, and these will be parallel and vertical for any body of moderate dimensions. It is not possible to incline the directions of the forces in this case, but the same effect may be produced by inclining the body. The weights of all particles will still be vertical, but their directions will be altered in relation to a fixed line AB in the body (Fig. 60 (*a* and *b*)). Supposing the line CD of the resultant weight W to be marked on the body in Fig. 60(*a*), and to be marked again as EF in Fig. 60(*b*), the intersection G of these lines of W would be the centre of the weights of the composite particles. The name **centre of gravity** is given to this point.

Centre of gravity by calculation. The general method of calculation will be understood by reference to Fig. 61. The body is supposed to be a thin sheet of material. Take two coordinate axes OX and OY. First let OX be horizontal; the weights of the particles being called w_1, w_2, w_3, etc., and their coordinates (x_1y_1), (x_2y_2), (x_3y_3), etc., we have, by taking moments about O,

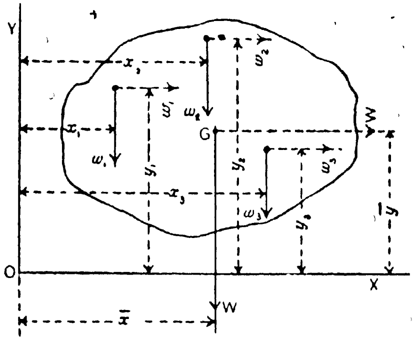


FIG. 61.—Centre of gravity of a thin sheet.

$$(w_1+w_2+w_3+\text{etc.})\bar{x} = w_1x_1+w_2x_2+w_3x_3+\text{etc.},$$

or,

$$\bar{x} = \frac{\Sigma wx}{\Sigma w}.$$

It is evident that Σw gives the total weight W of the sheet, hence

$$\bar{x} = \frac{\Sigma wx}{W}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now turn the sheet round until OY is horizontal; the lines of direction of the weights will be parallel to OX, and, by taking moments about O, we have

$$(w_1+w_2+w_3+\text{etc})\bar{y} = w_1y_1+w_2y_2+w_3y_3+\text{etc.},$$

$$\bar{y} = \frac{\Sigma wy}{\Sigma w}$$

$$= \frac{\Sigma wy}{W}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Draw a line parallel to OX, and at a distance $\bar{y}$ from it, and another parallel to OY at a distance $\bar{x}$; the intersection of these gives the centre of gravity G.

The position of the centre of gravity in certain simple cases may be seen by inspection. Thus for a slender straight rod or wire, it lies at the middle of the length. In a square or rectangular plate it lies at the intersection of the diagonals

A circular plate has G at its geometrical centre. The position of G in a triangular plate may be found by first imagining it to be cut into thin strips parallel to BC (Fig. 62). The centre of gravity of each strip will lie at its centre of length; hence all these centres will lie in DA, where D is the centre of BC, and hence DA contains the centre of gravity of the plate. In the same way, by taking strips parallel to AB, the centre of gravity will lie in CE, where E is the centre of AB. Hence G lies at the intersection of DA and CE, and it is easy to show by geometry that DG is one-third of DA. Hence the rule that G lies one-third up the line joining the centre of one side to the opposite corner.

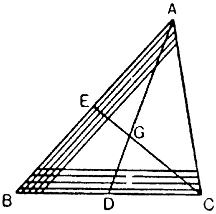


FIG. 62.—Centre of gravity of a triangle.

Advantage is taken of a knowledge of the position of G in thin plates having simple outlines in applying equations (1) and (2) above. The following examples will illustrate the method.

EXAMPLE 1. Find the centre of gravity of the thin uniform plate shown in Fig. 63.

Take axes OX and OY as shown and let the weight of the plate per square inch of surface be w. For convenience of calculation the plate is divided into three rectangles as shown, the respective centres of gravity being G_1, G_2 and G_3. Taking moments about OY, we have

$$w\{(6\times1)+(8\times1)+(3\times1)\}\bar{x}=w(6\times1\times3)+w(8\times1\times\tfrac{1}{2})+w(3\times1\times1\tfrac{1}{2}),$$

$$\bar{x}=\frac{26\cdot5}{17},$$

$$=\underline{1\cdot56}\text{ inches.}$$

Again, taking moments about OX, we have

$$17\bar{y}=(6\times1\times9\tfrac{1}{2})+(8\times1\times5)+(3\times1\times\tfrac{1}{2}),$$

$$\bar{y}=\frac{98\cdot5}{17}$$

$$=\underline{5\cdot8}\text{ inches.}$$

EXAMPLE 2. A circular plate (Fig. 64) 12 inches diameter has a hole 3 inches diameter. The distance between the centre A of the plate and

Take AB produced as OX, and take OY tangential to the circumference of the plate. It is evident that G lies in OX. Taking moments about OY, we may say that the moment of the plate as made is equal to that of the solid disc diminished by the moment of the material removed in cutting out the hole. Let w be the weight per square inch of surface, D the diameter of the plate and d that of the hole. Then

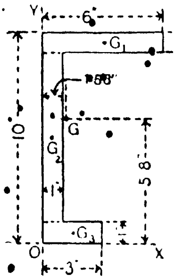


FIG. 63.

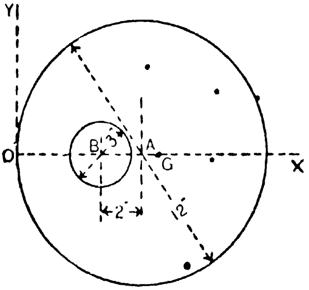


FIG. 64

$$\text{Weight of solid disc} \quad = w\frac{\pi D^2}{4}.$$

$$\text{Weight of piece cut out} = w\frac{\pi d^2}{4}.$$

$$\text{Weight of plate as made} = w\left(\frac{\pi D^2}{4} - \frac{\pi d^2}{4}\right)$$

$$= \frac{w\pi}{4}(D^2 - d^2).$$

Take moments about OY, and let $OG = \bar{x}$,

$$\frac{w\pi}{4}(D^2 - d^2)\bar{x} = \left(w\frac{\pi D^2}{4} \times 6\right) - \left(w\frac{\pi d^2}{4} \times 4\right),$$

$$(D^2 - d^2)\bar{x} = 6D^2 - 4d^2,$$

$$\bar{x} = \frac{6D^2 - 4d^2}{D^2 - d^2}$$

$$= \frac{828}{135} = \underline{6{\cdot}13} \text{ inches.}$$

Other cases of symmetrical solids which may be worked out by application of the same principles are given below.

Any uniform prismatic bar has its centre of gravity in its axis, at the middle of its length.

A solid cone or pyramid has the centre of gravity one-quarter up the line joining the centre of the base to the apex.

A cone or pyramid open at the base and made of thin sheet metal has its centre of gravity one-third up the line joining the centre of the base to the apex.

Graphical method for finding the centre of gravity. The following method of finding G by construction in the case of a thin sheet *abcd* (Fig. 65) is sometimes of service. Join *bd* and find the centres of gravity c_1 and c_2 of the triangles *abd* and *cbd*, join c_1c_2. Again, join *ac*, and find the centres of gravity c_3 and c_4 of the triangles *abc* and *adc*; join c_3 and c_4, cutting c_1c_2 in G, the centre of gravity of the sheet.

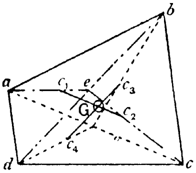


FIG. 65.—Centre of gravity found graphically.

States of equilibrium of a body. The equilibrium of a body will be **stable, unstable** or **neutral,** depending on whether it tends to return to its original position, to capsize, or to remain at rest when it is slightly disturbed from its original position. A body at rest under the action of gravity and supporting forces depends for its state of equilibrium on the situation of its centre of gravity. A cone gives an excellent example of all three states; when resting on its base on a horizontal table the equilibrium is stable (Fig. 66(*a*)), for on slightly disturbing it (Fig. 66(*b*)), R and W conspire to return it to its original position. If resting on its apex, the equilibrium is unstable (Fig. 66(*c*)); a slight disturbance (Fig. 66(*d*)) shows that R and W conspire to overthrow it. If resting on its curved surface on a horizontal table (Fig. 67), the equilibrium is

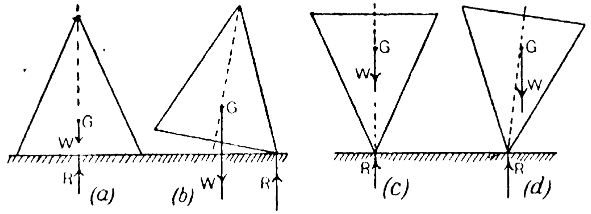


FIG. 66.—Stable and unstable equilibrium.

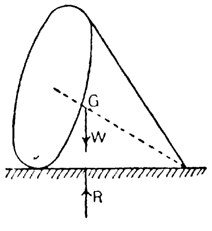


FIG. 67.—Neutral equilibrium.

neutral, for, no matter what the position may be, R and W act in the same vertical line, and so balance.

Reactions of the supports of a beam. In calculating the moment of the weight of a given body about a given axis, we may imagine that the whole weight is concentrated at the centre of gravity. This enables us to deal with problems on beams carrying distributed loads. The following example will make the method clear.

EXAMPLE. A beam is supported at A and B (Fig. 68). The section of the beam is uniform and its weight is 200 lb per foot run. It carries a load of 500 lb. per foot run uniformly distributed over 9 feet of the length as shown. Find P and Q.

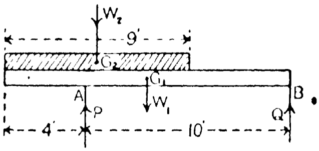


FIG. 68 — Reactions of the supports of a beam

The centre of gravity of the beam lies at G_1 at a distance of 7 feet from B. G_2 is the centre of gravity of the distributed load, and lies at $9\frac{1}{2}$ feet from B.

$$\text{Total weight of beam} = W_1 = 200 \times 14 = 2800 \text{ lb.}$$

$$\text{Total weight of load} = W_2 = 500 \times 9 = 4500 \text{ lb}$$

Apply W_1 at G_1 and W_2 at G_2, and take moments about B to find P:

$$P \times 10 = (W_1 \times 7) + (W_2 \times 9\tfrac{1}{2}),$$

$$P = \frac{(2800 \times 7) + (4500 \times 9\frac{1}{2})}{10}$$

$$= \underline{6235} \text{ lb.}$$

Again, take moments about A to find Q.

$$Q \times 10 = (W_1 \times 3) + (W_2 \times \tfrac{1}{2}),$$

$$Q = \frac{(2800 \times 3) + (4500 \times \frac{1}{2})}{10}$$

$$= \underline{1065} \text{ lb}$$

Checking the results, we have

$$P + Q = W_1 + W_2,$$

$$6235 + 1065 = 2800 + 4500,$$

$$7300 = 7300.$$

Parallel forces not in the same plane. In Fig. 69 are shown four bodies, one at each corner of the horizontal square ABCD. The weights of these bodies act vertically downwards, and hence are not

A cone or pyramid open at the base and made of thin sheet metal has its centre of gravity one-third up the line joining the centre of the base to the apex.

Graphical method for finding the centre of gravity. The following method of finding G by construction in the case of a thin sheet *abcd* (Fig. 65) is sometimes of service. Join *bd* and find the centres of gravity c_1 and c_2 of the triangles *abd* and *cbd*, join c_1c_2. Again, join *ac*, and find the centres of gravity c_3 and c_4 of the triangles *abc* and *adc*; join c_3 and c_4, cutting c_1c_2 in G, the centre of gravity of the sheet.

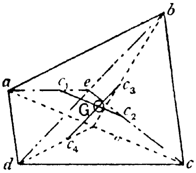


FIG. 65.—Centre of gravity found graphically.

States of equilibrium of a body. The equilibrium of a body will be **stable, unstable** or **neutral,** depending on whether it tends to return to its original position, to capsize, or to remain at rest when it is slightly disturbed from its original position. A body at rest under the action of gravity and supporting forces depends for its state of equilibrium on the situation of its centre of gravity. A cone gives an excellent example of all three states; when resting on its base on a horizontal table the equilibrium is stable (Fig. 66(*a*)), for on slightly disturbing it (Fig. 66(*b*)), R and W conspire to return it to its original position. If resting on its apex, the equilibrium is unstable (Fig. 66(*c*)); a slight disturbance (Fig. 66(*d*)) shows that R and W conspire to overthrow it. If resting on its curved surface on a horizontal table (Fig. 67), the equilibrium is

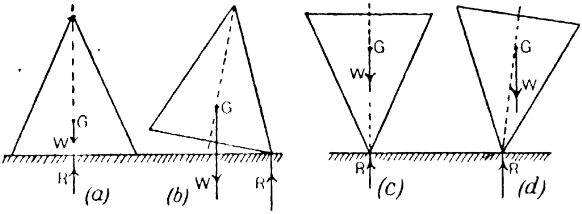


FIG. 66.—Stable and unstable equilibrium.

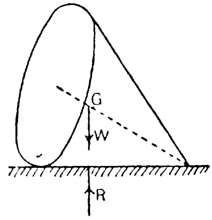


FIG. 67.—Neutral equilibrium.

It will be noticed that if one of the legs, say A, be lifted slightly, the table will rotate in a vertical plane about the line BC. This indicates that the pressure on A may be calculated by taking moments about BC. Draw GM and AN perpendicular to BC, and let P_A be the reaction of the leg A; then,

$$P_A \times AN = W \times GM,$$

$$P_A = W \times \frac{GM}{AN}.$$

In the same way, P_B may be found by taking moments about AC, and P_C by taking moments about AB. The results may be checked from

$$P_A + P_B + P_C = W.$$

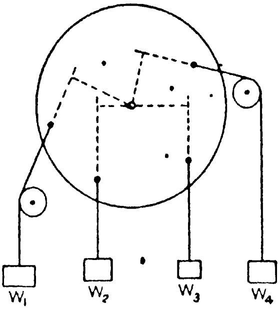


FIG. 71.—Apparatus for illustrating the principle of moments.

EXPT. 6.—**Principle of moments.** Fig. 71 shows a wooden disc which is free to rotate about its centre on a screw driven into a wall board. Attach cords to various points on the face of the disc and apply different forces by means of weights as shown. Let the disc come to rest under the action of these forces, and test the truth of the principle of moments by calculating the sum of the clockwise moments and also that of the anti-clockwise moments.

EXPT. 7.—**Reactions of a beam.** Fig. 72 shows an apparatus consisting of a wooden beam supported by means of two hanging spring balances. Apply various loads and calculate the reactions of the supports. Make allowance for the weight of the beam and also

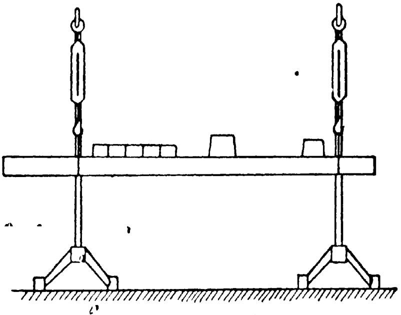


FIG. 72.—Apparatus for determining the reactions of a beam.

for any distributed loads by concentrating them at their respective centres of gravity. Repeat the experiment with altered loads and different positions of the points of support. Make a table showing in each case the calculated reactions and also those read from the spring balances.

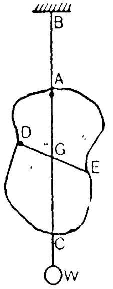


Fig. 73.—Experiment on the centre of gravity of a sheet.

Expt. 8.—**Centre of gravity of sheets.** The centre of gravity of a thin sheet may be found by hanging it from a fixed support by means of a cord AB (Fig. 73); the cord extends downwards and has a small weight W, thus serving as a plumb line. Mark the direction AC on the sheet and then repeat the operation by hanging the sheet from D, marking the new vertical DE. G will be the point of intersection of AC and DE. Carry out this experiment for the sheets of metal or millboard supplied.

Expt. 9.—**Centre of gravity of a solid body.** The centre of gravity of a body such as a connecting rod (Fig. 74) may be found by balancing it on a knife edge, which may be arranged easily by use of V blocks and a square bar of steel. G will lie vertically over the knife edge when the rod is balanced. Carry out this experiment on the bodies supplied, in each case making a sketch of the body and recording on the sketch the dimensions necessary for indicating the position of G.

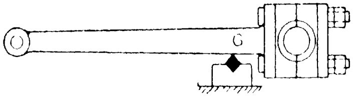


Fig. 74.—Experiment on the centre of gravity of a solid body.

EXERCISES ON CHAPTER III.

1. A uniform horizontal rod AB is pivoted at its centre C, and carries a load of 12 lb at D and another of 20 lb at E. D and E are on opposite sides of C, CD and CE being 8 inches and 12 inches respectively. If balance has to be restored by means of a 14 lb weight, find where it must be placed. What will be the reaction of the pivot?

2. A rod AB carries loads of 3 lb., 7 lb. and 10 lb. at distances of 2 inches, 9 inches and 15 inches respectively from A. Find the point at which the rod will balance. Neglect the weight of the rod.

3. Fig. 75 shows an arrangement of a right-angled bent lever ABC carrying a load of 40 lb. AB and BC are 12 inches and 3 inches respectively and AB is horizontal. C is connected by a horizontal link CE to a vertical lever DF, which is pivoted at D. DF and DE are 15 inches and 3 inches respectively. The arrangement is balanced by a cord FG passing over a pulley at G and carrying a load W. Find W, neglecting the weights of the various parts and also friction.

4. A lever safety valve for a steam boiler has the following dimensions: Diameter of valve, 3 inches; distance from fulcrum to valve centre, 4½ inches; weight of valve and its attachment to the lever, 4½ lb., distance from fulcrum to centre of gravity of the lever, 14 inches; weight of lever, 7 lb. The weight on the end of the lever is 90 lb. Find its distance from the fulcrum if the valve is to open with a steam pressure of 70 lb. per square inch.

5. A uniform beam 20 feet long weighs 1¼ tons, and is supported at its ends A and B. A uniformly distributed load of ½ ton per foot run extends over 10 feet of the length measured from A, and a concentrated load of 3 tons rests at a point 4 feet from B. Find the reactions of the supports by calculation.

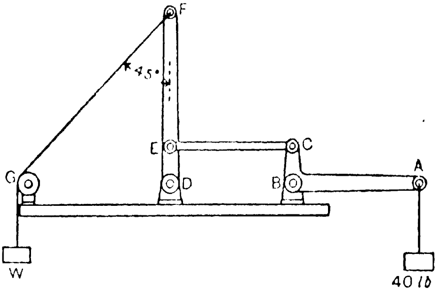


FIG. 75.

6. A beam 18 feet span carries loads of 2 tons, 4 tons and 8 tons at distances measured from one support of 3 feet, 8 feet and 12 feet respectively. Find graphically the reactions of the supports. Neglect the weight of the beam.

7. A uniform beam weighs 2 tons and is 24 feet long. It is supported at a point A 6 feet from one end, and at another point B 4 feet from the other end. There is a concentrated load of 1½ tons at each end and another of 3 tons at the middle of the beam. Find the reactions of the supports by calculation.

8. Three weights of 4 lb., 8 lb. and 12 lb. respectively are placed at the corners A, B and C of an equilateral triangle of 2 feet side. Find the centre of gravity.

9. A letter **L** is cut out of thin cardboard. Height, 3 inches; breadth, 2 inches; width of material, ½ inch. Find the centre of gravity.

10. A solid pyramid has a square base of 3 inches edge and is 5 inches high. It rests on its base on a board, one end of which may be raised. The edges of the base of the pyramid are parallel to the edges of the board, and slipping is prevented by means of a thin strip nailed across the board. Find, by drawing or otherwise, the angle which the board makes with the horizontal when the pyramid just tips over.

11. A flat equilateral triangular plate of 4 feet side is supported horizontally by three legs, one at each corner. A vertical force of 112 pounds is applied to the plate at a point which is distant 3 feet from one leg and 18 inches from another. Determine the compressive force in each leg produced by this load. (B.E.)

12. A scale-pan of a balance with unequal arms is weighted in such a way that the beam is horizontal when no masses are placed in the pans. A body when placed in the two pans successively is balanced by masses P and Q in the opposite pans. Prove that its mass is $\sqrt{PQ}$. (L U.)

13. A horizontal platform is supported on three piers ABC forming a triangle in plan. AB=6 feet; AC=8 feet; BC=8 feet. The centre of gravity of the platform and load carried is distant 5 feet from A and 4 feet from B Find the proportion of the load carried by each of the three piers. Show that, if there were four piers instead of three, the reactions could not be determined without further information (I.C.E.)

CHAPTER IV.

PROPERTIES OF COUPLES. SYSTEMS OF UNIPLANAR FORCES.

Moment of a couple. Consider the couple formed by the equal forces P_1 and P_2 (Fig. 76). Let d be the perpendicular distance, or **arm**, between the lines of the forces. It may be shown, by taking moments in succession about several points A, B, C, D, that **the moment of the couple is the same about any point in its plane, and is given by Pd.**

Thus, taking moments about A, we have

$$\text{Moment of the couple} = (P_1 \times 0) - (P_2 \times d)$$
$$= -P_2 d, \quad . \quad (1)$$

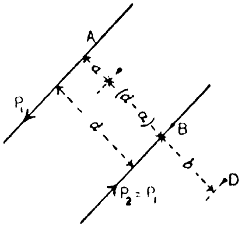


FIG. 76.—A couple has the same moment about any point in its plane.

the negative sign indicating an anti-clockwise moment.

Taking moments about B, we have

$$\text{Moment of the couple} = (P_2 \times 0) - (P_1 \times d)$$
$$= -P_1 d. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Taking moments about C gives

$$\text{Moment of the couple} = -(P_1 \times a) - P_2(d-a)$$
$$= -P_2 d. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Taking moments about D, we have

$$\text{Moment of the couple} = (P_2 \times b) - P_1(d+b)$$
$$= -P_1 d. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

As the forces are equal, the four results are identical, thus proving the proposition.

Equilibrant of a couple. It has been seen (p. 43) that no single force can be the resultant of a couple, hence no single force can equilibrate a couple. It will now be shown that **another couple of equal opposite moment applied in the same plane, or in a parallel plane, will balance a given couple.**

In Fig. 77 are shown two couples, one having equal forces P_1 and P_2, and the other couple having equal forces Q_1 and Q_2. Produce the lines of these forces to intersect at A, B, C and D, and let a and b be the arms of the P and Q couples respectively. From A draw AM and AN perpendicular to P_1 and Q_1 respectively. Then $AM = a$, and $AN = b$. The triangles AMC and AND are similar, hence

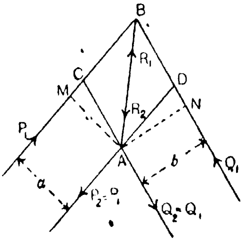


FIG. 77.—Two equal opposing couples are in equilibrium.

$$AC : AM = AD : AN,$$

$$AC : AD = AM : AN$$

$$= a : b. \quad \ldots \quad (1)$$

Now if the couples have equal moments, we have

$$Pa = Qb,$$

or

$$Q : P = a : b. \quad \ldots \quad (2)$$

Hence, AC and AD may be taken to represent Q and P respectively to some scale of force.

As ACBD is a parallelogram, it follows that the resultant of P_1 and Q_1 acting at B will be $R_1 = AB$.

Also the resultant R_2 of P_2 and Q_2 acting at A will be

$$R_2 = BA.$$

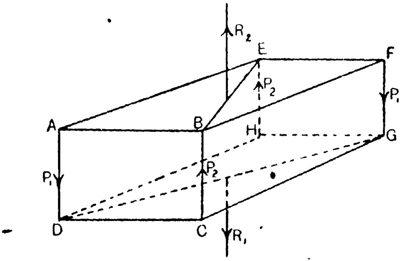


FIG. 78.—Equal opposing couples in parallel planes are in equilibrium.

As R_1 and R_2 are equal, opposite, and in the same straight line, they balance; hence, the given couples are in equilibrium. We may therefore state that couples of equal opposite moment acting in the same plane are in equilibrium, and either couple may be said to be the equilibrant of the other couple.

In Fig. 78 is shown a rectangular block having equal forces P_1 and P_2 applied to its vertical front edges AD and CB, and other equal forces P_1 and P_2 applied to the vertical back edges FG and HE. Let these forces be all equal, when the block will have a pair of equal opposite couples acting in parallel planes. That these couples balance may be seen by taking the resultant R_1 of the forces P_1, P_1, and also the resultant R_2 of the other equal pair P_2, P_2. These resultants are equal and opposite and act in the same straight line, and hence are in equilibrium.

Resultant of a couple. We have now seen that a couple can be balanced by the application in the same plane, or in a parallel plane, of a second couple having an equal opposite moment. Supposing the forces of the second couple to be reversed in sense, it is evident that the effect of this couple on the body will be identical with that of the first couple. We may say now that **either couple is the resultant of the other,** *i.e.* the effect on the body as a whole will be the same, no matter which couple be applied to it.

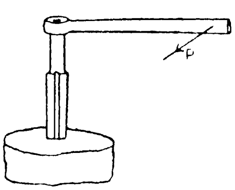


FIG. 79.—Single-handed tap wrench.

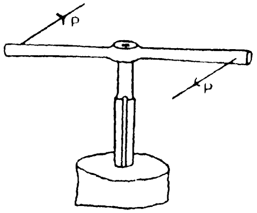


FIG. 80. Double handed tap wrench.

This proposition may be stated in a different way, viz. **a couple may be moved from any given position to another position in the same plane or in a parallel plane, without thereby altering its effect on the body as a whole.**

Owing to the equality of the forces forming a couple, the application of a couple to any body will not tend to move it in any direction, but will merely tend to set up rotation. For example, in tapping a hole, the use of a single-handed tap wrench (Fig. 79) will tend to bend the tap and to spoil the thread; a double-handed wrench enables a couple to be applied giving pure rotation to the tap (Fig. 80). It is evident that the same turning effort may be obtained by means of small forces and a large arm, or by larger forces and a smaller arm, a fact which we may state as follows: **The forces of a couple may be altered in magnitude provided the arm be altered so as to make the moment the same as at first.**

The case of a ship having screw propellers affords an example of the balancing of couples in parallel planes. Referring to Fig. 81, couples are applied to the shaft at A by the engines and, neglecting the friction of the bearings, these couples are balanced by an equal opposite couple produced by the resistance of the water acting on the propeller at E. The planes of these couples are perpendicular to that of the paper and hence are parallel. The distance of A from E is immaterial so far as the equilibrium of the couples is concerned; nor does the diameter of the propeller affect the problem of equilibrium.

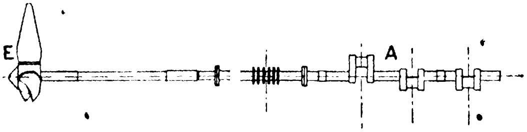


FIG. 81.—Screw propeller shaft.

The law that every force must have an equal opposite force may now be extended by asserting that **it is impossible for a couple to act alone; there must always be an equal opposite couple acting in the same plane, or in a parallel plane.**

Substitution of a force and couple for a given force. In Fig. 82 is shown a body having a force P_1 applied at A. We will suppose that it would be more convenient to have the force applied at another point B. Apply equal opposite forces P_2, P_2, to B, each equal to P_1 and in a line parallel to P_1; these will be self-balancing and will therefore not affect the equilibrium of the body. Let d be the perpendicular distance between P_1 and P_2. P_1 and the equal downward force P_2 at B form a couple, the moment of which is P_2d; this couple may be moved to any convenient situation in the plane, leaving the upward force P_2 at B. **A given force is therefore equivalent to a parallel equal force of like sense together with a couple having a moment obtained as above.**

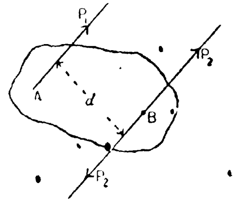


FIG. 82.—Transference of a force to a line parallel to the given line of action.

Substitution of a force for a given force and a given couple. In Fig. 83 we have given a force P acting at A, together with a couple Q, Q, having an arm d. The moment of the couple is Qd. Alter the forces of the couple so that each new force P′, P′ is equal to P, the new arm a being such that

$$P'a = Qd.$$

Apply the new couple so that one of its forces acts at A, in the same line as P, and in the opposite sense. These forces balance at A, leaving a single force P′ acting at a perpendicular distance a from the given force P.

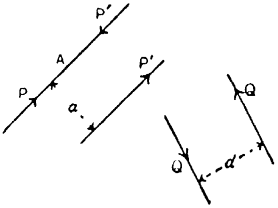


FIG. 83. Reduction of a given force and couple to a single force.

EXAMPLE 1. A single-handed tap wrench has a force of 30 lb. applied at a distance of 15 inches from the axis of the tap (Fig. 84). The centre line of the wrench is at a height of 5 inches above the face of the work being tapped. Find the moment of the couple acting on the tap and also the moment of the force tending to bend the tap.

Transferring P from A to B gives a force P acting at B, together with a couple having a moment given by

$$\text{Moment of couple} = P \times AB$$
$$= 30 \times 15 = \underline{450} \text{ lb.-inches.}$$

The force P acting at B tends to bend the tap about C. To calculate its moment we have

$$\text{Moment of } P = P \times BC$$
$$= 30 \times 5 = \underline{150} \text{ lb.-inches.}$$

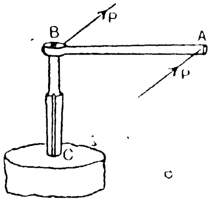


FIG. 84.

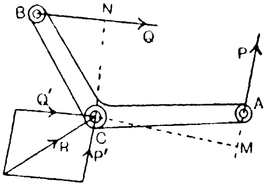


FIG. 85.

EXAMPLE 2. A bent lever ACB (Fig. 85) is pivoted at C, and has forces P and Q applied at A and B respectively. Find the resultant turning moment on the lever and also the resultant force on the pivot.

The solution may be obtained by drawing the lever to scale. Transfer

P and Q to C as shown, giving forces P′=P and Q′=Q acting at C, together with a clockwise couple Q×CN and an anti-clockwise couple P×CM, CN and CM being perpendicular to Q and P respectively. The resultant turning moment may be calculated by taking the algebraic sum of the couples, thus

$$\text{Turning moment} = (Q \times CN) - (P \times CM).$$

This moment will be clockwise if the result is positive.

To obtain the resultant force on the pivot, apply the parallelogram of forces as shown to find the resultant of P′ and Q′ acting at C. R gives the required force.

Equilibrium of a system of uniplanar forces. Any system of forces acting in the same plane will be in equilibrium provided (*a*) **there is no tendency to produce translational movement,** (*b*) **there is no tendency to rotate the body.** These conditions may be tested either by mathematical equations or by graphical methods. To obtain the necessary equations we may proceed as follows.

In Fig. 86, four forces P_1, P_2, P_3, P_4, are given acting in the plane of the paper at A, B, C and D respectively. Take any two rectangular axes OX and OY in the same plane and take components of each force parallel to these axes. Calling the angles made by the forces with OX α_1, α_2, α_3 and α_4, the components will be (see p. 23):

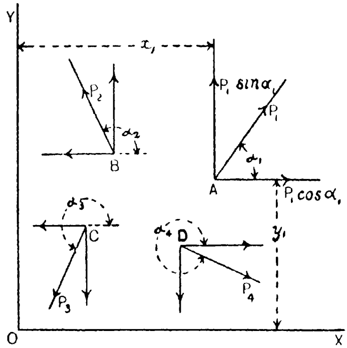


FIG. 86.—A system of uniplanar forces.

Components parallel to OX: $P_1 \cos\alpha_1$, $P_2 \cos\alpha_2$, $P_3 \cos\alpha_3$, $P_4 \cos\alpha_4$.
Components parallel to OY: $P_1 \sin\alpha_1$, $P_2 \sin\alpha_2$, $P_3 \sin\alpha_3$, $P_4 \sin\alpha_4$.

These components may be substituted for the given forces. Now transfer each component so that it acts at O instead of in its given position. This transference will necessitate the introduction of a couple for each component transferred. Let x_1 and y_1 be the coordinates of A, and describe similarly the coordinates of B, C and D. The couples required by the transference of the components of P_1 will be $(P_1 \cos \alpha_1)y_1$ and $(P_1 \sin \alpha_1)x_1$, the other couples may be written in the same manner, giving

Couples parallel to OX:

$$(P_1 \cos \alpha_1)y_1,\ (P_2 \cos \alpha_2)y_2,\ (P_3 \cos \alpha_3)y_3,\ (P_4 \cos \alpha_4)y_4.$$

Couples parallel to OY:

$$(P_1 \sin \alpha_1)x_1,\ (P_2 \sin \alpha_2)x_2,\ (P_3 \sin \alpha_3)x_3,\ (P_4 \sin \alpha_4)x_4.$$

The couples parallel to OX may be reduced to a single resultant couple, by adding their moments algebraically. Similarly, those parallel to OY may be reduced to a single couple, giving

Resultant couple parallel to OX $= \Sigma(P \cos \alpha)y$.

Resultant couple parallel to OY $= \Sigma(P \sin \alpha)x$.

Fig. 87 shows the reduction of the given system so far as we have proceeded, which now consists of a number of forces acting in OX

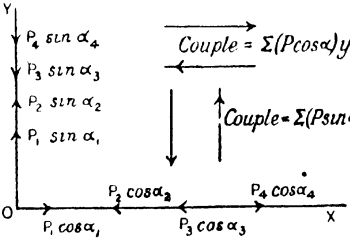


FIG. 87.—A system equivalent to that in Fig. 86.

and OY, together with two couples. For equilibrium, there must be no tendency to produce movement in a direction parallel to OY, hence the algebraic sum of the forces in OY must be zero. This condition may be written:

$$\Sigma P \sin \alpha = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

At the same time there must be no tendency to produce movement in a direction parallel to OX; hence the algebraic sum of the forces in OX must be zero, a condition which may be written:

$$\Sigma P \cos \alpha = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Further, there must be no tendency to produce rotation, a condition which may be secured provided (i) each of the couples is zero, in which case
$$\Sigma(P \sin \alpha)x + \Sigma(P \cos \alpha)y = 0 ;$$
or, (ii) the couples may be of equal moment and of opposite sense of rotation, in which case their algebraic sum will again be zero. Hence the complete condition of no rotational tendency may be written:
$$\Sigma(P \sin \alpha)x + \Sigma(P \cos \alpha)y = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

These equations (1), (2) and (3) being fulfilled simultaneously serve as tests for the equilibrium of any system of uniplanar forces. A little judgment must be exercised in the selection of the coordinate axes OX and OY in any particular problem so as to simplify the subsequent calculations.

EXAMPLE. A roof truss, 20 feet span, 5 feet rise (Fig. 88), has a resultant wind pressure of 2000 lb. acting at C, the centre of the right-hand rafter, in a direction perpendicular to that of the rafter. The truss is bolted down to the support at B, and rests on rollers at A, so that the reaction of the support at A is vertical. Find the reactions of the supports.

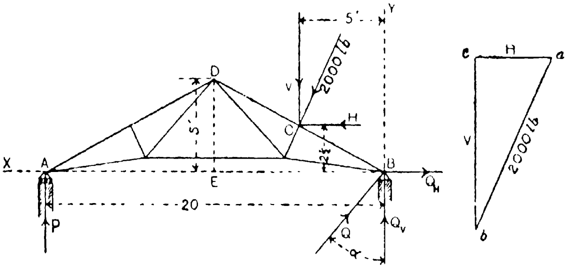


FIG. 88.—Reactions of the supports of a roof truss.

In this case, BX and BY are the most convenient coordinate axes. First find H and V, the components of the load parallel to these axes, by drawing the triangle of forces abc. This triangle is similar to the triangle DBE, hence
$$H : 2000 = 5 : \sqrt{125},$$
$$H = \frac{10000}{11 \cdot 18} = 895 \text{ lb.}$$
$$V : 2000 = 10 : \sqrt{125},$$
$$V = 1790 \text{ lb.}$$

the components of Q parallel to BX and BY respectively Then, from the equations of equilibrium, we have

$$\Sigma P \sin \alpha = 0, \quad \text{hence, } P + Q_V - V = 0,$$

or $$P + Q_V = 1790 \text{ lb} \quad \ldots\ldots\ldots\ldots (1)$$

$$\Sigma P \cos \alpha = 0; \quad \text{hence, } Q_H - H = 0,$$

or $$Q_H = 895 \text{ lb} \quad \ldots\ldots \quad (2)$$

$$\Sigma(P \sin \alpha)x + \Sigma(P \cos \alpha)y = 0, \text{ hence, } (P \times 20) - (V \times 5) - (H \times 2\tfrac{1}{2}) = 0.$$

It will be noted that the last equation is obtained by taking the algebraic sum of the moments of all the forces about B Reducing it, we have

$$20P = (1790 \times 5) + (895 \times 2\tfrac{1}{2}),$$

$$P = \frac{11187.5}{20} = \underline{559.4} \text{ lb} \quad \ldots\ldots\ldots\ldots (3)$$

Substitution of this value of P in (1) gives

$$559.4 + Q_V = 1790,$$

$$Q_V = 1230.6 \text{ lb} \quad \ldots\ldots\ldots\ldots (4)$$

To find Q, we have $$Q = \sqrt{Q_H^2 + Q_V^2}$$

$$= \sqrt{2311000}$$

$$= \underline{1520} \text{ lb} \quad \ldots\ldots\ldots\ldots (5)$$

To find the angle α which Q makes with the vertical, we have

$$\tan \alpha = \frac{Q_H}{Q_V}$$

$$= \frac{895}{1230.6}$$

$$= 0.727;$$

$$\alpha = 36^\circ\ 1'. \quad \ldots\ldots\ldots\ldots (6)$$

Graphical solution by the link polygon. A convenient method of determining graphically the equilibrant of a system of uniplanar forces will now be explained It is required to find the equilibrant of the given forces P_1, P_2 and P_3 (Fig. 89 (*a*)) Take any point A on the line of P_1, and proceed to balance P_1 by the application of any pair of forces p_1 and p_2 intersecting at A. The triangle of forces *ab*O (Fig. 89 (*b*)), in which

$$P_1 : p_2 : p_1 = ab : bO : Oa,$$

will determine the magnitudes of p_1 and p_2. Imagine p_1 and p_2 to be applied at A (Fig. 89 (*a*)) through the medium of bars, or links, one of which, AB, is extended to a point B on the line of action of P_2. To equilibrate this link, it must exert a pull p_2 at B equal and opposite to the pull it gives to A.

The forces p_2 and P_2 acting at B may be equilibrated by the application of a third force p_3 at B, p_3 being found in direction and magnitude from the triangle of forces O*bc* (Fig. 89 (*b*)), in which

$$p_2 : P_2 : p_3 = Ob \,.\, bc : cO.$$

Let p_3 be applied at B (Fig. 89 (*a*)) by means of a link BC, intersecting P_3 at C and exerting at C a force equal and opposite to that which it exerts on B.

The forces p_3 and P_3 acting at C are now equilibrated by means of a force p_4 applied at C, p_4 being found from the triangle of forces O*cd*, in which $$p_3 : P_3 : p_4 = Oc : cd : dO.$$

Let the force p_1 be applied at C by means of a link, and let this link and that in which p_1 acts intersect at D. Each link will exert a force at D equal and opposite to that which it communicates to A and C respectively. The forces p_1 and p_4 thus acting at D may be equilibrated by means of a third force E applied at D, E being found from the triangle of forces O*da*, in which

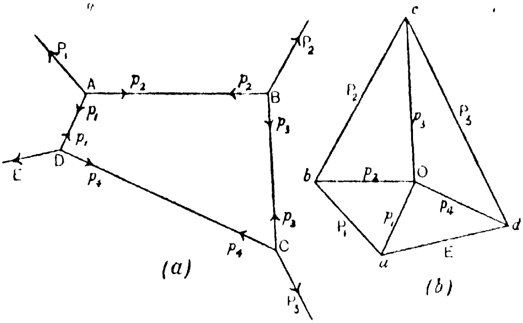


FIG. 89.—Graphical solution by the link polygon.

$$p_4 : E : p_1 = Od : da : aO.$$

It will now be seen, by reference to Fig. 89 (*a*), that **each of the given forces is balanced,** that **the closed link polygon** ABCD **is in equilibrium, and that the force** E **is also balanced.** It therefore follows that the forces P_1, P_2, P_3 and E are in equilibrium.

Reference to Fig. 89 (*b*) will show that *abcd* **constitutes a closed polygon of forces** for P_1, P_2, P_3 and E, and that the lines drawn from O to *a*, *b*, *c* and *d* are parallel respectively to the links in Fig. 89 (*a*). As we had a liberty of choice of the directions of the first two links, viz. DA and AB, and as these directions, once chosen, settled the position of the point O in Fig. 89(*b*), we infer that the position of O is immaterial, the only effect of varying its position being to change somewhat the shape of the link polygon without altering the final value or position of E. It should also be noted that each link in Fig. 89(*a*) is parallel to the line from O in Fig. 89(*b*) which falls between the sides of the force polygon representing the two forces connected by the link in Fig. 89(*a*). Thus, AB in Fig. 89(*a*) is parallel to O*b* in Fig. 89(*b*), the latter falling between *ab* and *bc* which represent P_1 and P_2 respectively.

In practice, Bow's notation is employed. Some examples are given to illustrate the method.

EXAMPLE 1. Given three forces of 3 tons, 4 tons and 2 tons respectively, find their equilibrant (Fig. 90(*a*)).

The principles on which the solution is based are, as has been found above, (*a*) the force polygon must close, (*b*) the link polygon must close.

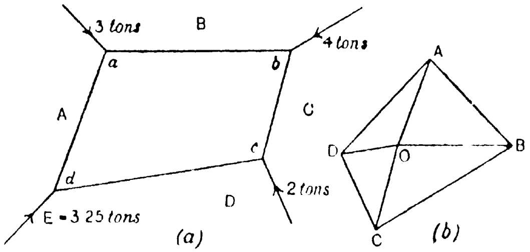


FIG. 90.—An application of the link polygon

Naming the spaces A, B and C, and placing D provisionally near to the force of 2 tons, draw the force polygon ABCD (Fig. 90(*b*)). The closing line DA gives the direction, sense and magnitude of the equilibrant. To find its proper position, take any pole O (Fig. 90(*b*)), and join O to the corners A, B, C, D of the force polygon. Choose any point *a* on the line of the 3 tons force. In space A draw a line *ad*, of indefinite length, parallel to OA in Fig 90(*b*). In space B draw a line *ab* parallel to OB; and, in space C a line *bc* parallel to OC. From *c* draw a line parallel to OD to intersect that drawn from *a* in the point *d*. Then E passes through *d*, and may now be shown completely in Fig 90(*a*).

EXAMPLE 2. Four forces are given in Fig. 91 (*a*); find their resultant. The method employed consists in first finding the equilibrant and then reversing its sense. This example is of slightly greater complication, but the working does not differ from that illustrated above. In Fig. 91 (*b*) ABCDE is the force polygon, the closing side EA represents the equilibrant, hence AE represents the resultant. The position of the equilibrant is found by drawing the link polygon *abcde*, having its sides parallel to the lines radiating from any pole O in Fig. 91 (*b*). The intersection of *ae* and *de* gives a point *e* on the line of action of R.

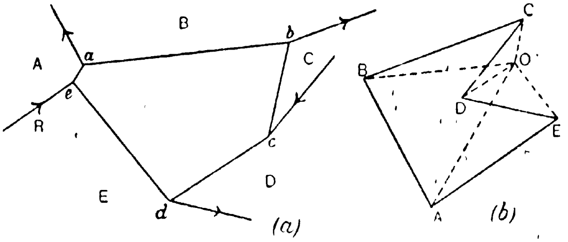


FIG. 91. The resultant determined by the link polygon

EXAMPLE 3. Given a beam carrying loads as shown (Fig. 92 (*a*)); find the reactions of the supports.

In this case, as all the forces are parallel, the force polygon becomes a straight line. The reactions AB and GA being unknown, begin in space B and draw the sides of the force polygon as BC, CD, DE, EF and FG. The corner A of the force polygon will fall on BG, and, its position having been determined, the segments GA and AB will give the magnitudes of the reactions. Choose any pole O and join it to the known corners of the force polygon, viz. B, C, D, E, F, G. Start constructing the polygon from a point *a* on the line of the left-hand reaction (Fig. 92 (*a*)) by drawing *ab* parallel to OB in the space lying between the reaction AB and the force BC. Then draw *bc*, *cd*, *de*, *ef* respectively parallel to OC, OD, OE, OF. From *f*, a point on the force FG, a line *fg* has to be drawn to intersect the reaction GA; as these forces are in the same straight line, it is clear that *fg* is of zero length, and that the link polygon will consequently have a side short. Complete the link polygon by drawing *fa*, and draw OA (Fig. 92 (*b*)) parallel to *fa*. The magnitudes of the reactions may now be scaled as AB and GA.

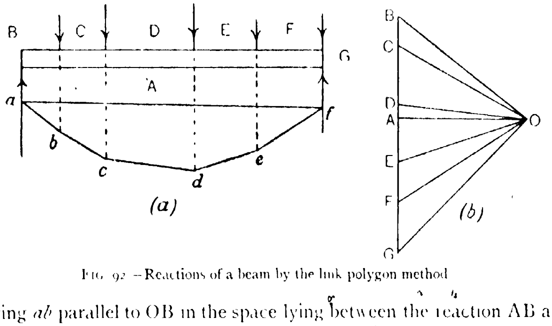


FIG. 92.—Reactions of a beam by the link polygon method

EXPT. 10.—**Equilibrium of two equal opposing couples.** In Fig 93 is shown a rod AB hung by a string attached at A and also to a fixed support at C. By means of cords, pulleys and weights, apply two equal, opposite and parallel forces P, P, and also another pair Q, Q. Adjust the values so that the following equation is satisfied:

$$P \times DE = Q \times FG.$$

Note that the rod remains at rest under the action of these forces.

Repeat the experiment, inclining the parallel forces P, P, at any angle to the horizontal, and inclining the parallel forces Q, Q, to a different angle, but arranging that the moment of the P, P, couple is equal to that of the Q, Q, couple. Note whether the rod is balanced under the action of these couples.

Apply the P, P, couple only, and ascertain by actual trial whether it is possible to balance the rod in its vertical position as shown in the figure by application of any single force.

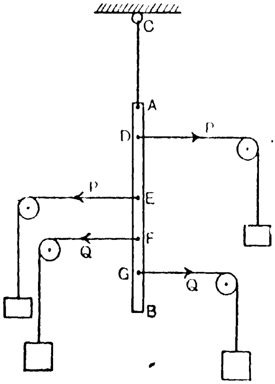


FIG. 93.—An experiment on couples

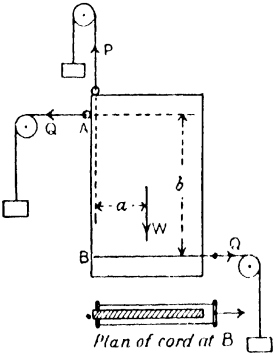


FIG. 94.—Couples acting on a door.

EXPT. 11.—**Couples acting on a door.** Fig. 94 shows a board which may be taken as a model of a door hung on two hinges. The equal forces W and P form a couple, which is balanced by the equal opposing couple Q, Q. Weigh the board, measure a and b, and calculate Q from

$$Qb = Wa.$$

Apply the forces as shown and note whether the door is in equilibrium.

EXPT. 12.—**Link polygon.** Fig. 95 (a) shows a polygon ABCDEA made of light cord and having forces P, Q, S, T and V applied as shown. Let the arrangement come to rest. Show by actual drawing (*a*) that the force polygon *abcdea* closes (Fig. 95(*b*)), its sides being drawn parallel and proportional to P, Q, S, T and V respectively; (*b*) that lines drawn from *a*, *b*, *c*, *d* and *e* parallel respectively to AB, BC, CD, DE and EA intersect in a common pole O.

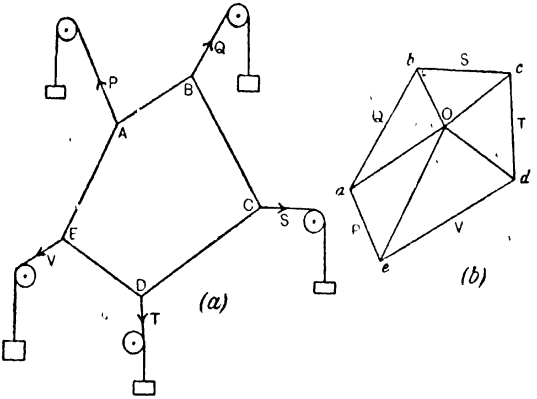


FIG. 95.—An experimental link polygon.

EXPT. 13.—**Hanging cord.** A light cord has small rings at A, B, C and D and may be passed over pulleys E and F attached to a wall board (Fig. 96(*a*)). Weights W_1, W_2, W_3 and W_4 may be attached to the rings, and P and Q to the ends of the cord. Choose any values for W_1, W_2, W_3 and W_4 and draw the force polygon for them as shown at *abcde*. Choose any suitable pole O, and join O to *a*, *b*, *c*, *d*

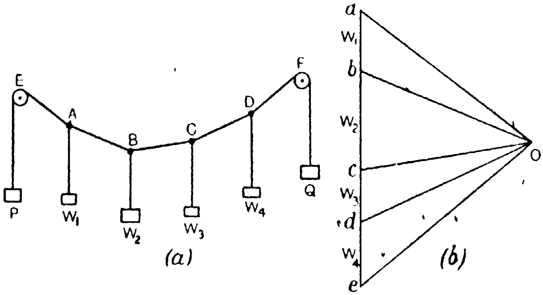


FIG. 96.—A hanging cord.

and *e*. O*a* and O*e* will give the magnitudes of P and Q respectively. Fix the ring at A to the board by means of a bradawl or pin; fix the pulley at E so that the direction of the cord AE is parallel to O*a*; fix the ring at B by means of a pin so that the direction of the cord AB is parallel to O*b*. Fix also the other rings C and D, and the pulley at F so that the directions of BC, CD and DF are parallel to *c*O, *d*O and *e*O respectively. Apply the selected weights W_1, W_2, W_3 and W_4, and also weights P and Q of magnitude given by O*a* and O*e*. Remove the bradawls and ascertain if the cord remains in equilibrium.

EXPT. 14.—**Hanging chain.** Fig. 97 (*a*) shows a short chain ACB in equilibrium under the action of forces V_1, V_2, H_1 and H_2 applied by means of cords, pulleys and weights. Find these forces by calculation, as indicated below, first weighing the chain, and apply them as shown in the figure so as to test for the equilibrium of the chain.

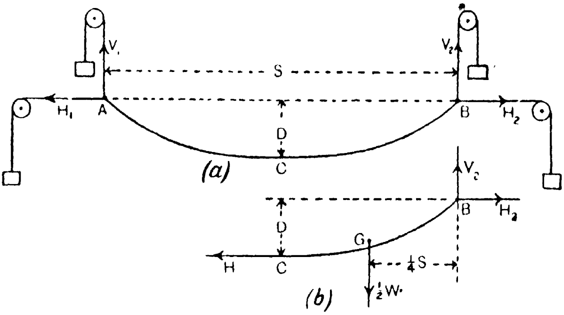


FIG. 97.—Equilibrium of a hanging chain.

Let D = the proposed dip or deflection of the chain in inches.
S = the span AB in inches.

It should be noted that D should not be too large when compared with S. Both may be measured conveniently by first stretching the chain between the two marked positions A and B on the wall board and then taking the required dimensions. It is assumed that A and B are on the same level.

Imagine the chain to be cut at its centre C, and consider the equilibrium of the right-hand half (Fig. 97 (*b*)). The weight of the whole chain being W lb., the weight of the half considered will be $\frac{1}{2}$W and will act at the centre of gravity G, which may be assumed to be at $\frac{S}{4}$ horizontally from B provided D is not too large. As a chain can only pull, the force H at C must be horizontal. Hence the portion BC is at rest under the action of two equal opposing couples, one formed by the equal forces V_2 and $\frac{1}{2}W$ and the other by the equal forces H and H_2. Hence

$$V_2 = \tfrac{1}{2}W \qquad \ldots\ldots(1)$$

and

$$H_2 \times D = \tfrac{1}{2}W \times \frac{S}{4},$$

or

$$H_2 = \frac{WS}{8D} \qquad \ldots\ldots(2)$$

EXERCISES ON CHAPTER IV.

1. A wooden gate weighs 100 lb., and has its centre of gravity situated 21 inches from the vertical axis of the hinges. The hinges are 24 inches apart vertically, and the vertical reaction required to balance the gate is shared equally between them. Calculate the magnitude and direction of the reaction of each hinge and show both reactions in a diagram.

2. A square plate of 2 feet edge has forces of 2, 3, 4 and 5 lb. applied as shown (Fig. 98). Find the force required in order to balance the plate.

3. A plate having the shape of an equilateral triangle of 3 feet edge has forces of 1, 2 and 3 lb. applied as shown (Fig. 99). Find the resultant force on the plate.

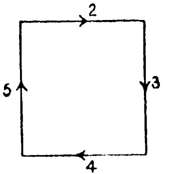


Fig. 98.

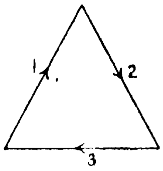


Fig. 99.

4. Suppose the plate in Question 3 to have equal forces of 2 lb. each applied along the edges in the same manner as before. What must be done in order to keep the plate in equilibrium?

5. A uniform beam 12 feet span and 18 inches deep weighs 900 lb. A load of 2 tons is applied to the top surface at 3 feet from the right-hand support at an angle of 45° to the horizontal (Fig. 100). Suppose the left-hand reaction to be vertical, and calculate the reactions of the supports.

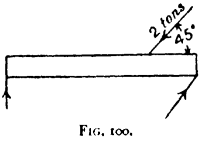


Fig. 100.

6. A beam AB rests against walls at A and B (Fig. 101). Vertical loads of 400 lb. and 600 lb. trisect the beam. Suppose the reaction at A to be horizontal, and calculate the reactions at A and B. Neglect the weight of the beam.

7. A triangular frame 15 feet span and 5 feet high (Fig. 102) carries loads of 400 lb. bisecting AC, 600 lb. at C and 800 lb. bisecting BC at right angles. The reaction at B is vertical. Find the reactions of the supports by calculation.

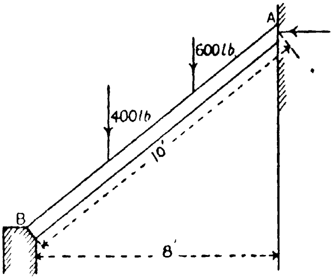


Fig. 101.

8. Prove that two couples of equal opposing moment, acting in the same plane, balance.

9. Show how a force acting at a given point may be moved to another point not in the original line of the force. Prove the method to be correct.

10. Choose any three forces not meeting at a point and not parallel to one another. Show how we can find, graphically, their resultant or their equilibrant. (B.E.)

11. Answer Question 10 in a manner suitable for calculation.

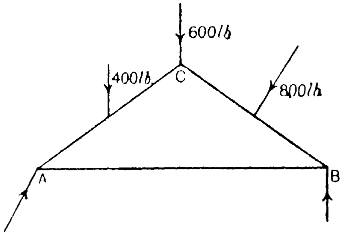


Fig. 102.

12. A number of forces act in a plane and do not meet in a point. Treating them graphically, what is the condition of equilibrium? Prove your statement to be correct. (You are expected to choose more than three forces.) (B.E.)

13. A uniform chain weighs 4 lb., and is hung from two points on the same level. The span is 4 feet and the central dip is 6 inches. Calculate the pulls at the ends of the chain, and show the directions of the chain at the ends.

14. A beam AB of 24 feet span is supported at the ends, and carries vertical loads of 1·5, 2, 3 and 4·5 tons at distances of 3, 6, 12 and 18 feet from the support at A. Use the link polygon method and find the reactions of the supports.

15. Answer Question 6 by construction.

16. Answer Question 7 by construction.

17. ABCD is a square of 2-inch side, BD being a diagonal. A force of 50 lb acts along BC from B towards C; a force of 80 lb acts along CD from C towards D; and a force of 60 lb. acts along DB from D towards B. Replace these forces by two equivalent forces, one of which acts at A along the line AD. Find the magnitude of both these forces and the line and direction of the second. (I.C.E.)

18. Prove that any system of coplanar forces may be replaced by a single force acting at any assigned point and a couple. Forces of 1, 2, 3, 4 lb weight act along the sides of a square taken in order. Find a point such that the forces may be replaced by a single force acting at that point. (L.U.)

CHAPTER V.

SIMPLE STRUCTURES.

Some definitions. A **structure** is an arrangement of various parts constructed in such a manner that no relative motion (other than the small amounts due to the straining of the parts) takes place when the structure is loaded. The simple framed structures considered in this chapter consist of bars assumed to be connected by pin joints and

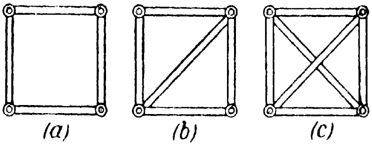


FIG. 103.—Classes of structures (*a*) deficient, (*b*) simply firm, (*c*) redundant.

carrying loads applied at these joints The bars under these conditions will be subjected to simple push or pull in the direction of their lengths, and our object will be to determine the magnitude of the force in each bar, and also whether the bar is under push or pull.

Structures may be **deficient, simply firm,** or **redundant.** Deficient structures are really mechanisms, that is, the parts are capable of considerable relative motion. Fig. 103 (*a*) shows an example of a deficient structure, consisting of four bars connected by pin joints. The arrangement may be made simply firm by the introduction of a single diagonal bar (Fig 103 (*b*)), and will now be capable of preserving its shape under the load. The introduction of a second diagonal bar (Fig. 103 (*c*)) produces a redundant structure In redundant structures, the length of any bar cannot be altered without either a corresponding alteration in the lengths of other bars of the structure, or the production of forces in the other bars Good workmanship is essential in redundant structures to ensure the accurate fitting together of all parts, otherwise some of the bars may require to be forced into position. Unequal heating causes unequal expansion in redundant structures, and therefore introduces forces in the various parts.

In simply firm structures, which form the subject of this chapter, the length of any part may be altered without thereby producing forces in the other parts. Consequently, the effects of unequal expansion are absent. A redundant structure may be converted into a simply firm structure by dropping out one or more of the redundant elements, or parts. Redundancy may be produced by stiff joints. For example, if the square in Fig. 103 (*a*) is made with one stiff joint (Fig. 104 (*a*)), the structure will now be simply firm. Two stiff joints (Fig. 104 (*b*)) will produce a redundant structure having one redundant element; three and four stiff joints in this example give structures of two and three elements of redundancy respectively.

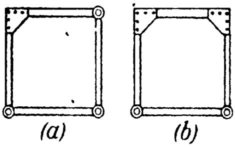


FIG. 104.—Effect of stiff joints.

Conditions of equilibrium. In solving problems concerning any structure, we may separate the forces into two groups, **external** and **internal.** The external forces include all forces applied as loads, or reactions, to the structure. Obviously these forces, acting on the structure as a whole, must be in equilibrium independently of the shape of the structure, or of the form or arrangement of its parts. This consideration enables us to apply the principles of the foregoing chapters to such problems as the determination of the reactions of the supports.

The internal forces include the pushes, or pulls, to which the various bars are subjected when the external forces are applied to the structure. Not only is the structure as a whole in equilibrium, but any bar, or any combination of selected bars in it, must be in equilibrium under the action of any external loads applied to the parts considered, together with the internal forces acting in the selected parts. Usually a joint is selected, when the principle just stated enables us to say that the forces acting at this joint, including external forces, if any, as well as the pushes or pulls of the bars meeting at the joint, are in equilibrium. Hence, the forces in these bars may be found by an application of the polygon of forces.

It should be remembered in applying the polygon of forces that the solution depends on there being not more than two unknowns; these may be either the magnitudes or the directions of two forces, or one magnitude and one direction. In cases where the forces do not abular statement of all inte… indicated on p… ere are three conditions of equilibrium to satisfy, …ther form of roof tru…be three unknowns

The …ss carrying spanning the reactions have been explained fully in the prev… apters; hence in some of the following cases the construction … calculations, for finding the reactions have been omitted.

Simple roof truss. Fig. 105 (*a*) shows a simple roof truss consisting of five bars. There are three loads applied as shown, together with two vertical reactions. To enable the forces to be named, letters are placed as shown for the application of Bow's notation. Thus, the left-hand reaction may be described as AB or BA, and the force in the vertical centre bar may be described as FG or GF, depending on the sense of rotation selected.

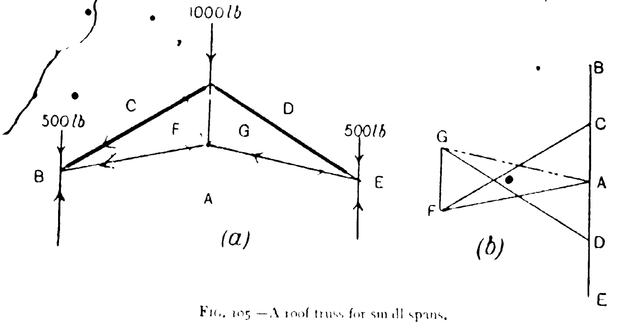


Fig. 105.—A roof truss for small spans.

TABLE OF FORCES.

Name of part	Force in lb		Name of part	Force in lb	
	Push.	Pull		Push	Pull
Reaction AB	1000	-	AF	---	1350
Reaction EA	1000		GA	--	1350
CF	1550		FG	--	575
DG	1550	-			

As all the external forces are vertical, the polygon of forces for the equilibrium of the truss as a whole will be a straight line. In drawing it, we may proceed round the truss either clockwise or anti-clockwise; but, once having settled on the direction, it should be preserved throughout the whole work of solution. Choosing a clockwise direction, the straight line ABCDEA (Fig. 105 (*b*)) will be the polygon for the external forces.

Selecting the joint at the left-hand sup[illegible] causes unequal exp[illegible]es, tw[illegible] of which are completely known, and oth[illegible] introduces forces [illegible]ections alone are known, viz. the forces CF and FA. [illegible] ygon of forces can be drawn. In Fig. 105 (*b*)), proceeding [illegible] of this [illegible] round the joint, AB and BC have been already drawn; draw [illegible]by parallel to the rafter and AF parallel to the tie-bar; these lines inters[illegible] in F and give the closed polygon of forces ABCFA. The force in the rafter may be scaled from CF and that in the tie-bar from FA. Taking these lines in order in relation to the joint under consideration, the sense of the force in the rafter in Fig. 105 (*a*) is CF in Fig. 105 (*b*), and hence is a push; that in the tie-bar has a sense FA, and hence is a pull.

Proceeding now to the top joint of the truss, we see that there are two unknowns, viz. the magnitudes of the forces in GF and DG, hence this joint may be solved by drawing the polygon of forces FCDGF (Fig. 105 (*b*))

Taking now the joint at the right-hand support, and drawing the polygon of forces GDEAG, we find that the closing line AG has its position fixed already on the diagram. This fact provides a check on the accuracy of the whole of the preceding graphical work; if on joining AG in Fig. 105 (*b*), it is found that this line is not parallel to the right hand rafter, some error has occurred, and in order to eliminate it the work must be repeated.

Rule for push or pull. The method of determining whether a bar is under push or pull may be simplified somewhat by developing the following rule from the principle explained above.

Select any bar such as FG, choose the joint at one end of it, say the lower; cross the bar in the same sense of rotation in relation to this joint as was chosen in drawing the force diagrams – in this case clockwise; name the spaces in this order, viz. FG. FG in Fig. 105 (*b*) gives the sense of the force acting at the lower joint. As the force is upwards, the bar is **pulling**.

It makes no difference in the application of the rule which end of the bar is selected. For example, choosing the top joint of the same bar and crossing it again clockwise as regards the upper end, the order is GF. GF in Fig. 105 (*b*) is downwards, hence the bar is **pulling** at the top joint.

It is desirable to indicate on the drawing of the truss which bars are under push and which under pull. Probably the best way of doing this is to thicken the lines of the bars under push. If the whole line is thickened, the direction of the bar will be lost, hence, as shown in Fig. 105 (*a*), a short piece at each end is left thin.

A tabular statement of the forces in the bars should be made in the manner indicated on p 79.

Another form of roof truss. Fig. 106(*a*) shows a common type of roof truss carrying symmetrical loads. There will be no difficulty in

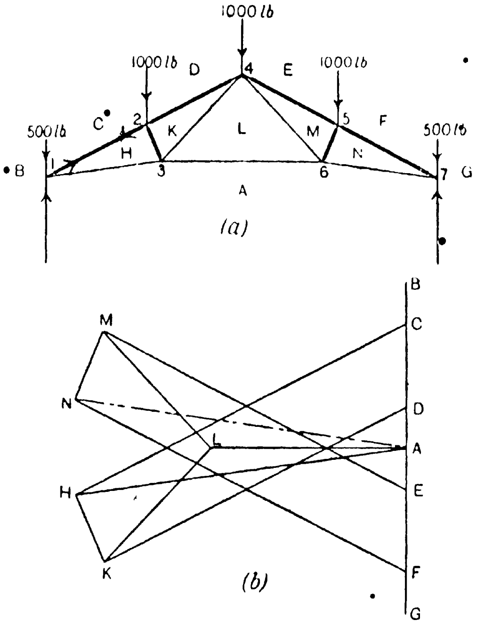


FIG. 106.—Forces in a common type of roof truss; weights only considered.

TABLE OF FORCES.

Name of part	Force in lb Push	Force in lb Pull	Name of part	Force in lb Push	Force in lb Pull
Reaction AB	2000	—	AH	—	4225
Reaction GA	2000	—	AL	—	2480
CH	4675	—	AN	—	4225
DK	4250	—	HK	900	—
EM	4250	—	KL	—	1960
FN	4675	—	LM	—	1960
			MN	900	—

following the diagram of forces (Fig. 106 (*b*)). The order in which the joints have been taken is indicated by the number placed against the joint. The sense of rotation employed is clockwise, and the closing check line is NA.

The effect of wind pressure on the right-hand side of this truss is determined in Fig. 107 (*a*) and (*b*). It is assumed that the wind load produces forces of 500 lb. at the top and bottom ends of the rafter, and of 1000 lb. at the middle, all three being perpendicular to the

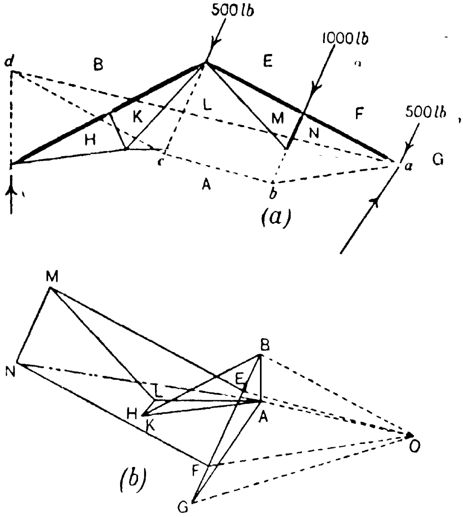


Fig. 107.—Wind acting on the right-hand side of the truss.

TABLE OF FORCES.

Name of part	Force in lb		Name of part	Force in lb	
	Push	Pull		Push	Pull
Reaction AB	560	—	AH	—	1550
Reaction GA, inclined	1510	—	AL	—	1360
			AN	—	3160
BH	1700	—	HK	0	—
BK	1700	—	KL	—	260
EM	2770	—	LM	—	1910
FN	2770	—	MN	1000	—

rafter. As an example of the use of the link polygon, the reactions of the supports have been determined by this method. The left-hand reaction has been assumed to be vertical when that of the right-hand support will be inclined. Wind pressure only has been taken account of in the working. It will be noted that there are three unknown elements in the reactions, viz the magnitude of the left hand reaction and both the magnitude and direction of the right hand reaction. In fact, all that is known of the latter reaction is that it acts through the point *a* Now, in drawing the link polygon, one link must fall between this reaction and the force FG. As the line of the reaction is unknown, it will be impossible to draw this link unless the artifice is adopted of starting the drawing of the link polygon at the point *a*. The effect of this will be that the link in question will have zero length.

First draw as much of the external force polygon as possible; this is shown by BEFG in the force diagram. A will lie in the vertical through B as the reaction AB is vertical. Taking a convenient pole O and joining OB, OE, OF and OG, we start drawing the link polygon by making *ab* (Fig 107 (*a*)), which falls between FG and EF, parallel to OF. *bc* falls between EF and BE, and is made parallel to OE. *cd* falls between BE and AB, and is made parallel to OB. The link parallel to OG is omitted, as it is of zero length, coinciding with *a*. Hence the closing line is *da*; drawing OA parallel to *ad* to intersect the vertical through B in A gives the left hand reaction as AB and the right hand reaction as GA.

The remainder of the diagram giving the internal forces is worked out in the usual manner, NA being the closing line.

An application of the method of graphical moments. The effect of wind pressure on the left-hand side of this truss is determined in Fig. 108. The student will have noted, in applying the link polygon to the problem of finding the reactions, that the lines of the polygon have a tendency to obscure the drawing of the truss. In the case now before us, the method of graphical moments (p. 46) is employed and involves the drawing of very few lines on the truss. The resultant of the three wind loads has been taken as a single force of 2000 lb. applied at *c*. Join *ab*, and with centre *b* and radius *bc* describe an arc cutting *ab* in *d*. Make *ae* equal to 2000 lb. to scale; join *be* and draw *df* perpendicular to *ab* and cutting *be* in *f*. Draw *fg* parallel to *ab*, when *ga* will be the vertical component of the right-hand reaction. The horizontal component of this reaction will be equal and opposite to the horizontal component of the force of 2000 lb. acting at *c*. Draw the triangle of forces *clm*, and make *ha* equal to *mc*; the right-hand reaction will be the resultant *ka* of the components represented by *ga* and *ha*.

The external force polygon (Fig. 108 (*b*)) may now be drawn

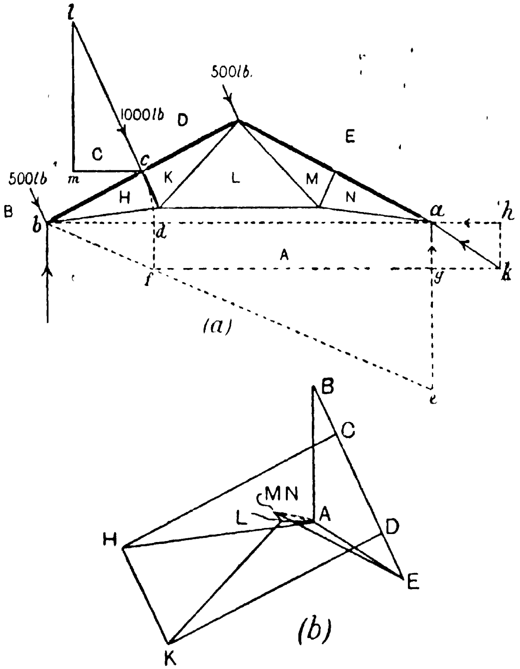


FIG. 108.—Wind acting on the left hand side of the truss.

TABLE OF FORCES.

Name of part.	Force in lb.		Name of part	Force in lb.	
	Push	Pull.		Push.	Pull
Reaction AB	1200	—	AH	—	1870
Reaction EA, inclined	1020	—	AL	—	320
			AN	—	400
CH	2300	—	HK	1000	—
DK	2300	—	KL	—	550
EM	1400	—	LM	—	100
EN	1400	—	MN	0	—

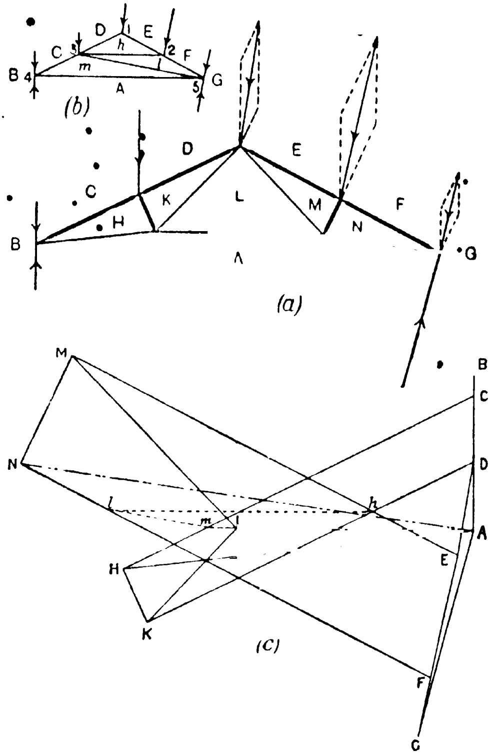


Fig. 109.—Combined dead and wind loads on the truss

TABLE OF FORCES.

Name of part.	Force in lb.		Name of part.	Force in lb.	
	Push	Pull		Push	Pull.
Reaction AB	2560	—	AH	—	5830
Reaction GA, inclined	3350	—	AL	—	3900
			AN	—	7450
CH	6300	—	HK	900	—
DK	5950	—	KL	—	2100
EM	7100	—	LM	—	3840
FN	7500	—	MN	1900	—

for the three wind loads and the two reactions, and is shown by ABCDEA. The internal force diagram is completed as before. Notice that when the wind blows on the right-hand side, no force is induced thereby in HK, and when acting on the left-hand side there is no force in MN. This arises from the fact that there is no external load at the joint in the two cases respectively. Two forces acting in the same straight line, as is the case in the two parts of the rafter, balance, and it is impossible to apply a single inclined force at the point of action without disturbing the equilibrium.

The total force in any bar of the frame due to the dead loads, *i.e.* the weights of the parts of the truss, and to the wind pressure jointly may now be determined by adding the results for the dead load (Fig. 106) and either those of Fig. 107 or of Fig. 108 depending on whether the wind is blowing on the right- or left-hand side.

Combined dead load and wind pressure. As a further example of another method of obtaining the reactions, a diagram has been drawn in Fig. 109 for the combined dead loads and wind load on the right-hand side. The two forces acting at each joint of the right hand rafter have been combined by the parallelogram of forces, and the resultant used as a single force at each joint (Fig. 109 (*a*))

To find the reactions of the supports, we may take advantage of the principle that the external forces balance independently of the arrangement of the parts of the truss. Hence, any other convenient arrangement may be substituted for that given without disturbing the values of the reactions. The substituted frame chosen is sketched in Fig 109 (*b*). It will be seen that it is possible to determine all the forces in its parts without first determining the reactions. Thus, starting at the top joint 1, where there are two unknowns only, we obtain DE*h* in the force diagram (Fig. 109 (*c*)). Proceeding to joint 2, we obtain EF*lh*, at joint 3, CD*hlm* is obtained, and at joint 4 we obtain BC*m*A, thus determining the point A on the force polygon, and hence the reactions AB and GA.

The internal force diagrams for the given arrangement of bars may now be proceeded with, the closing line being NA. It will be observed that the greater part of the lines drawn in the force diagram for the substituted frame are required for the actual frame, hence there has been but little wasted work.

Another form of structure In Fig. 110 (*a*) is shown a structure intended to carry a load at its upper end. Since there is but one vertical load, the reactions of the foundation must reduce to one vertical upward force equal to the load applied. Hence, the polygon or the external forces is completed by drawing AB downwards and BA upwards. It will be noted in this example that it is not necessary to determine the actual reactions of the foundations before finding he forces in the parts. A start can be made at the joint 1, as there are only two unknowns there. The order of solution of the other oints is indicated by the numerals. In drawing the various polygons

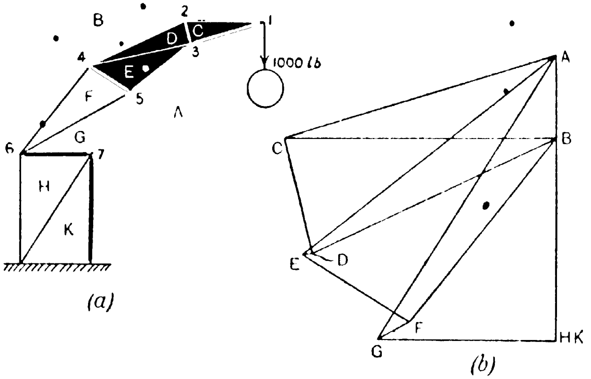


Fig. 110 A braced frame

TABLE OF FORCES

Name of part	Force in lb		Name of part	Force in lb	
	Push	Pull		Push	Pull
AC	3540		BC		3400
AE	3960		BD		3350
AG	4050		BF	—	2850
AK	3440		BH		2440
CD	1420		DE	—	120
EF	1550		FG		450
GH	2220	—	HK	—	0

(Fig. 110 (*b*)), anti-clockwise sense of rotation has been chosen. The student will observe that there is no force in HK, H and K coinciding in the force diagram. It is easy to see that this must be the case from consideration of the fact that the bars BH and AK are vertical, and therefore the vertical forces in them are capable of balancing the external load applied without any aid from the diagonal HK. In fact, the diagonal HK merely serves to steady the frame under the given loading. There would, of course, be a force in this bar if an inclined load were applied to the frame, or if there were a side effort caused by wind pressure.

A larger roof truss. In Fig. 111 is shown a roof truss of a larger type and having a different arrangement of parts from those dealt

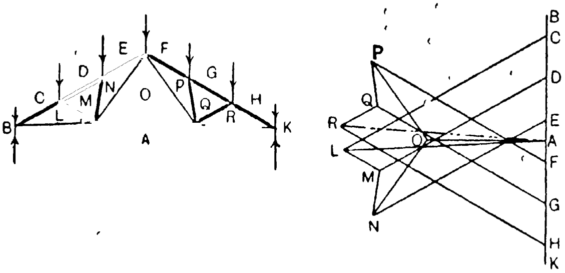


FIG. 111.—A larger roof truss

with previously. This case presents no difficulties, and is included as an example which the student can work out for himself.

The roof truss shown in Fig. 112 (*a*) presents a difficulty which arises frequently. The external force polygon is drawn easily, but

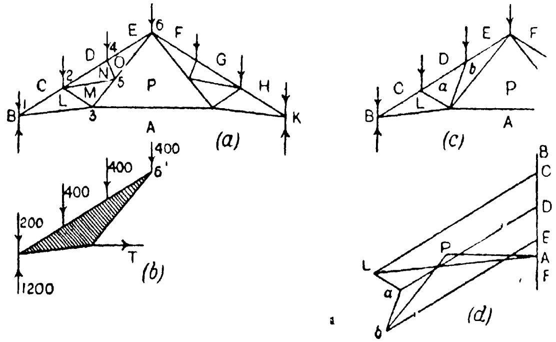


FIG. 112.—A more difficult example of truss.

in drawing the force polygon for the internal forces it will be found that it is impossible to proceed with the drawing after solving point 1. All other points such as 2 and 3 have more than two unknowns, hence the solution cannot be obtained by application of the ordinary methods. We may proceed by either of two methods.

(*a*) It will make no difference whatever in the forces in the remaining part of the truss if we imagine the left-hand portion (shown shaded in Fig. 112 (*b*)) to be solid. Separating this portion as shown, we may calculate T, the force in the bar AP, by taking moments about point 6. Thus, taking the loads as shown, let the half-span be 15 feet and let the perpendicular from point 6 to the line of T be 7·5 feet, then

$$(T \times 7{\cdot}5) + (400 \times 5) + (400 \times 10) + (200 \times 15) = 1200 \times 15.$$

$$T = \frac{18{,}000 - 2000 - 4000 - 3000}{7{\cdot}5}$$

$$= \frac{9000}{7{\cdot}5}$$

$$= \underline{1200}\ \text{lb.}$$

Having found T, the number of unknowns at the point 3 (Fig. 112 (*a*)), will now be found to be two only, hence this point may be solved. The solution for points 2, 5, 4, 6 may now be obtained in the usual manner.

(*b*) It will make no difference whatever in the force in the bar AP if, instead of imagining the triangular portion above considered to be solid, we imagine it to have a different interior arrangement of bars. Thus, in Fig. 112 (*c*) is shown this portion with a new arrangement of bars substituted for that given. The force diagram for this substituted frame may be drawn as in Fig. 112 (*d*), and stopped directly the force in AP is found. The original arrangement of bars is now restored and the force diagram completed in the usual manner. The result for the force in AP is found graphically in Fig. 112 (*d*) to be 1200 lb.

This method must be applied with caution. Care must be taken to ensure that the substituted arrangement of bars does nothing whatever to alter the force in the bar considered, viz. AP in Fig. 112 (*a*).

EXERCISES ON CHAPTER V.

In each of these exercises the forces should be tabulated, distinguishing carefully push and pull members.

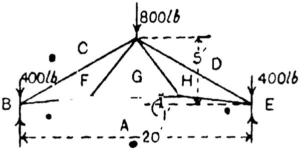


FIG. 113.

1. Find the forces in all the bars of the roof truss shown in Fig. 113. The bars AF, FG, GH and HA are equal.

2. Find the forces in all the bars of the truss given in Fig. 114. The loads are in lb. units.

3. Find the forces in the roof truss shown in Fig. 106; apply the same loads, with the exception of that at the centre of the right-hand rafter, which in this case is 2000 lb. Span 24 feet, rise 6 feet, rise of tie-bar 1 foot. Each rafter is bisected perpendicularly by the inclined strut.

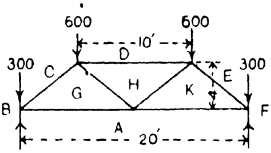


FIG. 114.

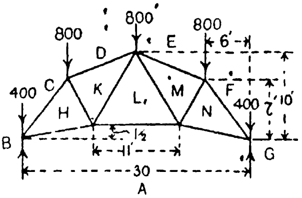


FIG. 115

4. Find the forces in all the members of the truss shown in Fig. 115. The loads are in lb. units.

5. Take again the roof truss given in Question 1. Remove all the loads and apply wind loads of 400 lb at each end of the right-hand rafter, acting at right angles to the rafter. Find the forces in all the parts due to wind only. The left-hand reaction is vertical

6. Answer Question 5, supposing that the wind loads are applied to the left-hand rafter only. The left-hand reaction is vertical

7. From the results obtained in answering Questions 1, 5 and 6, construct a table showing the maximum and minimum forces in each bar due to dead load and wind pressure combined.

8. Find the forces in all the bars of the roof truss given in Fig. 116. The loads are in lb. units.

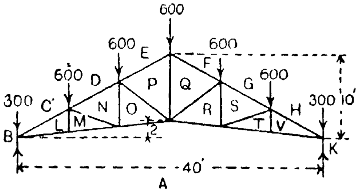


FIG. 116.

9. A roof truss similar to Fig. 111 has a span of 30 feet; the rise is 8 feet and the height of the central horizontal part of the tie bar above the supports is 18 inches. If the truss carries a symmetrically distributed load of 4 tons, find the force in AO by calculation.

10. In the roof truss given in Question 8, in addition to the stated loads, there are wind loads of 400, 800, 800 and 400 lb. applied at the joints of the right-hand rafter and perpendicular to the rafter. The left-hand reaction is vertical. Find the reactions of the supports, using the link polygon.

11. Answer Question 10 by application of the substituted frame method.

12. Answer Question 10 by calculation.

13. In Question 10 find the forces in all the parts due to combined dead load and wind pressure.

14. A loaded Warren girder is shown in Fig. 117. Find the forces in all the members. The loads are in lb. units.

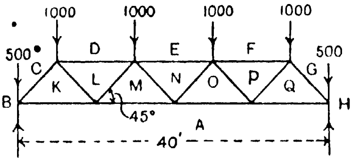


FIG. 117.

15. A frame secured to a vertical wall has dimensions as shown in Fig. 118. The bars AD, AF, AH and AL are each 5 feet in length. Find the forces in all the parts produced by the load of 1 ton.

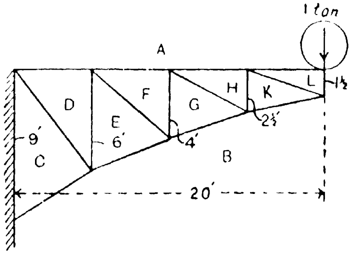


FIG. 118.

16. Answer Question 15 if the load is moved horizontally so as to be vertically over the middle joint of the top member of the frame.

17. Part of a pin-jointed frame, shown in Fig. 119, is loaded with a vertical dead load of 10,000 pounds and a normal wind pressure of 15,000

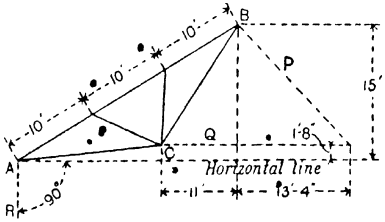


FIG. 119.

pounds, both being taken as uniformly distributed along AB. The supporting forces P, Q and R are shown by dotted lines. Find these forces and the forces in the bars which meet at C, indicating the struts and ties. (L.U.)

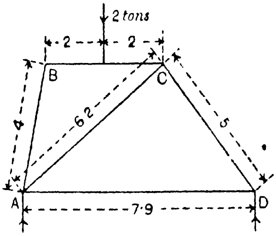


Fig. 120.

18. A frame is loaded with 2 tons and supported as shown in Fig. 120. Find the reactions at A and D and the forces in the members, indicating which are struts and which are ties. (I.C.E.)

CHAPTER VI.

SIMPLE STRESSES AND STRAINS

Stress. If any section in a loaded body be taken, it will be found in general that the part of the body which lies on one side of the section is communicating forces across the section to the other part, and is itself experiencing equal opposite forces. The name **stress** is given to these mutual actions. The stress is described as **tensile** or **pull** if the effect is to pull the portions of the body apart, **compressive** or **push** if they are being pushed together, and **shearing** or **tangential** if the tendency is to cause one portion of the body to slide on the other portion.

The stress is said to be distributed uniformly in cases where all small equal areas experience equal loads. Stress is measured by stating the force per unit area, the result being described as **unital stress**, or **stress intensity**, or often simply as **the stress**. In the case of a uniform distribution of stress, the stress intensity will be found by dividing the total force by the area over which it is distributed. Should the stress vary from point to point, its intensity at any point may be stated by considering that the forces acting on a very small area embracing that point will show a very small variation and may be taken as uniformly distributed. Thus, if a be a very small portion of the area and p the load on it, the stress intensity on a will be p/a.

Units of stress employed in practice are pounds or tons per square inch or per square foot, or in the metric system, grams or kilograms per square centimetre. One atmosphere is sometimes used as a unit, being a stress of 14·7 lb. per square inch; it is useful to remember that a stress of one kilogram per square centimetre is roughly equal to one atmosphere.*

---

kilogram per square centimetre = 2·205 lb. per $\frac{1}{6·45}$ square inch = 14·19 lb. per square inch.

EXAMPLE 1. A bar of circular cross section 2 inches in diameter is pulled with a force of 12 tons at each end. Find the tensile stress.

$$\text{Area of cross-section} = \frac{\pi d^2}{4} = 3{\cdot}1416 \text{ sq. inches.}$$

$$\text{Tensile stress intensity} = \frac{\text{load}}{\text{area}}$$

$$= \frac{12}{3{\cdot}1416}$$

$$= \underline{3{\cdot}82} \text{ tons per square inch.}$$

EXAMPLE 2. Suppose the same bar to be in two portions connected by means of a knuckle joint having a pin $1\frac{1}{2}$ inches in diameter (Fig. 121), and calculate the intensity of shearing stress on the pin.

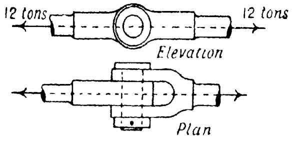


FIG. 121.

It will be observed that the pin would have to shear at two sections for the joint to fracture by failure of the pin, hence .

$$\text{Area under shear stress} = \frac{\pi d^2}{4} \times 2$$

$$= \frac{\pi}{4} \times (1\tfrac{1}{2})^2 \times 2$$

$$= 3{\cdot}53 \text{ square inches.}$$

$$\text{Shear stress intensity} = \frac{\text{load}}{\text{area}}$$

$$= \frac{12}{3{\cdot}53}$$

$$= \underline{3{\cdot}39} \text{ tons per square inch.}$$

Stresses in shells. A **shell** is a vessel constructed of plates the thickness of which is small compared with the overall dimensions of the vessel, for example, a boiler of the cylindrical type. Such vessels have generally to withstand internal fluid pressure, and the plates are put under tensile stress thereby. Owing to the thinness of the plates, the stress on any section may be considered to be distributed uniformly.

Taking a cylindrical shell (Fig. 122) in which there are no stays passing from end to end.

Let d = diameter of shell, inches,
p = fluid pressure, pounds per square inch,
t = thickness of plate, inches,
P = total pressure on each end of vessel, then

$$P = p \times \frac{\pi d^2}{4} \text{ lb.}$$

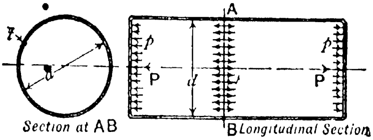


FIG. 122.—Stresses in a cylindrical shell.

Owing to the forces P, P, any section such as AB will be under tensile stress.

$$\text{Sectional area at AB} = \text{circumference of shell} \times t$$
$$= \pi dt.$$

$$\text{Tensile stress intensity on AB} = \frac{P}{\pi dt}$$
$$= \frac{p \times \frac{\pi d^2}{4}}{\pi dt}$$
$$= \frac{pd}{4t} \text{ lb. per square inch.}$$

The stress on a longitudinal section may be found in the following manner. Consider a ring cut from the shell by two cross-sections, one inch apart (Fig. 123). It may be assumed that all other such rings will be under similar conditions, provided they are not taken too near to the ends of the shells where the staying action of the ends would interfere. The fluid pressure on the ring is shown by arrows in Fig. 123, everywhere directed perpendicular to the curved surface of the ring, *i.e.* radial. Components of these being taken, parallel and perpendicular to a diameter AB, it will be seen that those parallel to AB equilibrate independently of the others. The upward and downward components perpendicular to AB will have resultants R_1 and R_2 respectively, which will have the effect of producing tensile stress on the sections at A and B. Clearly R_1 and R_2 will be equal; to obtain their magnitudes proceed thus:

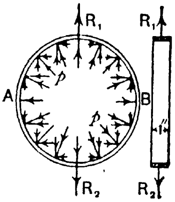


FIG. 123.—A ring cut from a cylindrical shell.

There will be no difference experienced in the equilibrium of the ring if we imagine it to be filled up to the level of AB with cement (Fig. 124). The pressure on the surface of the cement will be perpendicular to AB, and the resultant force due to this will be

$$Q = p \times \text{area of surface of AB}$$
$$= p \times d \times 1.$$

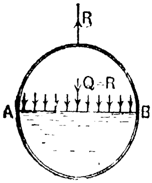


FIG. 124.—Resultant pressure on half of the ring.

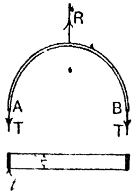


FIG. 125.—Stresses at A and B.

R and Q now preserve the equilibrium of the ring, and must therefore be equal, hence $R = pd.$

Imagine the material at A and B to be cut, and consider the equilibrium of the top half of the ring (Fig. 125). Forces T, T at A and B will be required, and are produced in the uncut shell by tensile stress at A and B. For equilibrium, we have

$$R = 2T,$$
$$T = \frac{R}{2} = \frac{pd}{2}.$$

Also,

Stress intensity at A or B $\times t \times 1 = T$;

∴ stress intensity on longitudinal section

$$= \frac{pd}{2t} \text{ lb. per square inch.}$$

Comparison of these results will show that the stress on a longitudinal section is double that on a circumferential section, a fact which explains why the longitudinal joints in boilers are made much stronger than the circumferential joints.

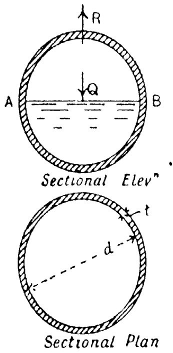


FIG. 126.—Stresses in a spherical shell.

A **spherical shell** may be worked out in a similar manner. Let the shell be filled up to the level of a horizontal diameter AB with cement (Fig. 126), then

$$Q = R = p \times \frac{\pi d^2}{4}.$$

The complete cross-section at AB is a ring of diameter d and thickness t, and is under tensile stress of intensity given by

$$\text{Tensile stress intensity} \times \text{area of cross-section} = \text{R},$$

$$\therefore \text{Tensile stress intensity} = \frac{\text{R}}{\pi dt}$$

$$= \frac{p \times \frac{\pi d^2}{4}}{\pi dt}$$

$$= \frac{pd}{4t} \text{ lb. per square inch.}$$

As before, p = fluid pressure in lb. per square inch,
d = diameter of sphere in inches,
t = thickness of plate in inches.

It will be noted that the stress intensity in a spherical shell is the same as that on the circumferential sections of a cylindrical shell of the same diameter and thickness, and subjected to the same fluid pressure. It will also be observed that a spherical shell is self staying on account of the fact that its shape does not tend to alter when it is exposed to the internal fluid pressure. The same is true for the cylindrical portion of an ordinary boiler shell, but the flat end plates are liable to be bulged outwards unless supported or stayed in some effectual manner.

Riveted joints. Plates may be connected permanently by means of riveted joints. In **lap joints** the edges of the plates overlap (Figs. 131 and 132) and are connected by one or two rows of rivets; in **butt joints** the plates are brought together edge to edge (Figs. 133 and 134) and cover plates pass along the seam on both sides or on one side only. As the strength of the joint depends to a considerable extent on the workshop methods employed, it is necessary to make brief reference to these methods.

Excepting in the case of very thin plates and small rivets, the rivets are heated before being inserted in the holes and are closed by the use of hand or pneumatic hammers, or by a hydraulic riveting machine. Owing to the great pressure exerted in the latter method, the rivets generally fill the hole better when finished and the plates are held together more firmly. In either method, the cooling of the rivet and consequent longitudinal contraction assist largely in binding the plates together, while at the same time the rivet is put under pull stress of an uncertain amount.

Rivet holes may be punched or drilled. Punching injures the metal by overstraining the material round the hole, a defect which may be remedied by annealing, or by punching the hole about $\frac{1}{16}$ inch smaller than the proper diameter, and then enlarging it to the size required with a reamer, thus getting rid of the overstrained material. The plates are punched separately, hence there is difficulty in ensuring that the holes shall come exactly opposite one another when the plates are brought together; drilling is effected with the plates together in position, and this method is to be preferred as giving fair holes, as well as producing no injury to the plates. Punched holes may be brought fair by bolting the plates together before reamering.

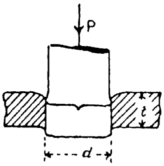


Fig. 127.—Stress on a punch.

There is a lower limit to the diameter of hole which may be punched in a plate of given thickness, depending on the value of the stress under which the punch will crush.

Let d = diameter of hole, inches (Fig. 127),

t = thickness of plate, inches,

q = shearing stress of material of plate, in tons per square inch.

p = crushing stress of material of punch, in tons per square inch.

Area under shear stress $= \pi d \times t$.

Force P required to shear the material $= q\pi dt$.

$$\text{Push stress on punch} = \text{P} \div \frac{\pi d^2}{4}$$

$$= q\pi dt \times \frac{4}{\pi d^2}.$$

Equating this to p will give the limiting value of d, thus

$$p = q\pi dt \times \frac{4}{\pi d^2},$$

$$d = \frac{4qt}{p}.$$

p for the material of the punch, tool steel, is about four times the value of q for mild steel, hence, the condition that the punch is on the point of crushing is

$$d = t,$$

showing that the minimum diameter of hole which may be punched is equal to the thickness of the plate. If d is less than t, p must have a value greater than $4q$ for punching to be possible.

Riveted joints should not be designed so as to load the rivets by tension, as the heads are not reliable under pull. The loading should be of the nature of pull or push along the direction of the plates, thus putting the rivets under shear stress. Lap joints (Fig. 128) and butt joints having a single cover plate (Fig. 129) are put under a bending action by reason of the forces being in parallel lines. Butt joints having double cover plates (Figs. 133 and 134) are free from this objection. In lap joints, the rivets will sustain equal shearing forces whether the plates be under pull or push; in butt joints under push, the forces will be communicated from plate to plate along the edges in contact without putting the rivets under shear stress at all, provided the fitting is perfect. The cover plates and rivets in this case serve only to prevent the plates getting out of the same plane. For these reasons, both compression and tension members are best fitted with butt joints having double cover plates.

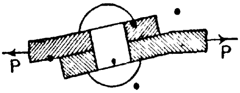


Fig. 128.

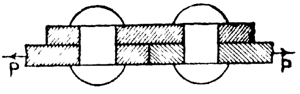


Fig. 129.

Methods of failure of riveted joints. These may be described by reference to Fig. 130, showing a single riveted lap joint.

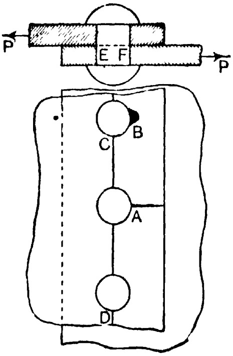


Fig. 130.—Methods of failure of riveted joints.

(*a*) If the hole is situated too near the edge of the plate, the material may open out as at A during punching, or by reason of the bursting pressure exercised by the hot, soft rivet while being closed. To prevent this happening, the distance from the centre of the hole to the edge of the plate should not be less than 1·5 times the diameter of the rivet.

(*b*) The material of the plate may crush at B owing to the rivet being too large in diameter. When the joint is loaded the rivet bears on one half of the cylindrical surface of the hole, producing a bearing stress which is calculated by dividing the load on the rivet by the "projected area" of the hole, the latter being calculated by taking the product of the diameter of the hole and the thickness of the plate. In girder work, the design of the riveted joints has to be based sometimes on the safe bearing stress; this stress ranges from $7\frac{1}{2}$ to 10 tons per square inch in practice.

(*c*) One of the plates may give way by tearing along the line CD.

(*d*) The rivets may shear at EF.

The most economical joint would be equally ready to fail by all four ways simultaneously. It is impossible to calculate (*a*) from first principles, but expressions giving the relations of the various quantities may be found by equalising the resistances of the joint to crushing, tearing and shearing. It is customary in this country to neglect the increase in strength owing to the frictional resistance to the plates sliding on one another. The precise conditions for any riveted joint cannot be stated definitely, hence empirical rules, or rules which are partly empirical, are often employed in practice.

Lap joints. Lap joints may be single or double riveted; it is rarely the case that there are more than two rows of rivets. The **pitch** is the distance from centre to centre of the rivets measured along the row. The strength of the joint may be considered by taking a strip equal in breadth to the pitch, as the conclusions arrived at for this piece may be assumed to be true for the entire joint.

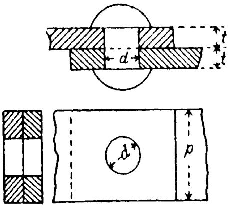


FIG. 131.—Strength of a single-riveted lap joint.

Let p = pitch of the rivets, inches;

d = diameter of the rivets, inches;

t = thickness of the plates, inches;

f_t = the ultimate tensile strength of the plates, tons per square inch;

f_s = the ultimate shearing strength of the rivet, tons per square inch;

f_b = the bearing stress, tons per square inch of projected area, when the joint is on the point of failing by crushing.

We have, for a single riveted lap joint (Fig. 131):

Least area of plate section under pull $= (p-d)t$,

Resistance of joint to tearing $= f_t(p-d)t$ tons. ....(1)

$$\text{Area of rivet section under shear} = \frac{\pi d^2}{4},$$

$$\text{Resistance of joint to shearing} = f_s\frac{\pi d^2}{4} \text{ tons. ....(2)}$$

Projected area $= dt$,

Resistance of joint to crushing $= f_b dt$ tons. ....(3)

Equating (1), (2) and (3) gives:

$$f_t(p-d)t = f_s\frac{\pi d^2}{4} = f_b dt.$$

Taking
$$f_s\frac{\pi d^2}{4} = f_b dt,$$

$$d = \frac{4t}{\pi}\cdot\frac{f_b}{f_s}$$

$$= 1\cdot27t\frac{f_b}{f_s}. \quad \dots \quad (4)$$

The diameter of the rivet may be found from this relation, and the pitch may then be calculated from

$$f_t(p-d)t = f_s\frac{\pi d^2}{4},$$

$$p - d = \frac{f_s}{f_t}\cdot\frac{\pi d^2}{4t},$$

$$p = \left(0\cdot785\frac{f_s}{f_t}\cdot\frac{d^2}{t}\right) + d. \quad \dots \quad (5)$$

In double riveted lap joints there will be two rivet sections per pitch under shear (Fig. 132); there will also be two bearing areas per pitch. Hence

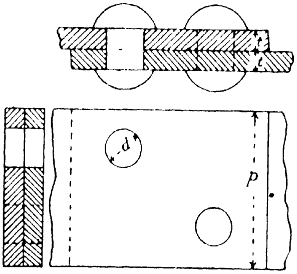


FIG. 132.—Strength of a double riveted lap joint.

$$f_t(p-d)t = 2f_s\frac{\pi d^2}{4} = 2f_b dt;$$

$$d = 1\cdot27t\frac{f_b}{f_s}, \quad \dots \quad (6)$$

$$p = \left(1\cdot571\frac{f_s}{f_t}\cdot\frac{d^2}{t}\right) + d. \quad \dots \quad (7)$$

Butt joints. The strength of butt joints may be calculated in a similar manner; it will be observed (Fig. 133) that, with two cover

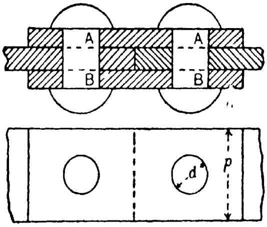


FIG. 133.—Strength of a single-riveted butt joint.

plates, the rivets are under double shear, *i.e.*, each rivet would have to shear at two sections A and B for the joint to fail by shearing. Each rivet will thus have a shearing area of $2\frac{\pi d^2}{4}$.

For a single riveted butt joint (Fig. 133),

$$f_t(p-d)t=2f_s\frac{\pi d^2}{4}=f_b dt,$$

$$d=0{\cdot}635t\frac{f_b}{f_s}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$p=\left(1{\cdot}571\frac{f_s}{f_t}\cdot\frac{d^2}{t}\right)+d. \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

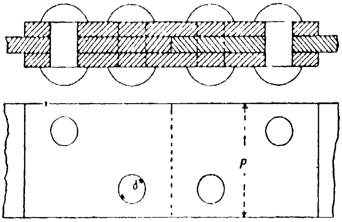


FIG. 134.—Strength of a double-riveted butt joint.

In the case of a double riveted butt joint (Fig. 134), we have

$$f_t(p-d)t=4f_s\frac{\pi d^2}{4}=2f_b dt,$$

$$d=0{\cdot}635t\frac{f_b}{f_s}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$p=\left(3{\cdot}142\frac{f_s}{f_t}\cdot\frac{d^2}{t}\right)+d. \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Data from experiments. The ultimate tensile strength of iron plates may vary from 21 to 26 tons per square inch, and for steel plates may vary from 27 to 32 tons per square inch. Iron and steel rivets have an ultimate shearing strength of about 23 tons per square inch. Owing to the difficulty of stating precisely what the actual conditions are in a finished riveted joint, these stresses should be used with caution. Experiments on actual joints with iron plates and iron rivets show that the ratio f_s/f_t is nearly 1 for drilled holes, and from 1·2 to 1·3 for punched holes which have been neither annealed nor reamered. For steel plates and steel rivets the values of the ratio appear to be about 0·75 for drilled holes and about 0·9 for punched holes neither annealed nor reamered. For either reamered or annealed punched holes the values are about the same as for drilled holes. Breakdown in experimental joints by crushing appears to take place for ratios of f_b/f_s of about 1·7 for rivets in single shear and about 2·35 for rivets in double shear Provision against crushing is often made by employment of an empirical rule for the diameter of the rivet. A good practical rule is

$$d = 1{\cdot}2\sqrt{t} \text{ to } 1{\cdot}4\sqrt{t}.$$

When this rule is used, the diameter of the rivet is calculated first, and the pitch is then determined by equating the resistances to tearing and shearing. Afterwards, the bearing stress should be calculated in order to ascertain that its value is not excessive.

In riveted joints designed under the Board of Trade rules, rivets under double shear are allowed $1\frac{3}{4}$ rivet sections per rivet only; this is owing to the probability of the rivets not all bearing equally. This rule is often disregarded in other joints.*

Efficiency of riveted joints. The efficiency of a riveted joint is the ratio of its actual strength to that of the solid plate. To calculate the efficiency, the ratios of the strength of the joint against tearing, shearing and crushing to the strength of the solid plate should be calculated separately, and the lowest value taken as the efficiency of the joint. It will be evident that all three ratios will be equal if he joint has been designed for equality of rupture by each of the hree ways of failure, and the efficiency may be obtained then by consideration of the tearing resistance only.

Resistance of joint to tearing $= (p - d)tf_t$.

Resistance of solid plate to tearing $= ptf_t$.

---

* For a full discussion of riveted joints, see *Machine Design*, Part I., by Prof. W. C. Unwin (Longmans, 1909).

$$\text{Efficiency} = \frac{(p-d)tf_t}{ptf_t},$$

$$= \frac{p-d}{p}.$$

Example 1. A double riveted butt joint with double cover plates is used to connect steel plates of 0·5 inch thickness; the holes are to be drilled. Find the diameter of the rivets from the empirical rule (p. 103), and also the pitch of the rivets, taking $f_s/f_t = 0{\cdot}75$.

$$d = 1{\cdot}2\sqrt{t}$$

$$= \tfrac{7}{8} \text{ inch, nearly.}$$

$$p = \left(3{\cdot}142\frac{f_s}{f_t}\frac{d^2}{t}\right) + d \quad \text{(p. 102)}$$

$$= (3{\cdot}142 \times 0{\cdot}75 \times (\tfrac{7}{8})^2 \times 2) + \tfrac{7}{8}$$

$$= \underline{4\tfrac{1}{2}} \text{ inches, nearly.}$$

Example 2. Calculate the efficiency of the above joint.

$$\text{Efficiency} = \frac{p-d}{p}$$

$$= \frac{4{\cdot}5 - 0{\cdot}875}{4{\cdot}5}$$

$$= 0{\cdot}805$$

$$= \underline{80{\cdot}5} \text{ per cent.}$$

Or the efficiency may be calculated by considering the resistance to shearing. Thus:

$$\text{Area per pitch under shear stress} = 4\frac{\pi d^2}{4}.$$

$$\text{Strength against shearing} = \pi d^2 f_s$$

$$\text{Efficiency against shearing} = \pi d^2 f_s \div ptf_t$$

$$= \frac{\pi d^2}{pt} \cdot \frac{f_s}{f_t}$$

$$= \frac{3{\cdot}142 \times 49 \times 0{\cdot}75}{4{\cdot}5 \times 0{\cdot}5 \times 64}$$

$$= 0{\cdot}802$$

$$= \underline{80{\cdot}2} \text{ per cent.}$$

Example 3. Calculate the bearing stress in the above joint when carrying a load which produces a stress of 4 tons per square inch in the solid plate.

$$\text{Area of solid plate per pitch} = pt$$

$$= 4{\cdot}5 \times 0{\cdot}5$$

$$= 2{\cdot}25 \text{ sq. inches.}$$

$$\text{Load per pitch} = 4 \times 2{\cdot}25$$

$$= 9 \text{ tons.}$$

This load is carried on the bearing surface of two rivets ; hence .

Projected bearing surface per rivet $= dt$.

$$\text{Bearing stress} = \frac{9}{2dt}$$

$$= \frac{9}{2 \times 0{\cdot}875 \times 0{\cdot}5}$$

$$= \underline{10{\cdot}3} \text{ tons per sq. inch.}$$

EXAMPLE 4 Two plates forming a tie-bar have to be connected end to end by a butt joint having double cover straps (Fig. 135). Each plate

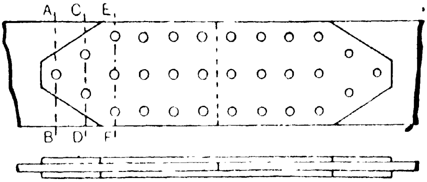


FIG. 135.—Riveted joint for a tie bar.

is 10 inches wide and $\frac{3}{4}$ inch thick, the rivets are $\frac{3}{4}$ inch in diameter. The stresses allowed are 6 tons per square inch pull, 4 tons per square inch shearing, and 10 tons per square inch bearing. Find the number of rivets required

Sectional area of each plate $= 10 \times \frac{3}{4} = 7{\cdot}5$ square inches.

Area abstracted by one rivet hole at the section AB $= \frac{3}{4} \times \frac{3}{4} = 0{\cdot}56$ sq. in.

Net sectional area of plate at AB $= 7{\cdot}5 - 0{\cdot}56$
$= 6{\cdot}94$ square inches.

Total safe pull on the plate $= 6{\cdot}94 \times 6 = 41{\cdot}64$ tons.

$$\text{Sectional area of one rivet} = \frac{\pi d^2}{4} = \frac{22}{7 \times 4} \times \frac{9}{16} = 0{\cdot}442 \text{ sq. in.}$$

Allowing $1\frac{3}{4}$ rivet sections for rivets under double shear, we have

Shearing resistance of one rivet $= 0{\cdot}442 \times 1{\cdot}75 \times 4$
$= 3{\cdot}09$ tons.

Projected area of one rivet $= \frac{3}{4} \times \frac{3}{4} = 0{\cdot}56$ square inch

Bearing resistance of one rivet $= 0{\cdot}56 \times 10 = 5{\cdot}6$ tons

As the shearing resistance is lower than the bearing resistance, the shearing resistance must be taken in calculating the number of rivets required. Let N be the number of rivets on each side of the joint ; then

Total safe pull on the plate = total shearing resistance of the rivets,

$$41{\cdot}64 = N \times 3{\cdot}09,$$

$$N = 14 \text{ rivets}$$

To obtain a good arrangement of rivets, 15 rivets have been placed on each side of the joint in Fig. 135

At the section AB, the safe load which can be applied is that calculated above as 41·64 tons. At CD, the tearing strength of the plate is less than at AB, but to this must be added the resistance of the rivet on the left-hand side of CD, as this rivet would have to shear simultaneously with the plate tearing at CD for the joint to fail in this way.

Sectional area of plate at CD $= 7{\cdot}5 - (2 \times 0{\cdot}56) = 6{\cdot}38$ square inches.

Resistance to tearing at CD $= 6{\cdot}38 \times 6 = 38{\cdot}28$ tons.

Adding the shearing resistance of one rivet to this, we have

Safe load with reference to the section CD $= 38{\cdot}28 + 3{\cdot}09$
$= 41{\cdot}37$ tons.

Considering the section EF, the three rivets on the left-hand side of EF would have to shear simultaneously with the plate tearing.

Resistance to tearing at EF $= \{7{\cdot}5 - (3 \times 0{\cdot}56)\}6 = 34{\cdot}92$ tons.

Shearing resistance of three rivets $= 3 \times 3{\cdot}09 = 9{\cdot}27$ tons.

Safe load with reference to the section EF $= 44{\cdot}19$ tons.

It is evident that the safe load with reference to any other section on the right-hand side of EF will have a greater value than that for the section EF. The minimum safe load is that calculated for the section CD, viz. 41·37 tons, which accordingly is the safe load which the joint will carry.

Strain. Strain refers to the alterations of form or dimensions which occur when a body is loaded or subjected to stress. Thus a pulled or pushed bar is found to have become longer or shorter after the load is applied, and is said to have **longitudinal strain.** This kind of strain is measured by taking the ratio of the change in length to the original length.

Let L = original length of bar,
ϵ = alteration in length, both in the same units.

$$\text{Longitudinal strain} = \frac{\epsilon}{L}.$$

Volumetric strain occurs when a body is subjected to uniform fluid pressure over the whole of its exposed surfaces. The volume will be changed somewhat under these conditions, and the volumetric strain is measured by taking the ratio of the change in volume to the original volume.

Let V = original volume of body,
v = change in volume, both in the same units.

$$\text{Volumetric strain} = \frac{v}{V}.$$

Shearing strain occurs when a body is subjected to shear stress. Such a stress is distinguished from the other two just mentioned in that it produces a change in the shape of the body, while pull, push, and hydrostatic stress produce no such change. We may obtain an idea of what happens by holding one cover of a thick book firmly on the table and applying a shearing force to the top cover (Fig. 136). The change in shape is evidenced by the square originally pencilled on the end of the book becoming a rhombus. A solid body would behave in the same manner under similar conditions of loading, only, of course, in a minor degree (Fig. 137). The shearing strain is measured by stating the angle in radians through which the vertical edge has rotated on application of the shearing stress.

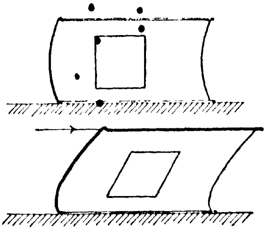


FIG. 136.—Shearing strain illustrated by a book.

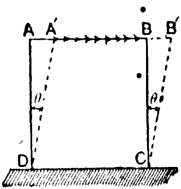


FIG. 137.—Measurement of shearing strain.

$$\text{Shearing strain} = \theta \text{ radians (Fig. 137).}$$

For metals θ is always very small, and it is often sufficiently accurate to write, referring to Fig. 137:

$$\text{Shearing strain} = \theta,$$
$$= \frac{BB'}{BC}.$$

Transverse strain. When a bar is pulled or pushed, not only is its length altered, but also its transverse dimensions. Thus a pulled bar becomes thinner, while a pushed bar becomes thicker. Such alterations are referred to as **transverse strains** and are measured in the same manner as longitudinal strains, viz. by taking the ratio of the alteration in transverse dimension to the original transverse dimension.

Let H = a transverse dimension of the bar,

h = the change in H when the bar is loaded.

$$\text{Transverse strain} = \frac{h}{H}.$$

For any given material, such as a metal, experiment shows that there is a definite ratio of longitudinal to transverse strain, ranging from 3 to 4 for common metals.

Let a = longitudinal strain,

b = transverse strain,

$$m = \frac{a}{b}.$$

The value of m depends on the kind of material; its reciprocal $\frac{1}{m}$ is called **Poisson's ratio.** Values of this ratio for common materials are tabulated on p. 683

Elasticity. **Elasticity** is that property of matter by virtue of which a body endeavours to return to its original form and dimensions when strained, the recovery taking place when the disturbing forces are removed. Strain takes place while the loads are being applied to a body, hence mechanical work (see p 325) is expended in producing strain, and is stored up, partly at any rate, in the body. The elasticity of any material is regarded as being perfect, provided the recovery of the original form and dimensions is perfect on removal of the loads, and provided also that the energy given out during recovery equals that expended while the body was being strained.

The elasticity of a large number of materials is practically perfect provided they are not stressed beyond a certain limit, which depends on the kind of material and also on the nature of the stress applied. If loaded beyond this **elastic limit** of stress, the recovery of original form and dimensions is incomplete and the body is said to have acquired **permanent set.**

Further, experiment shows that **the strains are proportional to the stresses producing them** provided that the elastic limit is not exceeded. This law was first discovered by Hooke, and bears his name. Most materials show slight divergencies from **Hooke's law,** but it is adhered to so closely in the case of common metals as to justify the assumption of its truth for nearly all practical purposes.

Modulus of elasticity. Assuming Hooke's law to be true, and selecting any elastic material to which loads may be applied.

Let p = the stress,

s = the strain produced by p.

Then p varies as s up to the elastic limit, hence the quantity $\frac{p}{s}$ will be constant for that material up to the elastic limit. The term **modulus of elasticity** is given to this quantity. The value of the modulus of elasticity depends firstly on the nature of the material, and in the second place on the nature of the stress. For any given material there are three moduli of elasticity which should be understood. In each case the measurement is made by taking

$$\text{Modulus of elasticity} = \frac{p}{s}.$$

The units of this expression will be governed by the unit of stress employed, as strain is simply a ratio.

Young's modulus for a pushed or pulled bar is obtained by dividing the push or pull stress intensity on a cross section at 90° to the axis of the bar by the longitudinal strain.

Let P = force of push or pull applied to the bar,
A = area of the cross section,
L = original length of the bar,
ϵ = change of length of the bar,
both the latter being in the same units.

Then, writing E for Young's modulus,

$$E = \frac{\text{stress}}{\text{strain}}$$

$$= \frac{P}{A} \div \frac{\epsilon}{L} = \frac{PL}{A\epsilon}.$$

The **bulk modulus** belongs to the case of a body subjected to hydrostatic stress, which produces volumetric strain.

Let p = the hydrostatic stress intensity,
V = the original volume of the body,
v = the change in volume,
both the latter being in the same units.

Then, writing K for the bulk modulus,

$$K = p \div \frac{v}{V} = \frac{pV}{v}.$$

The **rigidity modulus** refers to the case of a body under shearing stress, and consequently changing its shape by shearing strain.

Let q = the shearing stress intensity,
θ = the shearing strain, in radians.

Then, writing C for the rigidity modulus,

$$C = \frac{q}{\theta}.$$

The most convenient units to employ for the elastic moduli are tons or lb. per square inch in the British system, and kilograms per square centimetre in the metric system. A table of values will be found on p. 683.

Strains in a cylindrical boiler shell. It has been seen (p. 96) that the stresses in a cylindrical boiler shell on longitudinal seams and on circumferential seams are in the ratio of two to one. Suppose that in consequence of these stresses the circumference becomes greater by a small amount e. Let d be the original diameter of the shell, then the original length of the circumference will be πd, and the circumferential strain will be:

$$\text{Circumferential strain} = \frac{e}{\pi d}. \quad \ldots \ldots \ldots \ldots \ldots \quad (1)$$

Also, new length of the circumference $= \pi d + e$;

$$\therefore \text{ new length of the diameter} = \frac{\pi d + e}{\pi}.$$

Hence, change in the diameter $= \dfrac{\pi d + e}{\pi} - d$

$$= d + \frac{e}{\pi} - d$$

$$= \frac{e}{\pi},$$

$\therefore$ strain in the direction of a diameter

$$= \frac{e}{\pi d}. \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

Comparison of (1) and (2) shows that the diametral and circumferential strains are equal.

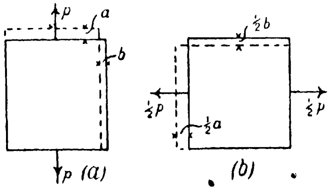


FIG. 138.—Strains in a boiler shell.

To obtain the circumferential strain, let p and $\frac{1}{2}p$ be the stresses on the longitudinal and circumferential seams respectively. If p were to act alone (Fig. 138 (a)), the circumferential strain would be a (extension) and the transverse strain would be b (contraction). If $\frac{1}{2}p$ were to act alone (Fig. 138 (b)), the longitudinal strain would be $\frac{1}{2}a$ (extension) and the circumferential strain would be $\frac{1}{2}b$

(contraction). Hence, when both stresses act together, the strains produced will be:

$$\text{Circumferential strain} = a - \tfrac{1}{2}b. \qquad \ldots\ldots \quad (3)$$

$$\text{Longitudinal strain} = \tfrac{1}{2}a - b. \qquad \ldots\ldots \quad (4)$$

Or, since

$$m = \frac{a}{b} \quad (\text{p. } 108),$$

$$b = \frac{a}{m}.$$

And

$$\text{Circumferential strain} = a - \frac{1}{2}\frac{a}{m}$$

$$= a\left(1 - \frac{1}{2m}\right). \qquad \ldots\ldots \quad (5)$$

$$\text{Longitudinal strain} = \frac{1}{2}a - \frac{a}{m}$$

$$= a\left(\frac{1}{2} - \frac{1}{m}\right) \qquad \ldots \quad (6)$$

Suppose m be taken equal to 4, then

$$\text{Circumferential strain} = a(1 - \tfrac{1}{8})$$

$$= \tfrac{7}{8}a. \qquad \ldots \quad (7)$$

$$\text{Longitudinal strain} = a(\tfrac{1}{2} - \tfrac{1}{4})$$

$$= \tfrac{1}{4}a. \qquad \ldots\ldots \quad (8)$$

Reference to Fig. 138 (a) shows that

$$E = \frac{p}{a}$$

or

$$a = \frac{p}{E}. \qquad \ldots\ldots \quad (9)$$

Hence:

$$\text{Circumferential strain} = \frac{7}{8}\frac{p}{E}. \qquad (10)$$

$$\text{Longitudinal strain} = \frac{1}{4}\frac{p}{E}. \qquad \ldots\ldots \quad (11)$$

EXAMPLE. A boiler shell 7 feet in diameter and 30 feet long is tested by hydraulic pressure (cold water) up to a stress of 6 tons per square inch on the longitudinal seams. Take $E = 13{,}500$ tons per square inch and $m = 4$, and find how much water will escape when a test cock on the top of the boiler is opened. Neglect any bulging of the ends.

[To answer this question, calculate the increase in volume of the shell while the pressure is being applied.]

$$\text{Circumferential strain} = \tfrac{7}{8} \times \tfrac{6}{13500} = \tfrac{21}{54000}.$$

The diametral strain is equal to this; hence:

$$\text{Change in diameter} = (7 \times 12) \times \tfrac{21}{54000}$$

$$= 0{\cdot}0327 \text{ inch.}$$

$$\text{Final diameter of shell} = 84{\cdot}0327 \text{ inches.}$$

Let D and d be the final and original diameters; then

$$\text{Increase in cross-sectional area of the water} = \frac{\pi}{4}(D^2 - d^2)$$
$$= \frac{\pi}{4}(D - d)(D + d)$$
$$= \frac{\pi}{4} \times 0{\cdot}0327 \times 2 \times 84 \text{ nearly}$$
$$= 4{\cdot}32 \text{ square inches.}$$

$\therefore$ increase in volume due to increase in sectional area $= 4{\cdot}32 \times 360$
$= 1555$ cub. in.

Again, Longitudinal strain $= \frac{1}{4} \times \frac{6}{13500} = \frac{3}{27000}$

$\therefore$ change in length of the shell $= 30 \times 12 \times \frac{3}{27000}$
$= 0{\cdot}04$ inch

Sectional area of the water $= \frac{\pi}{4} \times 84^2$ (nearly)
$= 5542$ sq. inches.

$\therefore$ increase in volume due to increase in length $= 0{\cdot}04 \times 5542$
$= 222$ cubic inches.

Total increase in volume $= 1550 + 222$
$= 1772$ cubic inches.

The change in volume which occurs when charging cylinders for holding compressed gases is sometimes taken as a test of the soundness of the material of which the cylinder is constructed. The test is made by having the cylinder immersed in water contained in a closed vessel fitted with an external glass tube connected to the water space. In charging, the expansion of the cylinder will displace some of the water, which will therefore rise in the glass tube. An increase in volume of more than a prescribed limit, as indicated by the tube reading, affords evidence of defects in the material of the cylinder.

Stresses in thick cylinders In Fig. 139 (a) is shown a cylinder of considerable thickness under external and internal fluid pressures. Let push stresses be denoted positive, and let the external pressure be greater than the internal pressure. Consider a ring of unit length, having an inner radius r and outer radius $(r + \delta r)$ (Fig. 139 (b)). Let the radial stress on its inner surface be p, and let that on the outer surface be $(p + \delta p)$. The resultants of these stresses on the half ring (Fig. 139 (c)) will be

$$P_1 = p \times 2r \text{ (see p. 96),}$$
$$P_2 = (p + \delta p) \times 2(r + \delta r).$$

The resultant P of P_1 and P_2 is

$$P = P_2 - P_1$$
$$= (p + \delta p) \times 2(r + \delta r) - p \times 2r$$

Let f be the tangential, or hoop stress on the ring; the area over which this stress is distributed is $\delta r \times 1$, and there are two horizontal sections, one at A and one at B (Fig. 139 (c)); hence,

$$P = 2f . \delta r,$$

$$\therefore \; 2f . \delta r = (p + \delta p) \times 2(r + \delta r) - 2pr,$$

or

$$f . \delta r = pr + r . \delta p + p . \delta r + \delta p . \delta r - pr$$

$$= r . \delta p + p . \delta r, \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

by neglecting the product of the small quantities δp and δr.

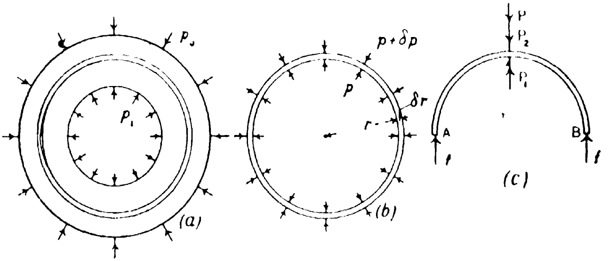


FIG. 139.—Stresses in a thick cylinder.

Another equation may be formed by consideration of the strains in the axial direction produced by p and f all over the cylinder. It may be assumed that cross sections of the cylinder remain plane when the fluid stress is applied, *i.e.* all fibres parallel to the axis of the cylinder lying between two cross sections change their lengths to the same extent. Hence the assumption that the axial strains are equal all over the cylinder.

$$\text{Axial strain produced by } p = \frac{p}{m\text{E}},$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad f = \frac{f}{m\text{E}}.$$

As both p and f are push stresses, both of these strains are extensions, and the total axial strain will be

$$\frac{p}{m\text{E}} + \frac{f}{m\text{E}} = \text{a constant},$$

or

$$p + f = \text{a constant}.$$

Writing $2a$ for the value of the constant, this gives

$$p + f = 2a. \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

From (2), $$f = 2a - p.$$

" (1), $$(2a - p)\delta r = r\,.\,\delta p + p\,.\,\delta r,$$

$$2a\,.\,\delta r - p\,.\,\delta r = r\,.\,\delta p + p\,.\,\delta r,$$

$$2a\,.\,\delta r = r\,.\,\delta p + 2p\,.\,\delta r.$$

Multiply each side of this equation by r, giving

$$2ar\,.\,\delta r = r^2\,.\,\delta p + 2pr\,.\,\delta r,$$

or in the limit, when δr becomes very small,

$$2ar = r^2\frac{dp}{dr} + 2pr.$$

The right-hand side is the differential coefficient of (pr^2), *i.e.*

$$\frac{d}{dr}(pr^2) = r^2\frac{dp}{dr} + 2pr \text{ (see p. 12).}$$

Hence, $$d(pr^2) = 2ar\,.\,dr.$$

Integrate, giving $$pr^2 = ar^2 + c.$$

$$p = a + \frac{c}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

and $$f = 2a - p$$

$$= a - \frac{c}{r^2} \quad \ldots\ldots\ldots \quad (4)$$

The solution of any particular problem may be obtained from (3) and (4) by first determining a and c from the given conditions. Take the ordinary case of a cylinder having an internal fluid pressure p_i, the external pressure being regarded as zero (Fig. 140). We have

$$p = p_i \text{ when } r = R_i\,; \quad \therefore\ p_i = a + \frac{c}{R_i^2} \quad \ldots\ldots\ldots \quad (5)$$

$$p = 0 \text{ when } r = R_o\,; \quad \therefore\ 0 = a + \frac{c}{R_o^2} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Hence, $$c = p_i\frac{R_i^2R_o^2}{R_o^2 - R_i^2},$$

$$a = -p_i\frac{R_i^2}{R_o^2 - R_i^2}.$$

Substitution of these values in (4) gives

$$f = -p_i\frac{R_i^2}{R_o^2 - R_i^2} - p_i\frac{R_i R_o^2}{R_o^2 - R_i^2}\cdot\frac{1}{r^2}$$

$$= -p_i\frac{R_i^2}{R_o^2 - R_i^2}\left(1 + \frac{R_o^2}{r^2}\right). \quad \ldots\ldots\ldots \quad (7)$$

This equation gives the hoop tension at any radius r; the maximum hoop tension will occur where r has its smallest value, *i.e.* at the inner skin, where $r = R_i$. Hence,

$$\text{Maximum } f = -p_i \frac{R_i^2}{R_o^2 - R_i^2}\left(1 + \frac{R_o^2}{R_i^2}\right)$$

$$= -p_i\left(\frac{R_o^2 + R_i^2}{R_o^2 - R_i^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

It will be noticed from equation (8) that the maximum hoop tension is always greater than the internal fluid stress p_i, independently of the thickness of the cylinder, hence, it is impossible to design a solid cylinder to withstand a fluid pressure greater than a certain value for a given material. The difficulty may be overcome by shrinking one cylinder on the top of another, or by winding wire under strong tension over the outside of the cylinder. The effect is to put the inner parts under initial push hoop stress, and gives a distribution of stress more nearly uniform when the fluid pressure is applied.

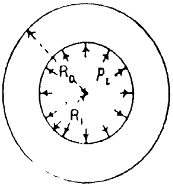


FIG. 140a.

Stresses produced by change in temperature. If a metal bar be heated, its length will increase by an amount proportional to the increase in temperature, and to a coefficient, the value of which depends on the kind of material; this is on the assumption that the bar is permitted to expand freely.

Let L = the original length, in inches;

t = the rise in temperature;

ϵ = the coefficient of expansion, *i.e.* the change in length per unit length produced by a rise in temperature of one degree.

Then, change in length $= Lt\epsilon$;

new length of the bar $= L + Lt\epsilon$

$$= L(1 + t\epsilon) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Suppose that the bar is now cooled to its original temperature, and that forces are applied to its ends so as to prevent it from returning to its original length. Evidently these forces will have the same value as those which would be required to produce an elastic extension $Lt\epsilon$ in the bar had its original temperature been kept unaltered.

Let P = the total force required in tons.

A = the cross sectional area in square inches.

$p = \dfrac{P}{A}$ = the stress produced by P in tons per square inch.

E = Young's modulus in tons per square inch.

Then Longitudinal strain $= \dfrac{Lt\epsilon}{L} = t\epsilon$.

Also,
$$E = \frac{P}{A} \div t\epsilon = \frac{P}{At\epsilon};$$
$$\therefore\ P = EAt\epsilon \text{ tons, } \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$p = Et\epsilon \text{ tons per square inch. } \quad \ldots\ldots\ldots\ldots(3)$$

EXAMPLE. If the bar be of steel for which $E = 13{,}500$ tons per square inch, and if the rise in temperature be 100° F., find the stress in the material under the conditions expressed above. Take

$$\epsilon = 0{\cdot}000007,$$
$$p = Et\epsilon$$
$$= 13{,}500 \times 100 \times 0{\cdot}000007$$
$$= \underline{9{\cdot}45} \text{ tons per square inch.}$$

Suppose now that the bar be heated and at the same time held rigidly between abutments which prevent entirely any change in the length. These conditions may be imagined as follows: first allow the bar to expand freely on heating; then apply forces to the ends and let these be sufficient to compress the bar back to its original length.

Length of the bar before applying the forces $= L(1 + t\epsilon)$.

Change in length produced by $P = Lt\epsilon$.

$$\text{Elastic strain produced by } P = \frac{Lt\epsilon}{L(1+t\epsilon)}$$
$$= \frac{t\epsilon}{1+t\epsilon}.$$

Now
$$E = \frac{p}{\text{strain}}$$
$$= p\left(\frac{1+t\epsilon}{t\epsilon}\right);$$
$$\therefore\ p = \frac{Et\epsilon}{1+t\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The denominator will be nearly unity, as $t\epsilon$ is usually very small; hence, (4) will have the same value nearly as (3).

Effects produced by unequal heating. Fig. 141 illustrates three bars A, B and C attached to rigid cross pieces D and E; E is fixed and D may rise or freely. B is centrally situated between A and C; A and C have ual sectional areas and B may have a different sectional area. A ree bars are of the same material

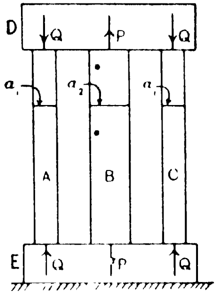


FIG. 141.—Stresses due to unequal heating.

If all three bars at the same temperature at first, and hey be raised through the same range temperature, all will attempt to expand ually in the direction of length, and no ess will be produced in any of them. pose, however, that B is raised to a certa temperature and that A and C are bo raised to the same higher temperature then B attempts to expand to a smalle xtent than A and C. The cross pieces D nd E will compel all three to come to th same length; hence, B will be under pu nd A and C will be under push. This is dicated in the figure by the forces P and Q. As no force whatever is required from the outside in order to alance the arrangement under the altered conditions of tempera e, it follows that

$$P = 2Q. \qquad \ldots \qquad (1)$$

Let the equal secti al areas of A and C be denoted by a_1 and the sectional area of B y a_2; then

$$\text{Stress in A} = \text{ress in C} = p_1 = \frac{Q}{a_1}; \quad \therefore\ Q = p_1 a_1.$$

$$\text{ress in B} = p_2 = \frac{P}{a_2}, \quad \therefore\ P = p_2 a_2.$$

Hence, from (1), $$p_2 a_2 = 2p_1 a_1,$$

or $$\frac{p_1}{p_2} = \frac{a_2}{2a_1}. \qquad \ldots \qquad (2)$$

This result indicates th t if all three bars have the same sectional area, then the stress in will be double that in A or C, irrespective of the actual values of the hanges of temperature.

To find the numerica alues of p_1 and p_2, proceed as follows:

Let L = the origin l length of each bar.
t_1 = change in temperature of A and C.
t_2 = change in temperature of B.
ϵ = the coeffici ent of expansion.
E = the comm n value of Young's modulus.

First assume that all three bars expand freely; then

$$\text{Extension of A} = \text{extension of C} = \mathrm{L}t_1\epsilon.$$

$$\text{Extension of B} = \mathrm{L}t_2\epsilon.$$

$$\text{New length of A or C} = \mathrm{L}(1 + t_1\epsilon)$$

$$= b_1\mathrm{L},$$

where

$$b_1 = 1 + t_1\epsilon.$$

$$\text{New length of B} = \mathrm{L}(1 + t_2\epsilon)$$

$$= b_2\mathrm{L},$$

where

$$b_2 = 1 + t_2\epsilon.$$

Let the bars now be compelled to come to the same final length L_4 by application of the forces P and Q.

$$\text{Shortening of A or C produced by Q} = b_1\mathrm{L} - \mathrm{L}_4.$$

$$\text{Extension of B produced by P} = \mathrm{L}_4 - b_2\mathrm{L}.$$

$$\text{Strain of A or C} = \frac{b_1\mathrm{L} - \mathrm{L}_4}{b_1\mathrm{L}}.$$

$$\text{Strain of B} = \frac{\mathrm{L}_4 - b_2\mathrm{L}}{b_2\mathrm{L}}.$$

Hence, for A or C,

$$\mathrm{E} = p_1\left(\frac{b_1\mathrm{L}}{b_1\mathrm{L} - \mathrm{L}_4}\right). \quad \text{.........} \quad (3)$$

And for B,

$$\mathrm{E} = p_2\left(\frac{b_2\mathrm{L}}{\mathrm{L}_4 - b_2\mathrm{L}}\right). \quad \text{............} (4)$$

$$\therefore p_1\frac{b_1\mathrm{L}}{b_1\mathrm{L} - \mathrm{L}_4} = p_2\frac{b_2\mathrm{L}}{\mathrm{L}_4 - b_2\mathrm{L}},$$

or

$$\frac{p_1}{p_2} = \frac{b_2}{b_1}\left(\frac{b_1\mathrm{L} - \mathrm{L}_4}{\mathrm{L}_4 - b_2\mathrm{L}}\right) \quad .. \quad (5)$$

As the ratio of p_1 and p_2 is known from (2), this result may be used for calculating L_4, the final distance between the cross pieces; substitution in (3) and (4) will then give the values of p_1 and p_2.

EXAMPLE. Take the following data for the arrangement shown in Fig. 141 and calculate the final distance between the cross pieces, also the stress in each mild steel bar.

$a_1 = 1$ square inch.

$a_2 = 3$ square inches.

$t_1 = 100°$ F.

$t_2 = 50°$ F.

L = 100 inches.

E = 30,000,000 lb. per sq. inch.

$\epsilon = 0{\cdot}000007$.

From (2),

$$\frac{p_1}{p_2} = \frac{a_2}{2a_1} = \frac{3}{2} = 1{\cdot}5.$$

Also, $b_1 = 1 + t_1\epsilon = 1 + (100 \times 0·000007) = 1·0007.$

$b_2 = 1 + t_2\epsilon = 1 + (50 \times 0·000007) = 1·00035.$

$b_1 L = 1·0007 \times 100 = 100·07$

$b_2 L = 1·00035 \times 100 = 100·035.$

From (5), $\dfrac{3·0021}{2·0007} = \dfrac{100·07 - L_F}{L_F - 100·035},$

whence $L_F = 100·04899.$

(Note, as the changes of length are calculated by taking the differences in the lengths of the bars, it is necessary in examples of this kind to use a larger number of significant figures than that employed usually.)

From (3), $30,000,000 = p_1 \dfrac{100·07}{100·07 - 100·04899},$

whence $p_1 = \underline{6298}$ lb. per square inch.

From (4), $30,000,000 = p_2 \dfrac{100·035}{100·04899 - 100·035},$

whence $p_2 = \underline{4195}$ lb. per square inch.

These stresses have the calculated ratio of 1·5.

This problem may be varied by using bars of different materials and raising the temperatures of all to the same extent. The differences in the elastic moduli will produce a similar effect to that caused by unequal heating, and the calculation is effected in a similar manner, making use of the proper values of the coefficients of expansion and of the elastic moduli.

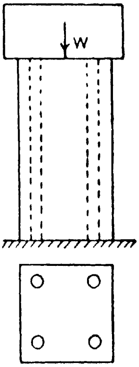


Fig. 142.—Reinforced concrete column.

Reinforced concrete column. In Fig. 142 is shown a concrete column reinforced by steel bars arranged as shown in the plan. Application of an axial load to the column will cause both steel and concrete to shorten to the same extent; as the lengths of both are equal, it follows that the strains are also equal. Using the suffixes c and s to denote the concrete and steel respectively, let

$s_s = s_c =$ the strains in the direction of the length of the column.

f_s and $f_c =$ the stresses in lb. per square inch.

E_s and $E_c =$ Young's moduli in lb. per square inch.

A_s and $A_c =$ the sectional areas in square inches.

Then,

$$E_s = \frac{f_s}{s_s}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$E_c = \frac{f_c}{s_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Dividing (1) by (2), we have

$$\frac{E_s}{E_c} = \frac{f_s}{f_c} \times \frac{s_c}{s_s}$$

$$= \frac{f_s}{f_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The ratio of E_s to E_c varies somewhat; the average value of 15 is usually taken. With this value, equation (3) shows that the stress in the steel will always be 15 times that in the concrete irrespective of the relation of the sectional areas of the concrete and steel. If 500 lb. per square inch be taken as a safe stress for the concrete, then the stress in the steel will be 7500 lb. per square inch.

Suppose W to be the load in lb. applied to the column; then

$$W = f_s A_s + f_c A_c$$

$$= 7500 A_s + 500 A_c, \quad \ldots\ldots \quad (4)$$

a result which enables the safe load to be calculated if the sectional areas of the steel and concrete are given.

The stresses produced in other composite bars under push or pull are calculated in a similar manner, making use of the proper values of Young's modulus. Such bars may take the form of a steel rod cased in some alloy such as gun-metal, or the arrangement may be as illustrated in Fig. 141, with A and C of one material and B of a different material. A central load applied to the top cross piece D will produce equal strains in all the bars, and the stresses will thus be proportional to the values of Young's modulus for the materials of the bars.

Classification of stresses. Stresses may be either **normal** or **tangential**; **oblique stress** is compounded of normal and tangential stresses. Stress is purely normal when its lines of direction are perpendicular to the surface over which it is distributed. Normal stresses may be either tensile or compressive. Stress is tangential or shearing when its lines of direction coincide with the surface over which it is distributed. Oblique stress may have its lines of direction inclined at any angle between 0° and 90° to the surface over which it is distributed. Normal tensile stress occurs in any section AB of a tie-bar subjected to axial pulls (Fig. 143), the section being perpendicular

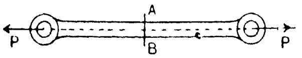


FIG. 143.—Stress in a tie bar

to the axis of the bar. The stress in this case will be uniformly distributed except for sections near the ends of the bar, and its intensity will be given by

$$p = \frac{P}{\text{area of section AB}}$$

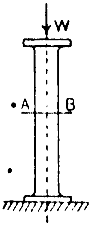


FIG. 144.—Stress in column

Normal compressive stress will be found on any horizontal section AB of a vertical column (Fig. 144) carrying a weight W. If the line of W coincides with the axis of the column, the stress will be uniformly distributed and of intensity given by

$$p = \frac{W}{\text{area of section AB}}$$

Relation of oblique stress with normal and tangential components. Let ABCD (Fig 145) represent the elevation of a cube of unit edge, the top face being subjected to normal stress p_n and also to tangential stress p_t. On the supposition that these stresses are uniformly distributed, we may substitute resultant forces P_N and P_T, acting at the centre O of the top face, in a plane parallel to that of the paper, the values of P_N and P_T being p_n and p_t as the face is of unit area. The resultant of P_N and P_T will be

$$R = \sqrt{P_N^2 + P_T^2},$$

and will act at an angle θ to the normal, the tangent of which is

$$\tan\theta = \frac{P_T}{P_N}.$$

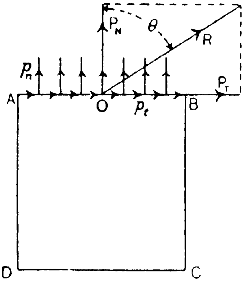


FIG. 145.—Relation of stresses.

Now R may be taken to be the resultant of an infinite number of forces having the same direction as R, and uniformly distributed over the top face of the cube (Fig. 146), these forces constituting an oblique stress r, the value of which will be

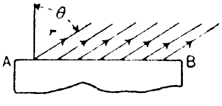


FIG. 146

$$r = \frac{R}{\text{area of top face}}$$

$$= \sqrt{P_N^2 + P_T^2}$$

$$= \sqrt{p_n^2 + p_t^2}. \qquad \text{(1)}$$

Other useful relations deduced easily from the figure are:

$$p_n = r \cdot \cos\theta, \tag{2}$$

$$p_t = r \cdot \sin\theta, \tag{3}$$

$$\tan\theta = \frac{p_t}{p_n}. \tag{4}$$

The angle θ is defined as the **angle of obliquity of the stress.**

Some examples of oblique stress. A useful method of determining the stresses on any section of a loaded body consists in first imagining that the body has been actually cut at the given section. One portion only of the body is then taken, and the resultant forces are determined which must be applied to the section in order to produce equilibrium in this portion. The stresses and their distribution may then be found.

Consider a column carrying a load P, the line of which coincides with the axis of the column (Fig. 147 (*a*)). Let the column be cut at a section AB and consider the upper portion (Fig. 147 (*b*)). For equilibrium, a resultant force P′ = P must be applied in the same line as P. This will give rise to a stress which will be seen afterwards to be uniformly distributed over AB. Let the area of the section AB be S; then

$$\text{Stress intensity on AB} = p = \frac{P}{S}. \tag{1}$$

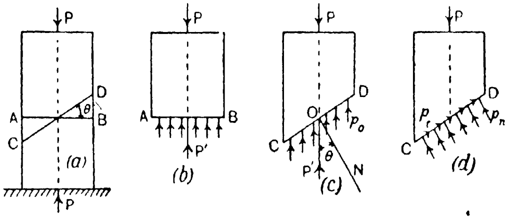


FIG. 147.—Normal and shear stresses in a column

Supposing the column to be cut along CD (Fig. 147 (*a*)), the angle between AB and CD being θ. Considering the equilibrium of the top portion (Fig. 147 (*c*)), we see that a resultant force P′ = P must be applied in the same line as P. P′ will give rise to an oblique stress uniformly distributed over the section CD; let ON be drawn normal to the section, when it will be evident that the angle between P′ and ON, which is the angle of obliquity, is equal to θ. To find the stress intensity, we have

$$\text{Stress intensity} = p_0 = \frac{P'}{\text{area of section CD}}.$$

Now
$$\frac{\text{area of section AB}}{\text{area of section CD}} = \cos\theta;$$

$$\therefore \text{ area of section CD} = \frac{S}{\cos\theta};$$

$$\therefore p_0 = P' \div \frac{S}{\cos\theta}$$

$$= \frac{P'}{S} . \cos\theta$$

$$= p \cos\theta. \qquad \ldots\ldots (2)$$

The intensity of the oblique stress on CD is therefore equal to the stress intensity on AB multiplied by the cosine of the angle between the two sections.

It is of interest to determine the components of p_0 normal and tangential to CD (Figs. 147 (*c*) and (*d*)). From equations (2) and (3), p. 122, we have

$$p_n = p_0 \cos\theta, \qquad \ldots\ldots (3)$$

$$p_t = p_0 \sin\theta. \qquad \ldots\ldots (4)$$

By substituting the value of p_0 from equation (2) above, we obtain

$$p_n = p . \cos^2\theta, \qquad \ldots\ldots (5)$$

$$p_t = p . \sin\theta \cos\theta. \qquad \ldots\ldots (6)$$

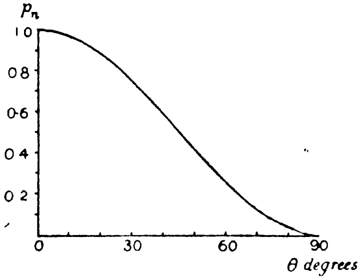


FIG. 148.—Variation of normal stress in a column

It will be easily seen, from equation (5), that p_n has its maximum value when θ is zero, the value being then p and the section AB in Fig. 147 (*a*). The value of p_n diminishes as θ is increased, being zero when $\theta = 90°$. Equation (6) may be written as

$$p_t = \tfrac{1}{2} p \sin 2\theta, \qquad \ldots\ldots (7)$$

an equation which shows that p_t has zero value when θ is zero, and that the value is again zero when θ is 90°. The maximum value will occur when 2θ is 90°, the value of the sine being then unity; θ will then be 45°, and the value of p_t will be

$$\text{Maximum value of } p_t = \tfrac{1}{2}p. \quad \ldots\ldots \quad \ldots \quad \ldots\ldots (8)$$

The fact that the section at 45° to the axis of the column has maximum intensity of shearing stress explains the reason why some materials, such as brick, stone or cement under compression, fracture along planes at 45° instead of simply crushing. Such materials are comparatively weak under shearing. The curves in Figs 148 and 149 have been plotted from equations (5) and (6), taking the maximum value of p_n as 1 ton per square inch, and illustrate the way in which p_n and p_t vary, depending on the angle at which the section is taken.

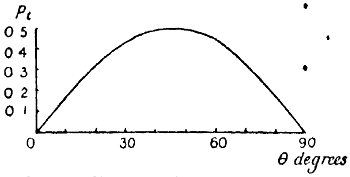


Fig. 149.—Variation of shear stress in a column.

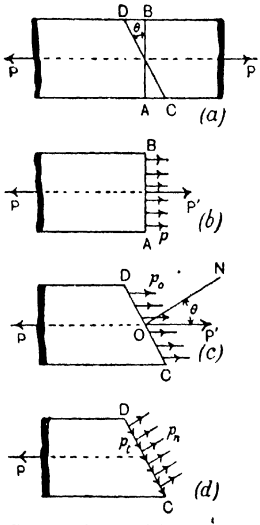


Fig. 150.—Normal and shear stresses in a tie bar.

The case of a rod under axial pulls may be worked out in a similar manner and the results will be identical, with the substitution of normal pull stress for the normal push stress which occurs in the column. Fig. 150 illustrates this case, and as it is lettered to correspond with the column diagrams there will be no difficulty in tracing the connection.

Stresses which are not uniformly distributed. A varying stress may be realised by considering a horizontal surface ABC (Fig. 151), having a number of slender vertical heavy rods of varying heights standing on it. Some of these rods are shown in the figure. The effect on the surface ABC, which is supposed to be covered entirely by the rods, will be to produce stress of varying intensity. There is, however, no difficulty in seeing that the resultant force on ABC will be the total weight of the rods, and that the line of the resultant force must pass through the centre of gravity of the whole of the rods taken together. We may deduce from this that, if a **stress figure** be drawn for a given section by erecting ordinates at all points of the section, of length to scale to represent the intensity of normal stress at each point, the resultant force will pass through the centre of volume of the stress figure. The magnitude of the resultant force may be found thus:

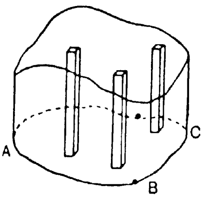


FIG. 151.—Representation of a varying stress.

Let p = stress intensity at a given point,
δa = a small area surrounding this point.

Then Resultant force = $\Sigma p \, . \, \delta a$, ... (1)

the summation being taken all over the section.

Equation (1) may be interpreted as meaning the volume of the stress figure, stress intensities being used for ordinates and square inches or other convenient units for units of area.

EXAMPLE. A rectangular surface ABCD is subjected to normal stress, which varies uniformly from zero along AD to 4 tons per square inch along BC (Fig. 152). AB is 4″ and BC is 3″. Find the resultant force, and show where it acts.

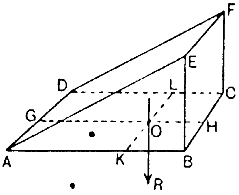


FIG. 152.—A uniformly varying stress.

The stress figure will be drawn in this case by erecting ordinates BE and CF, each to scale, representing 4 tons per square inch. Join EF, AE and DF, thus giving a stress figure of wedge shape. To find the magnitude of the resultant force, calculate the volume of the wedge by multiplying the area of the base by the ordinate of average height, viz. 2 tons per square inch. Thus,

$$\text{Resultant force} = R = 4 \times 3 \times 2$$
$$= \underline{24 \text{ tons.}}$$

The centre of volume of the wedge will lie vertically over a point O, found by the intersection of two lines GH and KL, G and H bisecting respectively AD and BC, and KB and LC being one-third of AB and CD respectively. R will then pass through O as shown.

It will be clear that, in the case of a uniform normal stress, the centre of volume of the stress figure lies in the normal drawn from the centre of area of the section. It therefore follows that, if a resultant normal force acts through the centre of area of a given section, a stress which will be distributed uniformly over the section will result.

In drawing stress figures, a useful convention is to draw the stress figure standing on one side of the section, for those parts of the section which are subjected to push stress, while pull stresses are represented by a stress figure standing on the other side of the section.

Shearing stress. In Fig. 153 (*a*) is shown a rectangular plate ABCD having shearing stress p_t distributed over its top edge. Let

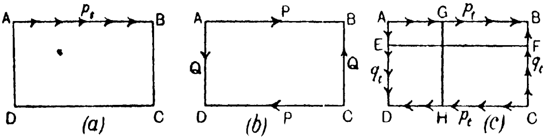


FIG. 153.—A plate under shearing stress.

the thickness of the plate from front to back be unity, then the total force along AB will be

$$P = p_t \times AB. \qquad (1)$$

Substituting P as shown in Fig. 153 (*b*), the plate may be equilibrated horizontally by the application along CD of an equal opposite force P; as P, P form a couple, equilibrium is completed by the application of equal opposite forces Q, Q along the edges AD and BC respectively, these forming a couple of moment equal and opposite to that of the first couple. For equilibrium we have

$$P \times AD = Q \times AB. \qquad (2)$$

Let all these forces be produced from shearing stresses applied to the edges of the plate (Fig. 153 (*c*)), and let q_t be the shearing stress which gives rise to Q, so that

$$Q = q_t \times AD. \qquad (3)$$

Substituting in (2), we have

$$p_t \times AB \times AD = q_t \times AD \times AB,$$

or

$$p_t = q_t. \qquad (4)$$

For the general equilibrium of the plate it is therefore necessary that equal shearing stresses be applied to all four edges.

Take any section EF of the plate as now stressed (Fig. 153 (*c*)), and consider the equilibrium of the portion ABFE (Fig. 154). From what has been said it will be seen by inspection of Fig. 154 that a shearing stress p_t must act along FE. Again, take another section GH (Fig. 153 (c)), and consider the equilibrium of the portion AGHD (Fig. 155). Inspection shows that a shearing stress p_t must act along GH. We conclude that if any rectangular block be subjected to shearing stresses, such stresses must be equal on all four edges, and there will be an equal shearing stress on any section which is parallel to any edge of the block.

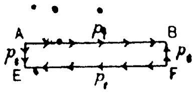


Fig. 154.

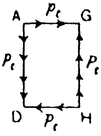


Fig. 155.

Cube under shear stress. For simplicity, consider a cube of unit edge, the elevation of which is ABCD (Fig. 156). Let shearing stresses p_t be applied as shown to those faces of the cube which are perpendicular to the paper. To find the stress on the diagonal section AC, cut the cube and consider the portion ABC (Fig. 157). The stresses along AB and CB produce forces p_t, p_t, acting at B; these will have a resultant r, acting at 45° to AB, and hence perpendicular to AC. The magnitude of r will be

$$r = p_t . \sqrt{2}.$$

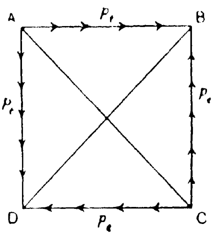


Fig. 156.—Cube under shear stress.

If r be produced it will evidently cut the diagonal AC at its middle point O, and may be balanced by an equal opposite force r applied at O as shown. Now r may be considered to be the resultant of a normal stress p_n uniformly distributed over the diagonal section AC, the intensity of this stress being

$$p_n = \frac{r}{\text{area of AC}}$$

$$= \frac{p_t . \sqrt{2}}{\text{AB} . \sqrt{2}},$$

$$= p_t.$$

This result shows that the diagonal AC is subjected to a normal pull stress of intensity equal to the given shearing stress. In the same way, by considering the portion ABD (Fig. 158), we may show that the diagonal BD is subjected to a normal push stress p_n of intensity also equal to the given shearing stress.

Supposing we have a rectangular plate ABCD (Fig. 159) having shearing stresses p_t applied to its edges. Consider any square portion

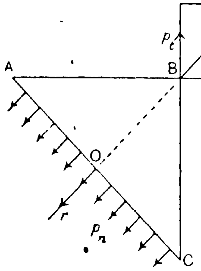


FIG. 157.—The diagonal AC is under pure normal pull stress.

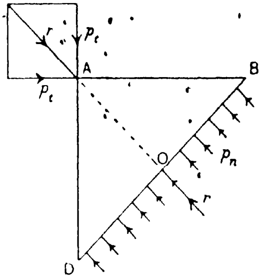


FIG. 158.—The diagonal BD is under pure normal push stress.

abcd having its edges parallel to the sides of the rectangle. We have already seen that these edges have equal shearing stresses p_t acting on them. Hence the diagonal sections of the square have normal pull stress on *ac* and normal push stress on *bd*, the intensity of each of these being p_t. We therefore infer that any section of the plate at 45° to an edge will have normal stress of push or pull acting on it of intensity equal to the given shearing stress.

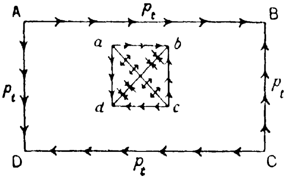


FIG. 159.—Stresses in a rectangular plate.

Two sections of a body intersecting at 90°, and having purely normal stresses acting on them, are called **principal axes of stress**; the stresses are called **principal stresses**.

[For laboratory experiments on stress and strain, see Chapter XIII.]

EXERCISES ON CHAPTER VI

1. A round rod has to carry a pull of 15 tons. Find the diameter if the safe stress is 6 tons per square inch.

2. A short hollow cast-iron column is 6 inches in external and 4½ inches in internal diameter. Calculate the safe load if the stress allowed is 7 tons per square inch.

3. Plates 0·5 inch thick are to be connected by a double-riveted lap joint. Find the principal dimensions of the joint. Take $d = 1{\cdot}2\sqrt{t}$; $f_t = 6$, $f_s = 5$, $f_b = 10$, in tons per square inch. Find the efficiency of the joint.

4. Answer Question 3 for a double-riveted butt joint with two cover-straps. The plates are $\frac{3}{4}$ inch thick. Allow 1·75 rivet sections per rivet under shear.

5. Two plates, each 16 inches by 0·5 inch thick, are to be connected by a butt joint having two cover straps. The joint is to be under pull. Take stresses as given in Question 3, and find the required number of rivets $\frac{7}{8}$ inch in diameter. What would be the safe load for the joint?

6. A cylindrical boiler shell is 7·5 feet in diameter, the working pressure is 150 lb. per square inch. If the efficiency of the longitudinal riveted joint is 75 per cent., find the thickness of the plate for a safe stress of 5 tons per square inch. What will be the stress on a longitudinal section of the plate at some distance from the joint? Find also the stress on a circumferential section of the plate.

7. A spherical vessel, 6 feet in diameter, is subjected to an internal gaseous pressure of 120 lb. per square inch. Find the thickness of plate required for a joint efficiency of 70 per cent. and a safe stress of 12,000 lb. per square inch.

8. A steel bar, 6 inches wide, 0·5 inch thick and 30 feet long, carries a pull of 18 tons. Find the extension in length and the contractions in width and thickness when the load is applied. Take E = 13,500 tons per square inch and *m* = 3·5.

9. A vertical square plate of steel, 6 feet edge and 0·75 inch thick, has shearing forces of 200 tons acting along each edge. Suppose the lower edge to be horizontal and to be fixed rigidly, what will be the horizontal movement of the top edge when the load is applied? Take C = 5500 tons per square inch.

10. A cylinder for storing compressed oxygen under a pressure of 120 atmospheres is 3 feet long and 5 inches diameter; the thickness of the steel plate of which it is constructed is $\frac{1}{4}$ inch. Find the alterations in diameter and length when the cylinder is being charged, and hence find the change in cubic capacity of the cylinder. Take E = 13,000 tons per square inch and *m* = 4.

11. A rod of brass 4 feet long and 0·5 inch diameter is cooled from 150° F. to 60° F. Find what forces are required in order to prevent any change in the length. Take E = 5700 tons per square inch and the coefficient of expansion = 0·00001.

12. A steel boiler tube is 15 feet long, 3 inches internal diameter and is made of metal 0·3 inch thick. Supposing that half its natural expansion due to a range of temperature of 240° F. is prevented, what forces will the tube exert in the direction of its length? What will be the stress in the tube? Take E = 13,500 tons per square inch and ϵ = 0·000007.

13. A tube of copper 1·5 inch bore and 4 feet long, of metal 0·1 inch thick, has an internal steel rod 0·5 inch diameter, having swelled ends to which the tube is brazed. Suppose there to be no self-stressing at first, what will be he stresses in the copper and in the steel if both are raised in temperature o an extent of 100° F.? Take E_s = 13,500 and E_c = 6200 tons per square nch; coefficient of expansion of steel = 0·000007 and of copper = 0·0000096.

14. A reinforced concrete column has a square section of 15 inches edge, and has four reinforcement bars of steel 1·5 inches diameter. Find the safe load if the stress in the concrete is 500 lb. per square inch. How much of this load is carried by the steel? Take the ratio of Young's modulus for the steel and for the concrete to be 15.

15. A tie bar has a rectangular section 4 inches by 1·5 inches, and carries a pull of 30 tons. Find the normal and tangential stresses on sections making angles of 0°, 30°, 45°, 60° and 90° with the axis of the bar. Plot curves showing the relation of the stresses and angles.

16. Draw the stress figure for a rectangular section 30 feet by 1 foot; there is a normal push stress of 4 tons per square foot at one short edge, and the stress varies uniformly to a normal push stress of 0·5 ton per square foot at the opposite edge. What is the resultant force on the section? Show where it acts.

17. A ferro-concrete column is 14 inches square in cross section; the main reinforcement consists of four longitudinal 2-inch diameter round steel rods, one rod being placed close to each angle of the cross section. The value of E (Young's modulus) for the steel is 29,000,000 lb. per square inch and for the concrete 3,000,000 lb. per square inch. If a gross compressive load of 60 tons is supported by this column, what is the gross load and the compressive stress per square inch in (*a*) the concrete, (*b*) the reinforcing bars? (B.E.)

18. A column which carries a load of 300,000 lb. rests on a foundation whose area is 10 square feet; find the normal and tangential components of the stress on a plane in the foundation, whose inclination to the horizontal is 15°. Find also the inclination of the plane on which the tangential stress is a maximum, and calculate this maximum value. (L.U.)

19. The London Building Act, 1909, allows stresses in steel of $5\frac{1}{2}$ tons per square inch in shear and 11 tons per square inch of bearing area, but limits the shearing strength of a rivet in double shear to 1·75 times that of a like rivet in single shear. Prepare a table of rivet strengths, with these stresses, for 1-inch rivets in single and double shear with plates of $\frac{3}{8}$ inch, $\frac{1}{2}$ inch, $\frac{5}{8}$ inch, $\frac{3}{4}$ inch and $\frac{7}{8}$ inch in thickness. (I.C.E.)

20. A cylinder, 8 inches external and 4 inches internal diameter, has an internal fluid pressure of 2000 lb. per square inch. Find the maximum and minimum hoop tensions.

CHAPTER VII.

STRENGTH OF BEAMS.

Some definitions. **Beams** are parts of a structure, usually supported horizontally, for the purpose of carrying loads applied transversely to their lengths. The term **beam** or **joist** is understood generally to refer to a structure of moderate size and constructed of one piece of material, such as the timber beams or joists used for supporting floors, or rolled steel beams also often used for floors. Beams of larger size and constructed of several parts secured together are called **girders**.

Any beam will bend when loaded, owing to the strains which take place in the material. If straight initially, it will take the shape of some curve; if curved initially, it will alter its curvature. The theory of the strength and stiffness of beams may be developed from the fundamental principles that (*a*) the beam as a whole is in equilibrium under the action of the **external forces**, which term embraces the applied loads and the reactions of the supports; (*b*) any portion of the beam lying between two sections is in equilibrium under the action of any external forces applied to that portion, together with the stresses communicated across the sections from the other parts of the beam.

Pure bending occurs when the following conditions are complied with. (*a*) There must be no resultant push or pull along the beam due to the action of the external forces; this condition will be realised in the case of a horizontal beam carrying vertical loads and so supported that the reactions are vertical. (*b*) The external forces must be all applied in the plane in which the beam bends. In connection with the latter condition, it

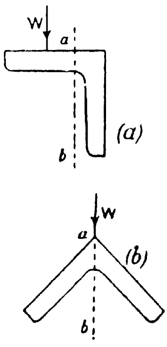


FIG. 160.—Unsymmetrical and symmetrical angle sections.

may be explained here that it does not follow necessarily that a beam carrying vertical loads will bend in a vertical plane. Side or horizontal bending as well as vertical bending will occur if the beam section be not symmetrical about a vertical line passing through the

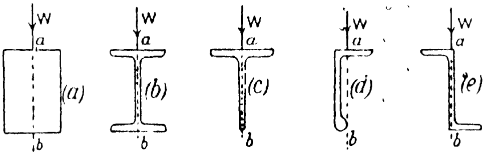


FIG. 161.—Examples of symmetrical and unsymmetrical sections

centre of area of the section. For example, the angle section shown in Fig. 160 (*a*) is not symmetrical about the vertical *ab*, and hence pure bending cannot occur with vertical loads. If the angle be situated as in Fig. 160 (*b*), symmetry about *ab* is secured, and pure bending will occur, *i.e.* the beam when loaded vertically will bend in the vertical plane, of which *ab* is the trace. Figs. 161 (*a*), (*b*) and (*c*) show other examples of symmetrical sections. An unsymmetrical bulb angle and **Z** bar are shown in Figs. 161 (*d*) and (*e*). Pure bending alone will be considered.

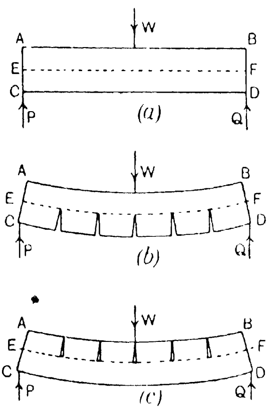


FIG. 162.—Longitudinal tension and compression in a loaded beam

Nature of the stresses in a beam. In practice, the problem which has to be solved first is generally that of finding the reactions of the supports for given loading. In simple cases of pure bending, in which the beam rests on two supports, but is not fixed, the solution may be obtained by the methods given in Chaps. III. and IV. We now proceed to examine the stresses in the material of a loaded beam. The nature of these may be understood by consideration of the beam shown in Fig. 162 (*a*), which carries a single load W, and is supported at its ends. Supposing a number of saw cuts to be made in the lower portion of the beam (Fig. 162 (*b*)), it is evident that these will tend to open out on the beam being loaded. Had the saw cuts been made in the upper portion (Fig. 162 (c)), it is clear that these would tend to close on loading the beam. We are therefore justified in concluding that longitudinal fibres situated in the lower portion of this beam are under pull, while those lying in the upper portion are under push.

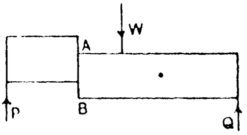


Fig. 163.—Shearing tendency in a loaded beam

Again, it will be evident that if a vertical section AB be taken (Fig. 163), there is a tendency for the left-hand portion to slide upwards and for the right hand portion to slide downwards, indicating that there must be shear stresses acting on the section.

Bending moment and shearing force. Let the beam shown in Fig. 164 (*a*) be cut at any section AB, and consider the problem of restoration to equilibrium of the left hand portion (Fig. 164 (*b*)). In general, the external forces will not be in equilibrium unaided, hence stresses will be required at the section AB. Whatever may be the magnitudes and directions of these stresses, they may be resolved into components along and perpendicular to AB, and their resultant forces X, Y and S substituted for the actual stresses. The problem may now be solved by application of the equations (p. 64), denoting horizontal and vertical forces by the suffixes x and y respectively

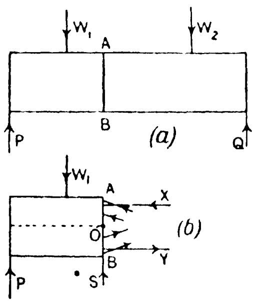


Fig. 164.—Bending moment and shearing force at a beam section

$$\Sigma P_x = 0, \quad \ldots\ldots (1)$$

$$\Sigma P_y = 0, \quad \ldots\ldots (2)$$

$$\Sigma P_x y + \Sigma P_y x = 0. \quad \ldots\ldots (3)$$

Since there are no forces other than X and Y acting along the beam, it follows from equation (1) that these are equal, and hence they form a couple.

Equation (2) shows that the algebraic sum of the forces parallel to AB must be zero, and hence S must be equal to the algebraic sum of the external forces applied to the portion of the beam under consideration. S is called the **shearing force**, and will produce shear stress distributed in some manner over the section AB.

The meaning of equation (3) may be ascertained by taking moments about any axis in the section AB, the axis being perpendicular to the plane of bending and indicated by O in Fig. 164 (*b*). The second term clearly refers to the resultant moment of the external forces applied to the portion of the beam considered (notice S has no moment about this axis); the first term refers to the moment of the couple produced by the equal forces X and Y. The equation shows that these moments must be equal. The resultant moment of the external forces is termed the **bending moment,** and the moment of the couple is termed the **moment of resistance.**

Equation (3) may thus be read:

Bending moment at AB = moment of resistance at AB.

It will be evident, since the forces X, Y and S are communicated as stresses from the right-hand portion to the left-hand portion of the beam, and hence are mutual interactions, that their values would be unaltered had the calculation been performed by considering the right-hand portion of the beam instead of the left-hand portion. Hence the bending moment and shearing force at any section may be calculated from the loads and reactions applied to either portion of the beam. If the calculations be made for both portions the results should agree, thus affording a check on the accuracy of the work.

Rules for bending moment and shearing force. The bending moment at any section of a beam means the tendency to rotate either portion of the beam about that section, and is calculated by taking the algebraic sum of the moments about the section of all the forces acting either on one or other portion of the beam.

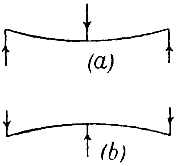


FIG. 165.—Positive and negative bending.

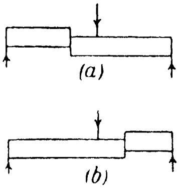


FIG. 166.—Positive and negative shear.

The shearing force at any section of a beam means the tendency of one portion of the beam to slide on the other portion, and is calculated by taking the algebraic sum of all the forces acting either on one or other portion of the beam.

It is usual to call bending moments positive when the tendency is to cause the beam to become convex downwards, as in Fig. 165 (*a*). Fig. 165 (*b*) shows a case of negative bending moment. Shearing forces are denoted as positive if the tendency to slide is that shown in Fig. 166 (*a*), and negative if that in Fig. 166 (*b*).

Bending-moment and shearing-force diagrams. Such diagrams are often required in the solution of beam and girder problems, and may be drawn by first calculating the values of the bending moments and shearing forces at a sufficient number of sections of the given beam. A horizontal datum line is chosen of length to scale to represent the length of the beam; the calculated values are then set off as ordinates, above or below the datum line according as they are positive or negative. The ends of the ordinates being joined by straight lines, or a curve depending on the circumstances, the result gives complete representations of the bending moments and shearing forces throughout the beam.

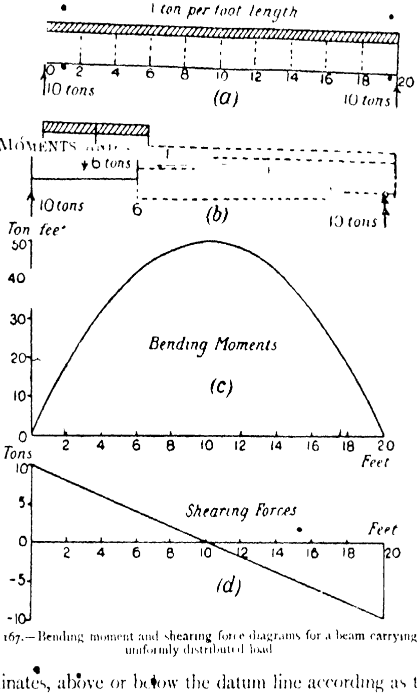


FIG. 167.—Bending moment and shearing force diagrams for a beam carrying a uniformly distributed load

EXAMPLE. A beam of 20-feet span is supported at its ends and carries a uniformly distributed load of 1 ton per foot length (Fig. 167 (*a*)). Draw bending-moment and shearing-force diagrams.

To do this, first calculate the bending moments and shearing forces at sections 2 feet apart throughout the length of the beam. The reaction of each support will be 10 tons. Sample calculations are given below for the section 6 feet from the left-hand support, together with a complete table of the results from which the diagrams in Fig. 167 (c) and (d) have been plotted. As the loading is continuous, it is evident that both the bending moment and shearing force vary continuously; hence neither diagram shows any break or sudden change in direction.

For section 6 (Fig. 167 (b)),

$$\begin{aligned}\text{Bending moment} &= (10\times 6)-(6\times 3)\\ &= 60-18\\ &= 42 \text{ ton feet, positive.}\\ \text{Shearing force} &= 10-6\\ &= 4 \text{ tons, positive}\end{aligned}$$

Section	Bending moment, ton feet	Shearing force, tons	Section	Bending moment, ton feet	Shearing force tons
0	0	+10	10	+50	0
2	+18	+8	12	+48	−2
4	+32	+6	14	+42	−4
6	+42	+4	16	+32	−6
8	+48	+2	18	+18	−8
10	+50	0	20	0	−10

Diagrams of bending moment and shearing force for four important cases are given in Fig. 168. These cases are of constant occurrence in practice, and should be worked out independently by the student.

Shearing force at a concentrated load. Any difficulty which may occur in dealing with the shearing force at a concentrated load will disappear if it is remembered that there is never any case of a load being concentrated on a geometrical point, or line. This arises from the fact that such would produce an infinitely great stress, the area being zero. All loads are distributed really over a small portion of the length of the beam. In Fig 169 (a), a load W is shown resting on a beam, and it may be convenient for some purposes, such as the calculation of the reactions, to speak of it as concentrated at its centre of gravity C; actually it is distributed over a short length DE of the beam. The shearing force at any section lying between A and D will be positive and equal to P; for any section between B and E the shearing force will be negative and equal to Q. For sections lying between D and E, the shearing force will be +P at D, and will gradually diminish to zero, then will change sign to negative, and will increase numerically to $-Q$ at E. The section at which zero shearing force occurs may be determined

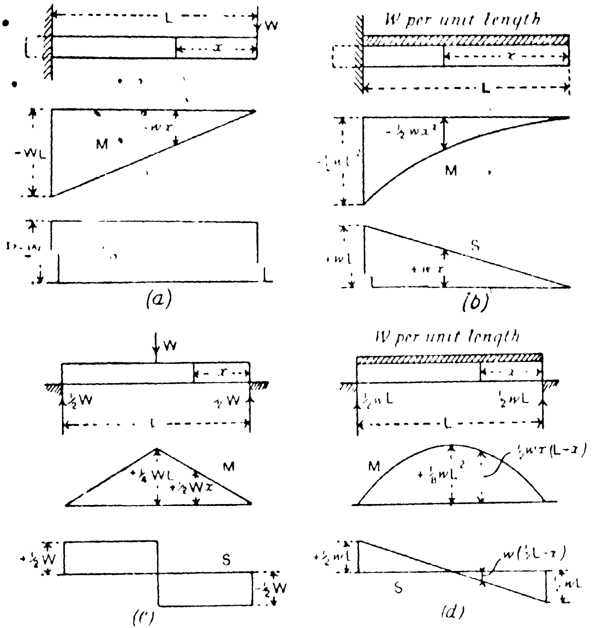


FIG. 168.—Bending moment and shearing force diagrams for four important cases.

from the consideration that the portion of W lying to the left of the section must be equal to P. Thus,

$$P \times AB = W \times CB,$$

$$\therefore P = \frac{CB}{AB} . W. \quad \ldots\ldots\ldots \quad (1)$$

Let F be the section of zero shear, then

$$DF : P = DE : W;$$

$$\therefore P = \frac{DF}{DE} . W. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Hence

$$\frac{DF}{DE} = \frac{CB}{AB}$$

or

$$DF : DE = CB : AB. \quad (3)$$

We infer from this result, that the section of zero shear divides the load into segments which are inversely proportional to the segments into which the centre of gravity of the load divides the beam. The shearing-force diagram for this case is shown in Fig. 169 (*b*)).

In solving problems of this character, it is usually sufficient to state the shearing forces on each side of the load given concentrated.

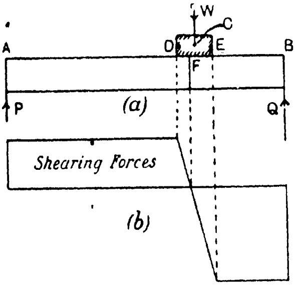


Fig. 169.—Shearing force at a load.

Example.—Draw the bending-moment and shearing-force diagrams for the beam shown in Fig. 170 (*a*).

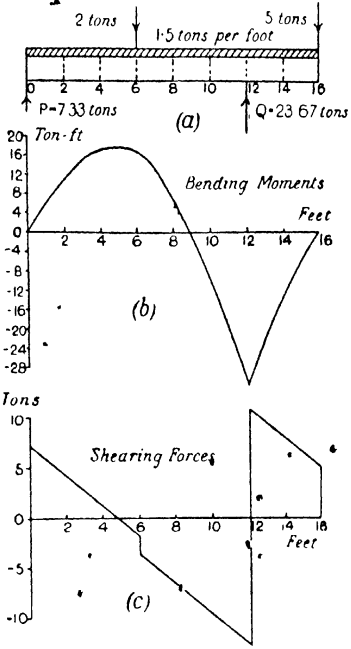


Fig. 170.—Bending moment and shearing force diagrams for a loaded beam.

Sections at 2-feet intervals have been chosen, and the calculations have been made in each case by considering the left-hand portion. Clockwise moments have been considered as positive and anti-clockwise as negative, thus giving the proper sign for the results of the bending-moment calculations. Forces acting upwards have been taken as positive and downward forces as negative, giving the proper sign for the shearing-force results. The calculations are given in the table, and the diagrams have been plotted from the results as shown in Fig. 170(*b*) and (*c*). Two results are given for the shearing force at the 6 feet and the 12 feet sections; the first is that immediately to the left of the section, the second is that just to the right of the section. The shearing force at 16 feet from P is that immediately to the left of the 5-ton load.

BENDING MOMENTS AND SHEARING FORCES FOR A LOADED BEAM.

Distance of section from P, feet	Bending moment, ton feet.	Shearing force, tons
0	0	P = +7·33
2	(P × 2) − (3 × 1) = +11·67	P − 3 = +4·33
4	(P × 4) − (6 × 2) = +17·33	P − 6 = +1·33
6	(P × 6) − (9 × 3) = +16·99	P − 9 = −1·67
		P − 9 − 2 = −3·67
8	(P × 8) − (12 × 4) − (2 × 2) = +6·67	P − 12 − 2 = −6·67
10	(P × 10) − (15 × 5) − (2 × 4) = −9·67	P − 15 − 2 = −9·67
12	(P × 12) − (18 × 6) − (2 × 6) = −32	P − 18 − 2 = −12·67
		P − 18 − 2 + Q = +11·0
14	(P × 14) − (21 × 7) − (2 × 8) + (Q × 2) = −13	P − 21 − 2 + Q = +8·0
16	(P × 16) − (24 × 8) − (2 × 10) + (Q × 4) = 0	P − 24 − 2 + Q = +5·0

Graphical methods of obtaining the bending-moment diagram In Fig. 171 (*a*) is shown a beam carrying two loads W_1 and W_2. The reactions of the supports P and Q have been determined by means of the force polygon shown in Fig. 171 (*b*), and the link polygon, Fig. 171 (*c*), as has been explained on p. 70. It will happen usually that the closing link *ab* of the link polygon is not horizontal, and it is convenient for our present purpose that it should be so. To obtain this result, the pole O of the force polygon in Fig. 171 (*b*) has been moved vertically to O′ in the horizontal line through A. A new link polygon (Fig. 171 (*d*)) is then drawn, having its sides parallel to the dotted lines radiating from O′ in Fig. 171 (*b*); *a′b′* will now be horizontal.

The triangles *a′ed′* and O′AB are similar; hence

$$\frac{d'e}{a'e} = \frac{\mathrm{AB}}{\mathrm{O'A}}$$

or

$$d'e \times \mathrm{O'A} = \mathrm{AB} \times a'e. \quad \ldots\ldots\ldots\ldots (1)$$

Now AB represents the reaction P, hence AB × $a'e$ represents the moment of P about the section at W_1, *i.e.* represents the bending moment at W_1. Therefore the ordinate $d'e$ of the link polygon, when multiplied by the horizontal polar distance O'A, gives the bending moment at W_1

In the same way, from the similar triangles $c'fb'$ and DAO' we may show that $c'f$ × O'A represents the bending moment at W_2. Therefore, the link polygon $a'd'c'b'$ is the bending-moment diagram for the whole beam.

To obtain the scale of the diagram, it will be noted that both $d'e$ and AB in (1) above should be measured to the scale of force used in drawing the force diagram, Fig. 171 (*b*), also, both $a'e$ and O'A should be measured to the scale of length used in drawing the beam in Fig. 171 (*a*). Let these scales be *p* tons per inch height of DB in Fig 171 (*b*) and *l* feet per inch length in Fig. 171 (*a*). Then, if any ordinate *y* of Fig. 171 (*d*) be measured in inches, and if O'A be measured also in inches, the bending moment at the section of the beam vertically above *y* will be given by

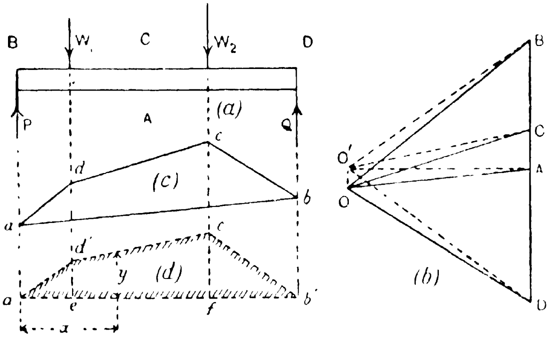


FIG. 171 —Bending-moment diagram by the link polygon method

$$M_x = y \,.\, O'A \,.\, pl \text{ ton feet.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Another useful graphical method of obtaining the bending-moment diagram is illustrated in Figs. 172 (*a*) and (*b*). A base line OA is selected of length equal to that of the beam. Choosing a convenient scale of moments, AB is set off equal to PL, and is divided at E by setting off BE equal to W_1a_1. The remainder EA of BA will evidently be equal to W_2a_2, as is shown by the equation of moments about the right-hand support, viz.:

$$PL = W_1a_1 + W_2a_2 \qquad \ldots\ldots \qquad \ldots\ \ldots (1)$$

Join OB cutting the vertical through W_1 in C, join CE cutting W_2 produced in F, join FA. Then OCFA is the bending moment diagram for the complete beam

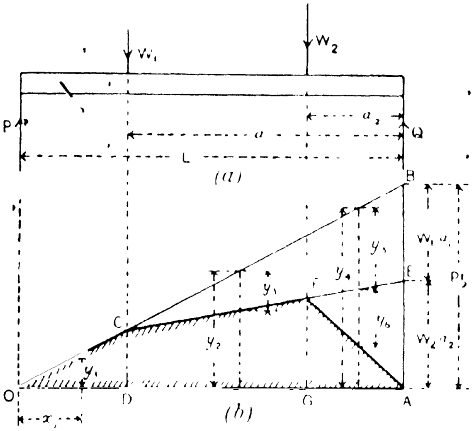


FIG. 172 Bending moment diagram by the method of graphical moments

To prove this, take any ordinate y_1 From similar triangles, we have

$$\frac{AB}{OA} = \frac{y_1}{x_1},$$

$$y_1 = \frac{AB}{OA} \cdot x_1.$$

Now $AB = P \times L$ and $OA = L$, hence

$$y_1 = P x_1,$$

that is, y_1 represents the moment of P about the section of the beam vertically over y_1: hence OCD is the bending moment diagram for the portion of the beam lying between P and W_1 In the same way, it may be shown that y_3 represents the moment of W_1 about the section vertically over y_3; y_2 represents the moment of P about the same section, and has the opposite sign to that of W_1, hence $(y_2 - y_3)$ is the bending moment for this section. Similarly $(y_4 - y_5 - y_6)$ is the bending moment for the section vertically over y_4

In applying either of these graphical methods to the case of distributed loads, these loads may be cut up into portions of short length and the weight of each concentrated at its centre of gravity. The result will give a nearly equivalent system of concentrated loads.

Bending of a beam. Suppose we have a beam consisting of a number of planks of equal lengths laid one on the other, and supported at the ends. A load W, applied at the centre of the span, will cause all the planks to bend in a similar fashion, and, as their lengths will remain equal, the planks will overlap at the ends as shown (Fig. 173 (*a*)). Strapping the planks firmly together will prevent this occurring, and the beam will now bend as a whole, the ends of the planks remaining in one plane (Fig. 173 (*b*)). The upper planks have become shorter and the lower planks longer; hence, one intermediate plank will be unaltered in length. Assuming the middle plank to remain the same length as at first, it is clear that all planks above the middle must have become shorter, and all below the middle, longer than at first. It will also be evident that the change of length, and consequently the longitudinal strain, of any plank will depend on its distance above or below the middle, being greater as the distance is increased.

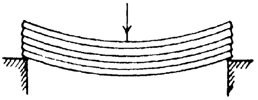


FIG. 173 (*a*).—Plank beam.

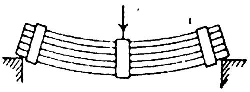


FIG. 173 (*b*).—Strapped plank beam.

For ordinary practical beams, it is assumed that no section is warped when loads are applied; thus transverse sections which were plane in the unloaded beam remain plane when the loads are applied. While this assumption is justified on appeal to experiment, it must be noted that it is no longer true if the beam has been loaded excessively so that the elastic limit of the material has been exceeded.

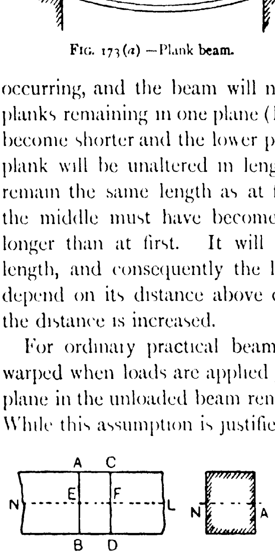


FIG. 174.—Neutral axis of a beam section.

Some important definitions. In Fig. 174 is shown a portion of an unloaded beam. We have seen already that there will be one longitudinal section which will not suffer change of length when the beam is loaded; let NL represent this section, which is called the **neutral lamina**. Any plane transverse section, such as AB or CD, will intersect the neutral lamina in a straight line, which is shown by NA in the cross section; this line is called the **neutral axis** of the section.

Longitudinal strains. In Fig. 175 (*a*) is shown a portion of a bent beam. Two adjacent and originally parallel sections AB and CD have been altered in position by the bending to A′B′ and C′D′. *ab* is any longitudinal fibre parallel to the neutral lamina NL, and has been changed in length from *ab* to *a′b′*, the change being one of shortening if *ab* lies on the concave side of NL and of extension if *ab* lies on the convex side. The actual change of length is made up of the two pieces *aa′* and *bb′*. It is clear from the geometry of the figure that the combined length of these pieces will be proportional to the distance of *ab* from NL; thus

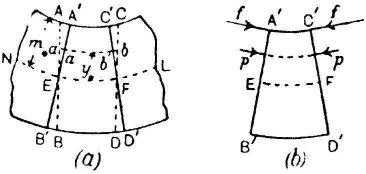


FIG. 175 —Longitudinal strains and stresses in a beam.

$$(aa' + bb') : (AA' + CC') = Ea : EA.$$

The strain of *ab* will be given by

$$\text{Strain of } ab = \frac{aa' + bb'}{ab}.$$

Also,

$$\text{Strain of AC} = \frac{AA' + CC'}{AC}.$$

Now all fibres lying between AB and CD were originally of equal lengths, viz. EF, hence their strains are proportional simply to their changes in length, and hence to the distances of the fibres from NL. We may therefore write, taking *y* and *m* to be the distances respectively of *ab* and AC from NL:

$$\text{Strain of } ab : \text{strain of AC} = y : m,$$

or

$$\frac{\text{strain of any fibre}}{\text{distance of fibre from NL}} = \text{a constant}.$$

Longitudinal stresses. Changes of length of any fibre must have been brought about by longitudinal stresses of push or pull, depending upon whether shortening or extension has been produced. Thus *a′b′* in Fig. 175 (*a*) must be under longitudinal push; any fibre lying on the convex side of NL will be under longitudinal pull. Assuming the elastic limit not to be exceeded, these stresses will be proportional to the strains. Hence, from what has been said above regarding the strains, the longitudinal stress on any fibre will be proportional to its distance from NL.

Let $f =$ longitudinal stress on A'C' (Fig. 175,(*b*)),

$p =$ " " $a'b'$.

Then $f : p = m : y$,

or $\frac{f}{m} = \frac{p}{y} =$ a constant.

The student will observe that fibres under longitudinal push stress not only shorten, but also expand laterally, while those under pull stress contract laterally. The ordinary theory of beams assumes that such lateral changes take place freely, the justification being that calculations based on the ordinary theory agree very closely with experimental results. The effect of the lateral changes on the section of a beam bent convex downwards will be understood by reference to Fig. 176, in which the lateral contractions of the lower fibres and the lateral expansions of those above the neutral lamina have the effect shown of causing the cross section apparently to be bent convex upwards, *i.e.* in the opposite sense to that of the length of the beam. The transverse curvature is called **anticlastic**, and may be observed very well if a rubber beam be experimented upon. The interference of anticlastic bending with the ordinary theory of beams will be most marked with a very broad beam of little depth, a strip of clock spring, for example.

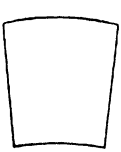


FIG. 176.—Anticlastic curvature

Moment of resistance. Knowing the nature of the distribution of the stresses over the cross section, we may now proceed to find an expression for the moment of resistance. Referring to Fig. 177, showing a part side elevation and section of a loaded beam under pure bending, let a be the cross-sectional area of any fibre.

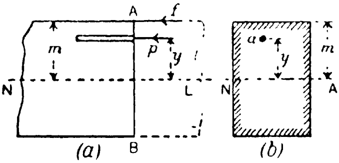


FIG. 177.—Moment of resistance of a beam.

Stress on $a = p$.

Now $p : f = y : m$;

$$\therefore\ p = \frac{y}{m} f \quad \ldots \ldots \ldots \ldots \ldots (1)$$

Also,

$$\text{Force on } a = pa = \frac{y}{m}fa$$

$$= \frac{f}{m}ay \quad . \quad . \quad . \quad . \quad (2)$$

The force on any other fibre would be obtained in a similar manner, and, as these forces will be both push and pull when taken over the whole section, we may obtain the resultant force by summing algebraically. Thus:

$$\text{Resultant force on section} = \Sigma\frac{f}{m}ay$$

$$= \frac{f}{m}\Sigma ay. \quad . \quad . \quad . \quad . \quad (3)$$

The factor Σay simply means the moment of area of the whole section about N.A., and, as in pure bending there is no resultant force along the length of the beam, we may equate equation (3) to zero. Now $\frac{f}{m}$ will not be zero, hence

$$\Sigma ay = 0. \quad . \quad . \quad . \quad . \quad (4)$$

This latter result can only be true provided N.A., the axis about which moments of area are to be taken, passes through the centre of area of the section and is perpendicular to the plane of bending. Hence, we have a simple rule for the position of the neutral axis of any section. The methods of finding the centres of gravity of thin sheets, discussed in Chapter III., may be applied.

Again, taking moments about N.A., and using equation (2) for the force on a, we have

$$\text{Moment of the force on } a = \frac{f}{m}ay \times y$$

$$= \frac{f}{m} \cdot ay^2. \quad . \quad . \quad (5)$$

A similar expression would give the moment of the force on any other fibre, and it will be noticed that all such moments will have the same sign independent of that of y, as the y has been squared in each case. The total moment may be obtained by summation, thus:

$$\text{Total moment of resistance} = \frac{f}{m}\Sigma ay^2. \ldots\ldots\ldots\ldots (6)$$

In this result, Σay^2 may be termed the **second moment of area** of the section, thus distinguishing it from the first moment, which would be Σay. The name **moment of inertia** is applied more commonly to Σay^2, arising on account of its similarity to the expression used in calculating the moment of inertia of a thin plate.

The moments of inertia of many simple sections may be calculated easily by application of the methods of the integral calculus. Rolled sections are dealt with more easily by a graphical process, which will be explained later. Writing I_{NA} for the moment of inertia of the section with reference to the neutral axis, and making use of what has been said on p. 134, we have

Bending moment = moment of resistance,

or
$$M = \frac{f}{m} I_{NA} \quad \ldots\ldots\ldots\ldots \quad (7)$$

This expression may be applied by first calculating the bending moment at the given section of the beam. It is useful to choose m as the greatest ordinate of the section, using NA as a datum line, when f, which is the stress on the fibre at a distance m from NA, will be the maximum value of the stress on the section. An example will render the method clear.

EXAMPLE. A beam of 12-feet span carries a uniformly distributed load of 0·5 ton per foot run, together with a load of 2 tons at 3 feet from one end (Fig. 178). Given that the moment of inertia of the rectangular section is 180 in inch units, find the greatest stress on the section at the middle of the span, which is 10 inches deep.

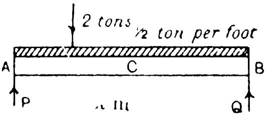


Fig. 178.

To find the reactions, take moments about B (Fig. 178):

Total distributed load = 6 tons.

$$P \times 12 = (6 \times 6) + (2 \times 9) = 54,$$
$$P = \frac{54}{12} = 4·5 \text{ tons.}$$

As a check, take moments about A.

$$Q \times 12 = (6 \times 6) + (2 \times 3) = 42,$$
$$Q = \frac{42}{12} = 3·5 \text{ tons}$$
$$P + Q = 4·5 + 3·5$$
$$= 8 \text{ tons} = \text{total load.}$$

Now find the bending moment at C, thus

$$M_C = (P \times 6) - (2 \times 3) - (3 \times 3)$$
$$= 27 - 6 - 9$$
$$= 12 \text{ ton-feet}$$
$$= 144 \text{ ton-inches.}$$

Or
$$M_C = (Q \times 6) - (3 \times 3)$$
$$= 21 - 9$$
$$= 12 \text{ ton-feet, as before.}$$

Again, taking $m = 5$ inches, we have

$$M_c = \frac{f}{m} I_{NA},$$

$$144 = \frac{f}{5} \cdot 180,$$

$$f = \frac{5 \times 144}{180}$$

$$= 4 \text{ tons per square inch.}$$

In solving beam problems it is advisable to take all dimensions for bending moments and resisting moments in inches.

Modulus of a beam section. The modulus of a beam section may be defined as the quantity by which the stress intensity at unit distance from the neutral axis must be multiplied in order to give the moment of resistance of the section. Taking the equation,

$$\text{Moment of resistance} = \frac{p}{y} I_{NA},$$

let y be unity, and let p_1 be the stress corresponding to this value of y.

Then $$\text{Moment of resistance} = p_1 I_{NA}$$
$$= p_1 Z_1,$$

where Z_1 is a modulus of the section.

Another modulus may be obtained by making use of the maximum stress form of the equation for the strength of a beam, viz.

$$\text{Moment of resistance} = \frac{f}{m} I_{NA}$$
$$= f \frac{I_{NA}}{m}$$
$$= fZ,$$

where Z is the modulus of the section, and is found from

$$Z = \frac{I_{NA}}{m}.$$

The latter is the more useful form of modulus in practice; its value differs numerically from that of Z_1. It will be noted that only sections which are symmetrical above and below the neutral axis will have equal values of m and f for tension and compression. Such sections have one value only for the modulus, all others having two values, one corresponding to the maximum tensile stress, the other to the maximum compression stress.

Let f_t = maximum tensile stress,
m_t = distance of f_t from the neutral axis,
f_c = maximum compressive stress,
m_c = distance of f_c from the neutral axis.

Then, since the bending moment M at any section equals the moment of resistance at that section, we may write

$$M = \frac{f_t}{m_t} I_{NA}$$

$$= f_t Z_t,$$

where $Z_t = I_{NA}/m_t$ is the tension modulus.

Also,

$$M = \frac{f_c}{m_c} I_{NA}$$

$$= f_c Z_c,$$

where $Z_c = I_{NA}/m_c$ is the compression modulus.

These results may be written

$$f_t = \frac{M}{Z_t},$$

$$f_c = \frac{M}{Z_c},$$

from which it may be inferred that the given safe stresses in tension and compression respectively must not be exceeded by the values obtained by dividing the bending moment at any section by the tension or compression modulus of the section.

Graphical method of finding the neutral axis and moment of inertia of a section. Advantage is taken of the fact that the neutral axis passes through the centre of area of the section. To illustrate the method, reference is made to Fig. 179, in which is given an irregular figure, and it is required to draw a line through the centre of area parallel to OX, and also to find the moment of inertia of the figure about OX.

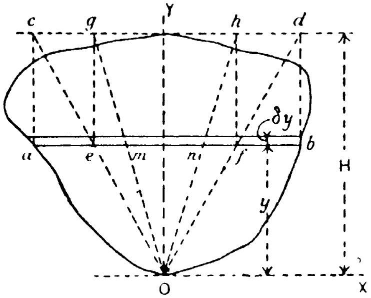


FIG. 179.—Graphical method of finding the neutral axis and moment of inertia.

Draw any convenient axis OY perpendicular to OX, and take any narrow strip ab parallel to OX. Let the breadth of ab be δy and let y be the distance of ab from OX. The area of the strip will be $(ab \,.\, \delta y)$ and its moment of area about OX will be

$$\text{Moment of area of strip} = ab \,.\, \delta y \,.\, y. \quad \ldots \quad \ldots\ldots \quad (1)$$

Draw *cd* parallel to OX through the highest point on the figure; draw *ac* and *bd* parallel to OY and join *c*O and *d*O, cutting *ab* in *e* and *f* respectively. Then, from similar triangles, we have

$$\frac{cd}{ef} = \frac{H}{y},$$

or

$$\frac{ab}{ef} = \frac{H}{y},$$

$$\therefore\ ab = \frac{H}{y} \cdot ef \quad \ldots\ldots \quad (2)$$

Substituting in (1) gives

$$\text{Moment of area of strip} = \frac{H}{y}\, ef\, \delta y\, y$$

$$= ef \,.\, \delta y \,.\, H. \quad \ldots\ldots \quad (3)$$

Now ($ef\ \delta y$) is the area of the strip *ef*, hence, if the whole section were cut into strips such as *ab*, and the construction repeated for each strip, the total moment of area would be given by the sum of the areas of the reduced strips such as *ef* multiplied by the constant factor H. In practice, a few breadths only are taken; the reduced breadth for each is found by application of the above construction, and a fair curve is drawn through the ends. The area inclosed by this curve when multiplied by H will give the moment of area about OX of the given figure. Now the moment of area may also be found by taking the product of the area of the given figure and the distance of its centre of area from OX.

Let A_1 = the area of the given figure, in square inches.
A_2 = the area of the reduced figure, in square inches.
$\bar{y}$ = the distance of the centre of area from OX, in inches.
H = the height of the figure, in inches.

Then $A_1\bar{y} = A_2H$,

$$\bar{y} = \frac{A_2}{A_1}H. \quad \ldots \quad (4)$$

Fig. 180 shows the application of this method to a T section. The area A_1 of the section and the shaded area A_2 of the reduced figure were found by use of a planimeter. The neutral axis NA is drawn parallel to OX and at a distance $\bar{y}$ from it.

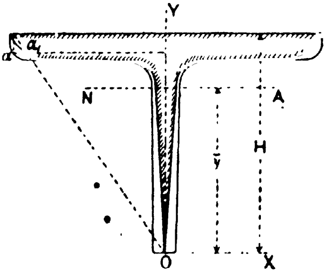


FIG. 180.—Neutral axis of a T section.

Referring again to Fig. 179, draw *eg* and *fh* parallel to OY, and join *g*O and *h*O, cutting *ab* in *m* and *n* respectively. Then, from similar triangles:

$$\frac{gh}{mn} = \frac{H}{y},$$

or

$$\frac{ef}{mn} = \frac{H}{y},$$

$$\therefore\ ef = mn\,\frac{H}{y}. \qquad \ldots\ldots\ldots\ldots (5)$$

Now, from the definition,

$$\begin{aligned} I_{OX} \text{ of strip } ab &= \text{area of strip} \times y^2 \\ &= ab\,.\,\delta y\,.\,y^2 \\ &= \left(\frac{H}{y}\,.\,ef\right).\,\delta y\,.\,y^2 \quad \text{(from (2), p. 149)} \\ &= H\,.\,ef\,.\,\delta y\,.\,y \\ &= H\,.\,mn\,.\,\frac{H}{y}\,.\,\delta y\,.\,y \quad \text{(from (5) above)} \\ &= mn\,.\,\delta y\,.\,H^2 \end{aligned} \qquad (6)$$

Again, $(mn\,.\,\delta y)$ is the area of the strip *mn*, hence the total moment of inertia may be obtained by multiplying the sum of the areas of all such strips by the constant factor H^2. Choose a number of strips and repeat the construction on each, thus finding a number of points such as *m* and *n*. Draw a fair curve through them, when its area A_3, multiplied by H^2, will give the total moment of inertia.

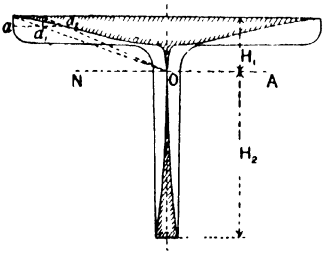


FIG. 181.—Moment of inertia about N.A of a **T** section

The moment of inertia of the same **T** section is worked out in Fig. 181. Greater accuracy is secured by using the neutral axis instead of OX in Fig. 180, thus producing two reduced figures, one for the original area above N.A and another for that below N.A.

Let

A_3' = shaded area of reduced figure above N.A, in square inches.

A_3'' = shaded area of reduced figure below N.A, in square inches.

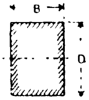
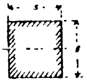
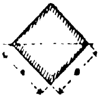
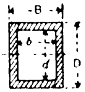
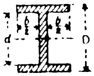
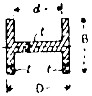
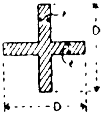
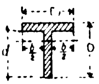
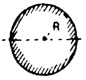
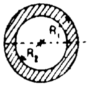
H_1 = height above N.A, in inches.

H_2 = height below N.A, in inches.

Then,

$$\text{Total moment of inertia about N.A} = A_3'H_1^2 + A_3''H_2^2. \quad \ldots\ldots (7)$$

PROPERTIES OF SECTIONS.

Name of section	Section	Area	Distance of N A from bottom	I_{NA}	k^2_{NA}
Rectangle		BD	$\frac{1}{2}D$	$\frac{BD^3}{12}$	$\frac{D^2}{12}$
Square		s^2	$\frac{1}{2}s$	$\frac{s^4}{12}$	$\frac{s^2}{12}$
Square		s^2	$\frac{s}{\sqrt{2}}$	$\frac{s^4}{12}$	$\frac{s^2}{12}$
Box		$BD - bd$	$\frac{1}{2}D$	$\frac{BD^3 - bd^3}{12}$	$\frac{BD^3 - bd^3}{12(BD - bd)}$
I		$BD - bd$	$\frac{1}{2}D$	$\frac{BD^3 - bd^3}{12}$	$\frac{BD^3 - bd^3}{12(BD - bd)}$
I on side		$2Bt + dt$	$\frac{1}{2}B$	$\frac{2tB^3 + dt^3}{12}$	$\frac{2tB^3 + dt^3}{12(2Bt + dt)}$
Cruciform		$2Dt - t^2$	$\frac{1}{2}D$	$\frac{tD^3 + (D - t)t^3}{12}$	$\frac{tD^3 + (D - t)t^3}{12(2Dt - t^2)}$
T		$BD - bd$	$\frac{BD^2 - bd^2}{2(BD - bd)}$	$\frac{(BD^2 - bd^2)^2}{12(BD - bd)}$ $\frac{4BDbd(D - d)^2}{12(BD - bd)}$	$\frac{I_{NA}}{BD - bd}$
Circle		πR^2	R	$\frac{\pi R^4}{4}$	$\frac{R^2}{4}$
Hollow circle		$\pi(R_1^2 - R_2^2)$	R_1	$\frac{\pi}{4}(R_1^4 - R_2^4)$	$\frac{R_1^2 + R_2^2}{4}$

Radius of gyration. The radius of gyration of a section may be defined thus: Let k be such a quantity that the product of the area A of the section and k^2 is equal to the moment of inertia of the section with reference to a given axis; thus:

$$I = Ak^2.$$

Then k is called the radius of gyration of the section with reference to the stated axis. The square of its value may be found in any given case by first ascertaining the moment of inertia and then dividing by the area of the section. There are many cases where the use of k in preference to I is advantageous in the working of problems.

Some commonly occurring sections and their properties are given in the Table, p 151. No fillets or tapers have been taken into account in the tabulated results, which will therefore be of service in obtaining approximate solutions only in the case of ordinary practical sections.

A rule, by use of which may be calculated the moment of inertia about an axis OX parallel to another axis CX passing through the centre of area, is expressed in the equation

$$I_{OX} = I_{CX} + Ad^2,$$

where A is the area of the section and d is the distance between the parallel axes.

Proportional laws for the strength of beams. Suppose we have two beams of rectangular sections, both supported at the ends and carrying central loads, but of differing dimensions, the following equations will hold for the sections at the middle of the span:

$$M_1 = \frac{W_1 L_1}{4} = \frac{f_1}{m_1} I_1 = \frac{f_1}{\frac{1}{2}d_1} \cdot \frac{b_1 d_1^3}{12} = \frac{f_1 b_1 d_1^2}{6},$$

or

$$W_1 = \frac{2}{3} \frac{f_1 b_1 d_1^2}{L_1}.$$

In the same way,

$$W_2 = \frac{2}{3} \frac{f_2 b_2 d_2^2}{L_2}.$$

Hence

$$\frac{W_1}{W_2} = \frac{f_1 b_1 d_1^2 L_2}{f_2 b_2 d_2^2 L_1}.$$

If the beams are made of the same material, the safe stress f_1 will be equal to f_2, and we may write

$$\frac{W_1}{W_2} = \frac{b_1 d_1^2 L_2}{b_2 d_2^2 L_1}.$$

Measuring the strengths of the beams by the central loads which they can carry safely, we may state this result as follows: The strengths of beams of rectangular sections and of the same material are proportional to their breadths, to the squares of their depths, and are inversely proportional to their lengths.

Proportional laws for beams of other sections may be obtained in a similar manner. Thus the strengths of solid circular sections are proportional to the cubes of the diameters, and are inversely proportional to the lengths.

Approximate calculation for beams of I section. The following simple method is often used and has sufficient accuracy for many practical purposes. Fig. 182 shows the section and part side elevation of a rolled beam of **I** section. The approximate moment of resistance is obtained by considering the maximum stress intensity f due to bending to be distributed uniformly over the flanges only, the web being neglected excepting for its resistance to shearing. The width of each flange being b and the thickness t, the total stress P on each flange will be obtained by taking the product of f and the flange area.

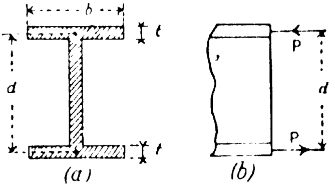


Fig. 182.—Approximate moment of resistance for a beam of **I** section.

Thus:

$$P = fbt.$$

Assuming each force P to act as though concentrated at the centre of area of the flange (Fig. 182 (*b*)), and that the distance between the centres is d, the moment of the couple formed by P, P, will give the moment of resistance. Thus:

$$\begin{aligned}\text{Moment of resistance} &= Pd\\ &= fbtd.\end{aligned}$$

This method may be used with fair results for rolled sections, and is used more extensively for built-up girders. To obtain the area of flange required at any section in such girders, the bending moment at the section is first calculated. Let this be M, then

$$\begin{aligned}M &= fbtd\\ &= fd \times \text{area of flange};\end{aligned}$$

$$\therefore \text{ area of flange} = \frac{M}{fd}.$$

In order to secure the most economical results in built-up girders, f should be constant throughout the girder. This result may be obtained by either of two methods: (*a*) d may be made proportional to M, in which case the area of the flange will be uniform throughout the length; (*b*) d may be constant and also the breadth b of the flanges; in this case the thickness of the flanges is increased by using two or more plates riveted together and extending along a portion of the length of the girder, more plates being used where the bending moment is greatest.

It may be shown that, in beams of **I** section, the distribution of shear stress is practically uniform over the web, hence, if S is the shearing force in tons and A_w is the area of the web in square inches, then

$$\text{Shearing stress} = q = \frac{S}{A_w} \text{ tons per square inch.}$$

Beams of uniform strength. A beam is said to have uniform strength when the maximum stress intensity is the same for all cross sections. Considering the equation

$$M = \frac{f}{m} I,$$

or

$$f = M\frac{m}{I},$$

m and I depend on the dimensions and shape of the section, and if these are constant throughout the beam, the only condition under which uniform maximum stress intensity f will occur is that M must be constant. Uniform bending moment may be produced in a portion of a beam by the application of couples. For example, if P and W be equal in the carriage axle shown on p. 186, then the bending moment throughout CD will be equal to the moment of the couple W × BD, and hence will be constant.

More usually M is not constant, in which case uniform f may be obtained by varying the section in such a manner that $M\frac{m}{I}$ is constant.

EXAMPLE 1. In Fig. 183, let AB be a cantilever carrying a load W at B. Supposing that the cantilever has a rectangular section of uniform depth d, what must be the profile in the plan in order that uniform strength may be obtained?

Here

$$m = \frac{d}{2},$$

$$I = \frac{bd^3}{12};$$

$$\therefore \frac{m}{I} = \frac{6}{bd^2}.$$

Again, the bending moment at any section distant x from B is

$$M = Wx.$$

For uniform strength, $$M\frac{m}{I} = \text{a constant};$$

$$Wx\frac{6}{bd^2} = \text{a constant},$$

or $$\frac{x}{b} = \text{a constant};$$

$$b = x \times \text{a constant}.$$

The required profile in the plan will therefore be triangular (Fig. 183).

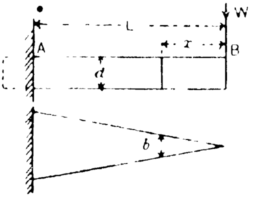


Fig. 183.

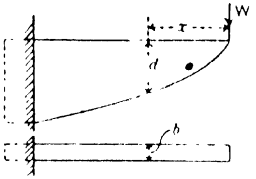


Fig. 184.

EXAMPLE 2. Supposing in Example 1 that the breadth had been uniform, and that it is required to find the profile in the elevation for uniform strength. As before, we have (Fig. 184)

$$Wx\cdot\frac{6}{bd^2} = \text{a constant},$$

$$\frac{x}{d^2} = \text{a constant},$$

$$d^2 = x \times \text{a constant},$$

or $$d = \sqrt{x} \times \text{a constant}.$$

Hence the profile is parabolic (Fig. 184).

EXAMPLE 3. Suppose in Example 1 that the load is uniformly distributed and that the breadth is uniform. Find the profile in the elevation for uniform strength (Fig. 185).

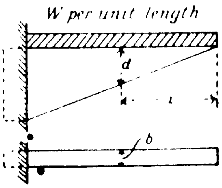


Fig. 185.

$$M_x = \tfrac{1}{2}wx^2;$$

hence $$\tfrac{1}{2}wx^2\cdot\frac{6}{bd^2} = \text{a constant},$$

or $$\frac{x^2}{d^2} = \text{a constant};$$

$$d = x \times \text{a constant}.$$

The profile in the elevation is therefore triangular (Fig. 185).

Other cases the student may work out easily for himself. It should be noted that, for practical reasons, the profile is often modified somewhat from that given by calculation.

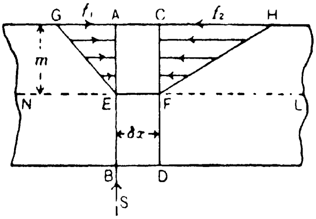


FIG. 186.—Stress figures due to bending

Distribution of the shearing stress over a beam section. In Fig. 186, AB and CD are two cross sections of a loaded uniform beam, separated by a small distance δx. Let the shearing force at AB be S and let M_1 and M_2 be the bending moments at AB and CD respectively; also, let M_2 be greater than M_1. Whatever may be the numerical value of the bending moment at AB, that at CD will be greater by an amount equal to the moment of S about any point on CD. Hence,

$$M_2 - M_1 = S \,.\, \delta x \quad \ldots\ldots\ldots\ldots\ldots (1)$$

This result will not be affected by any load which the beam may be carrying on AC, as the distance δx is supposed to be taken of too small a value to permit either the magnitude of the load, or its arm in taking moments about any point on CD, to attain an appreciable value. The reader is here reminded again that all loads must be distributed over a definite area, hence no concentrated load can be applied to AC.

Owing to the bending moments M_1 and M_2, there will be push stresses f_1 and f_2 at A and C respectively. Let EF be a portion of the neutral layer and let m be the distance EA or FC; then

$$M_1 = \frac{f_1}{m} I, \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$M_2 = \frac{f_2}{m} I \quad \ldots\ldots\ldots\ldots\ldots (3)$$

As I and m have the same value for both sections, and since M_2 is greater than M_1, f_2 will be greater than f_1. The stress figures will be AEG and CFH for the portions of the sections AE and CF respectively. It is clear that there will be a resultant force acting on CF which will be greater than that acting on AE; hence the net tendency will be to push the block AEFC towards the left. This block is shown separately in Fig. 187 in order that the question of restoring its equilibrium may be examined.

When the block forms a part of the beam it is clear that the only place where horizontal stresses may be applied in order to balance the resultants F_1 and F_2 is the horizontal section EF. Let Q be the total force produced by these stresses; then, for equilibrium,

$$Q = F_2 - F_1. \quad \text{.................} \quad (4)$$

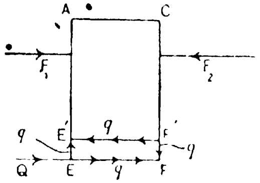


FIG. 187.—Equilibrium of the block AEFC.

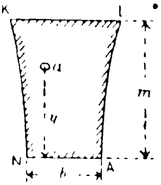


FIG. 188.—Cross-section of the block.

To find the values of F_2 and F_1, let a be a small portion of the sectional area of AE (Fig. 188) situated at a distance y from the neutral axis and let p be the stress on a; then

$$\frac{f_1}{m} = \frac{p}{y},$$

or,

$$p = \frac{y}{m} f_1.$$

Also,

$$\text{Force on } a = pa = \frac{y}{m} f_1 a$$

$$= \frac{f_1}{m} ay;$$

$$\therefore \text{ total force on AE} = \frac{f_1}{m} \Sigma ay,$$

or

$$F_1 = \frac{f_1}{m} AY, \quad \text{. . . .} \quad (5)$$

where A is the area of the portion of the section lying above the neutral axis, and Y is the distance of its centre of area from the neutral axis. AY will be the moment of area about the neutral axis of that portion of the section lying above the neutral axis. In the same way:

$$F_2 = \frac{f_2}{m} AY. \quad \text{.............} \quad (6)$$

Hence,

$$Q = F_2 - F_1$$

$$= AY\left(\frac{f_2}{m} - \frac{f_1}{m}\right). \quad \text{.................} \quad (7)$$

Now, from (2) and (3),

$$\frac{f_1}{m} = \frac{M_1}{I}$$

and

$$\frac{f_2}{m} = \frac{M_2}{I}.$$

Substitution of these in (7) gives

$$Q = AY\left(\frac{M_2}{I} - \frac{M_1}{I}\right)$$

$$= \frac{AY}{I}(M_2 - M_1). \quad \ldots \ldots \ldots (8)$$

Hence, from (1),

$$Q = \frac{AY}{I} \times S \, . \, \delta x. \quad \ldots \ldots \quad \ldots (9)$$

Let b be the breadth of the section at NA (Fig. 188); then the area of the horizontal section over which Q is distributed is $(\delta x \times b)$; hence, from (9),

$$\text{Shear stress on EF} = \frac{Q}{\delta x \, . \, b}$$

$$= \frac{SAY}{bI} \quad \ldots \ldots \quad \ldots (10)$$

This expression gives the intensity of shearing stress along the neutral layer; it also gives the shearing stress at points on the vertical sections AB and CD (Fig. 186) lying on the neutral axis. This may be understood by considering the thin rectangular block EFF′E′ (Fig. 187); if there is a shear stress q on its lower face, there must be equal shear stresses on all its faces perpendicular to the paper (p. 126).

UV (Fig. 189) is another horizontal section of the block AEFC. The shearing stress on this section arises from the fact that the stress figure CHXV for CV has a greater volume than the stress figure AGWU for AU. The determination of the intensity of shear stress on this section, and hence at the points U and V on the vertical sections, is proceeded with in the same manner as has been detailed above. AY in equation (5) will now mean the moment of area about the neutral axis of that portion of the section which lies above UV. b will be the breadth at U and V, and the final result will be

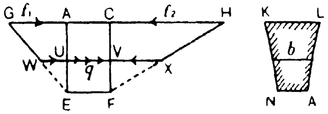


FIG. 189.—Shear stresses at U and V.

$$q = \frac{SAY}{bI}, \quad \ldots \quad \ldots \ldots \ldots (11)$$

where I, as before, is the moment of inertia of the whole section.

The student will observe that if there is no variation in the bending moment, *i.e.* if **M** is constant, between two sections of a beam, there can be no shearing force and hence no shear stress on the sections.

EXAMPLE. A beam has a rectangular section 4 inches broad and 12 inches deep, and has a shearing force of 6000 lb. (Fig. 190). Find the

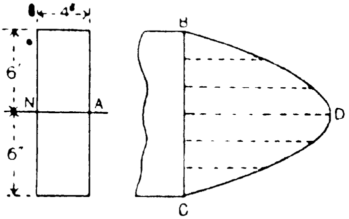


FIG. 190.—Distribution of shear stress on a rectangular section.

shearing stress at the neutral axis and at intervals of 2 inches from the neutral axis.

$$I = \frac{BD^3}{12}$$

$$= \frac{4 \times 12 \times 12 \times 12}{12} = \underline{576} \text{ inch units.}$$

At the neutral axis,

$$A = 6 \times 4 = 24 \text{ square inches,}$$

$$Y = 3 \text{ inches,}$$

$$q_1 = \frac{SAY}{bI}$$

$$= \frac{6000 \times 24 \times 3}{4 \times 576}$$

$$= \underline{187 \cdot 5} \text{ lb. per square inch.}$$

At 2 inches from the neutral axis,

$$A = 4 \times 4 = 16 \text{ square inches,}$$

$$Y = 4 \text{ inches,}$$

$$q_2 = \frac{6000 \times 16 \times 4}{4 \times 576}$$

$$= \underline{166 \cdot 7} \text{ lb. per square inch.}$$

At 4 inches from the neutral axis,

$$A = 2 \times 4 = 8 \text{ square inches,}$$

$$Y = 5 \text{ inches,}$$

$$q_3 = \frac{6000 \times 8 \times 5}{4 \times 576}$$

$$= \underline{104 \cdot 2} \text{ lb. per square inch.}$$

At 6 inches from the neutral axis,

$$A = 0;$$

$$\therefore q = 0.$$

These values have been used in constructing the diagram BCD (Fig. 190), the horizontal breadths of which show the shearing stress at any point of the section. The diagram is parabolic in outline. Fig. 191 shows the diagram of shear stress distribution for an I section. The quantities required for drawing it may be calculated by the same method. The result indicates the justification of ignoring the flanges and assuming that the web supplies the whole of the shearing resistance by means of a uniform shear stress (p. 154).

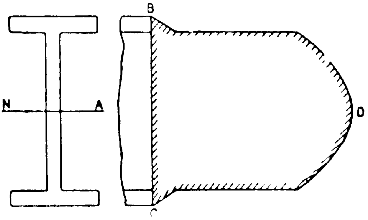


FIG. 191. Distribution of shear stress on an I section

EXERCISES ON CHAPTER VII.

1. A beam 20-feet span, supported at its ends, carries a load of 4 tons at the centre, another of 6 tons at 4 feet from one end, and a third load of 2 tons at 6 feet from the other end. Calculate the bending moments and shearing forces at each load, and draw the diagrams of bending moment and shearing force.

2. A beam AB, 16 feet long, rests on a support at A and on another support at C, which is four feet from B. The beam carries a uniformly distributed load of 0·5 ton per foot run, together with a load of 4 tons at 6 feet from A and another of two tons at C. Calculate the bending moments and shearing forces at intervals of 2 feet, and draw diagrams of bending moment and shearing force.

3. A beam AB, 10-feet span, supported at its ends, carries a distributed load which varies uniformly from 100 lb. per inch run at A to 200 lb. per inch run at B. Find the bending moments and shearing forces at intervals of 2 feet, and draw diagrams of bending moment and shearing force.

4. Making use of a graphical method, draw the bending-moment diagram for the beam given in Question 1. State your scale clearly.

5. Draw the bending-moment diagram for the beam given in Question 2, using a graphical method. Give the scale of your diagram.

6. Find by calculation the neutral axis of a T section $4\frac{1}{2}$ inches broad, 5 inches deep, metal $\frac{1}{2}$ inch thick. Neglect any fillets.

7. A cast-iron beam has an **I** section, in which the top flange is 3 inches broad, the bottom flange is 7 inches broad and the depth is 10 inches over all. The metal has a uniform thickness of 0·75 inch. Neglect fillets and calculate the position of the neutral axis.

Draw the section given in Question 6 as it would be made in practice. Find the neutral axis and moment of inertia, using a graphical method

9. Answer Question 7 in the manner directed in Question 8, giving the neutral axis the moment of inertia.

10. A timber beam of rectangular section, 3 inches broad by 9 inches deep by 12 feet span, carries a uniformly distributed load Find the load if the stress due to bending is limited to 400 lb. per square inch

11. A flat steel bar, section 2 inches by 1 inch, is 20 feet long, and is stored in a rack in which the two supports are each 4 feet from the end of the bar. Find the stress due to bending (*a*) at the middle of the length of the bar, (*b*) at the supports Suppose the bar to be resting on its edge, what would be these stresses? Take the weight of the material to be 0 28 lb per cubic inch

12. A beam of **I** section 10 inches deep, 6 inches wide, thickness of flanges $\frac{3}{4}$ inch, thickness of web $\frac{1}{2}$ inch, has a span of 15 feet and rests on the supports If a load of 2 tons is carried at the centre, find the maximum stress due to bending (*a*) by an approximate method, (*b*) by first calculating the moment of inertia Assuming the shearing force to be carried by the web and to be distributed uniformly, find the shear stress on the web Neglect the weight of the beam

13. A pipe 24 inches internal diameter is constructed of mild steel plate $\frac{3}{8}$ inch thick, and is full of water, the ends are closed by blank flanges If the pipe is supported at its ends, find the maximum span if the stress due to bending is not to exceed 5 tons per square inch Take the weight of steel to be 0 28 lb per cubic inch and of water to be 62 5 lb per cubic foot

14. A timber beam of rectangular section, supported at its ends, carries a uniformly distributed load, and has been made to a certain drawing. Another timber beam has been made to the same drawing by simply altering the scale, so that span, breadth and depth are each multiplied by a constant factor *n* Suppose both beams to be able to carry the same maximum stress due to bending, what will be the ratio of the uniformly distributed loads which may be applied?

15. A cast-iron bar of rectangular section is used as a beam of 3-feet span, supported at the ends, and carries a central load of 3000 lb The stress due to bending is not to exceed 1·5 tons per square inch. The bar is to have uniform strength (*a*) Draw the profile in the elevation if the breadth is uniform and equal to 1 5 inches (*b*) Suppose the depth to be uniform and equal to 3 inches, draw the profile in the plan.

16. Take the data of Question 10, and find the maximum shearing stress in the beam

17. A beam of **I** section, 10 inches deep, 5 inches wide, metal $\frac{5}{8}$ inch thick, has a maximum shear stress at a certain section of 1 ton per square inch Find the shear stress at places 1, 2, 3, 4 and $4\frac{1}{2}$ inches from the neutral axis. Plot a shear stress diagram

18. How is the section-modulus and radius of gyration of a section of a bar obtained, and how is this applied when ascertaining the strength of a beam? Calculate the section-modulus and radius of gyration of the section given in Fig. 192 about the axis YY (I C E)

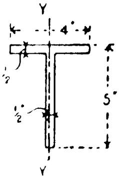


FIG. 192.

19. A girder AB, 25 feet long, carries three loads of 6, 11 and 7 tons respectively, placed at distances of 7, 16 and 21 feet from the end A. Find the reactions at either end and the bending moment at the centre.
(I.C.E.)

20. Fig. 193 represents a station roof, the centre pillars being 25 feet apart. The dead load can be taken as evenly distributed over the roof

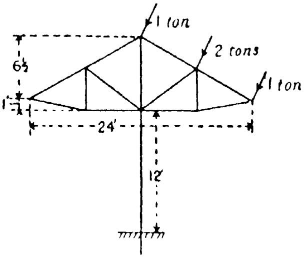


Fig 193

and of magnitude 15 lb. per square foot of projected plan area. The wind pressure is to be taken as shown. Find the magnitude and direction of the resultant force on the roof, and give the bending moment at the base of the pillar.
(L.U.)

CHAPTER VIII.

DEFLECTION OF BEAMS.

Curve assumed by a loaded beam. Any beam when loa bend; if the neutral lamina is straight, as seen in elevation in the unloaded beam, it will assume some curve when the loads are applied; any initial curvature of the neutral lamina will be altered

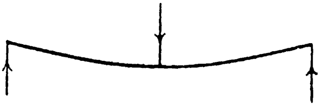


FIG. 194.—Curve of a beam supported at ends and loaded at middle.

to a new curvature on applying the loads. A useful way of studying the curves of a loaded beam is to employ a thin steel knitting needle; this may be laid on a sheet of drawing paper and "loaded" by means of drawing pins pushed into the board. Figs. 194–196 show some curves produced in this way.

Examining Fig. 196, which represents the curve of a cantilever carrying a load at its free end, and taking two points P and P_1 lying

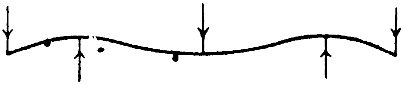


Fig. 195.—Curve of a beam overhanging the supports.

close together, two normals drawn from P and P_1 will intersect in O. It is evident that a short piece PP_1 of the curve could be drawn as a circular arc struck from O as centre with radius OP. If P and P_1 are taken very close together, O is called the **centre of curvature** for the curve at P, and $OP = R$ is called the **radius of curvature.** It can be seen readily in Fig. 196 that the radii of curvature for points near A are smaller than for others near B. In fact, as we shall see

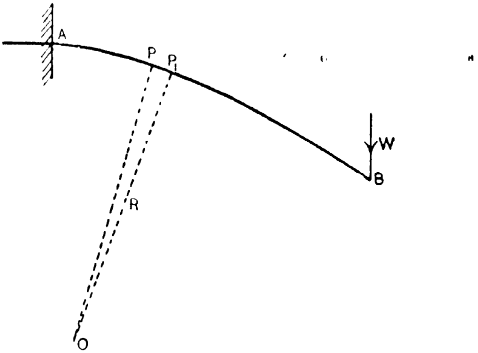


FIG. 196.—Curve of a cantilever loaded at the free end.

presently, the radius of curvature at any place is inversely proportional to the bending moment at that place.

Curvature is a term used by mathematicians to express the rate of change of direction of a curve. Referring to Fig. 197, and taking points P and P_1 lying close together, O will be the centre of curvature and $R = OP$. Draw tangents PT and P_1T_1. The direction of the curve at P is along PT, and that at P_1 is along P_1T_1; the change of direction between P and P_1 will be the angle α in the figure. It will be evident that the angle P_1OP is equal to α, and stating its value in radians,

$$\alpha = \frac{PP_1}{R}.$$

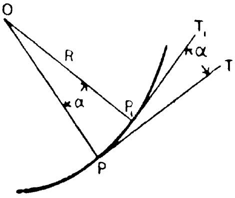


FIG. 197 —Curvature

The rate of change of direction may be expressed by dividing the change in direction by the distance PP_1 along the curve in which the change is effected; hence

$$\text{Curvature} = \text{rate of change of direction} = \frac{\alpha}{PP_1}$$

$$= \frac{PP_1}{R \times PP_1}$$

$$= \frac{1}{R}.$$

Curvature at a given point may therefore be stated as being the reciprocal of the radius of curvature. The units for curvature will be change of direction in radians per foot, or per inch, length of the curve according as R is in feet or inches.

It will be understood that, for ordinary beams which are straight when unloaded, the radius of curvature at any place when the beam is loaded will be very large and that the curvature will be very small.

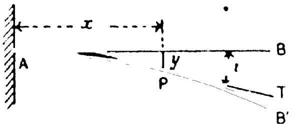


FIG. 198.—Slope and deflection of a cantilever.

Fig. 198 shows again the curve AB' of a loaded cantilever. Taking any point P on the curve and drawing a tangent PT, the angle i which PT makes with the original direction AB, is called the **slope** at P; i should be stated in radians. P is at a distance y below AB, and y is called the **deflection** at P. For our purposes it is sufficient to be able to state R, i and y for any point.

Curvature of a beam. Fig. 199 shows a portion of a loaded beam. Two cross sections occupying originally the positions AB and CD have been strained to A'B' and C'D'. Assuming that they lie close together, the point of intersection O will be the centre of curvature for the portion EF of the neutral lamina. Bisecting EF in M and joining OM cutting AC in K, we have similar triangles OME and EAA'. Hence, using a similar method to that on p. 143,

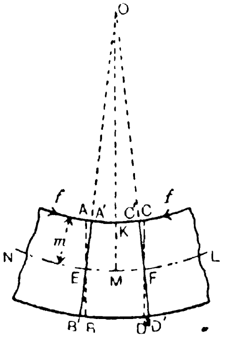


FIG. 199.—Curvature of a beam.

$$\frac{EM}{MO} = \frac{AA'}{EA'}$$

or

$$\frac{AK}{R} = \frac{AA'}{m},$$

$$\frac{1}{R} = \frac{AA'}{AK} \cdot \frac{1}{m} \quad \ldots\ldots\ldots \quad (1)$$

Again, $\text{Strain of AK} = \frac{AA'}{AK}.$

Also, $E = \frac{f}{\text{strain of AK}};$

$$\therefore \text{ strain of AK} = \frac{f}{E} = \frac{AA'}{AK}.$$

Substituting in (1), we have

$$\frac{1}{R}=\frac{f}{E}\cdot\frac{1}{m}$$

$$=\frac{f}{m}\cdot\frac{1}{E}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Again, $M_{AB}=\frac{f}{m}\cdot I_{NA}$;

$$\therefore \frac{f}{m}=\frac{M_{AB}}{I_{NA}}.$$

Substituting in (2) gives $\frac{1}{R}=\frac{M_{AB}}{EI_{NA}}$. ……………… (3)

We see therefore that the curvature at any place on the neutral lamina is proportional to the bending moment and inversely proportional to the moment of inertia of the section at that place.

Mathematical expressions for the slope and curvature of a curve, such as that shown in Fig. 198, are:

$$i=\frac{dy}{dx}, \quad \ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{1}{R}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}. \quad \ldots\ldots\ldots(5)$$

dy means the change in deflection as we pass along the beam by a small amount dx. For curves which are very flat, and hence for all beams, equation (5) simplifies by the denominator becoming unity; thus

$$\frac{1}{R}=\frac{d^2y}{dx^2}. \quad \ldots\ldots\ldots\ldots\ldots(6)$$

Hence, from (3) and (6), $\frac{d^2y}{dx^2}=\frac{M_{AB}}{EI_{NA}}$. ……………… (7)

Slopes and deflections may be calculated from (7) by first evaluating the bending moment; integration of both sides will then give the slope; further integration of both sides will give the deflection. The method is rather complicated, excepting in cases where the conditions of loading and supporting are simple.

The following examples of an easier method are given as leading to a graphical solution which is simple in its application. It will be assumed that the bending is pure and that the beam is of uniform section unless the contrary be stated; the latter assumption is made in order that I_{NA} may be constant throughout the length of the beam.

Cantilever having a load at the free end. In Fig. 200 (*a*) is shown a cantilever of length L and of uniform cross section, so that I_{NA} is constant. We may consider for a moment that the whole of the material is perfectly rigid, excepting the portion lying between the two adjacent transverse parallel sections AB and CD. Supposing a load W to be applied to the free end (Fig. 200 (*b*)), deflection of this end will take place by reason of the strains in the portion ABDC. AE and C′F′ will remain straight as at first, but C′F′ will now be inclined at an angle i to its original position. C′D′, the new position of CD, will be still perpendicular to C′F′, so that the angle made by C′D′ with its original position CD will also be equal to i. Let the deflection of F under these conditions be δ, and let NP be a portion of the neutral lamina. As both δ and i will be exceedingly small in any practical beam, we may write

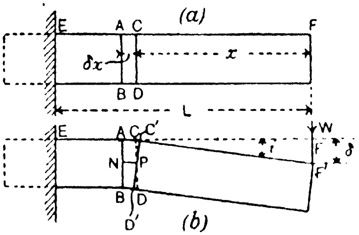


FIG. 200.—A rigid cantilever having a small elastic portion ABDC.

$$i = \frac{\delta}{x} \text{ radians,}$$

or

$$\delta = ix \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The strain of AC, produced by a tensile stress f induced by the bending moment M_x, will be CC′ divided by AC; hence we have

$$E = \frac{f}{\frac{CC'}{AC}}$$

$$= f \cdot \frac{AC}{CC'} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Again, from the general expression for the strength of a beam (p. 146) we have, noting that f is the stress intensity at a distance CP from the neutral plane,

$$M_x = \frac{f}{CP} \cdot I,$$

or

$$Wx = \frac{f}{CP} . I ;$$

$$\therefore f = CP . \frac{Wx}{I} . \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{3}$$

Substitution in (2) gives

$$E = CP . \frac{Wx}{I} . \frac{AC}{CC'}$$

$$= \frac{CP}{CC'} . \frac{Wx}{I} . \delta x$$

$$= \frac{1}{i} . \frac{Wx . \delta x}{I} ;$$

$$\therefore i = \frac{Wx . \delta x}{EI} . \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{4}$$

Hence, from (1),

$$\delta = \frac{W}{EI} . x^2 . \delta x . \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{5}$$

Had any other portion been taken similar in properties to ABCD, we should have obtained a similar expression for the deflection due to its strains. Hence the total deflection Δ of F will be obtained by integrating (5) between the limits $x = 0$ and $x = L$.

$$\Delta = \frac{W}{EI} \int_0^L x^2 . dx$$

$$= \frac{WL^3}{3EI} . \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{6}$$

The slope i_{max} at the end F may be obtained by integrating result (4), which gives the slope produced by the strains of the small portion ABDC of the cantilever.

$$i_{max} = \frac{W}{EI} \int_0^L x . dx$$

$$= \frac{WL^2}{2EI} \text{ radians.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{7}$$

Cantilever having a distributed load. The case of a uniform cantilever having a uniformly distributed load w per unit length may be worked out in a similar manner, the only difference being that

$$M_x = \frac{wx^2}{2} .$$

Inserting this value in place of Wx in (4) and (5) gives

$$i = \frac{wx^2 . \delta x}{2EI} , \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{8}$$

$$\delta = \frac{w}{2EI} . x^3 . \delta x . \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{9}$$

Integrating (9) to obtain Δ, we have

$$\Delta = \frac{w}{2EI}\int_0^L x^3\,dx$$

$$= \frac{wL^4}{8EI} \qquad (10)$$

The slope at the free end may be obtained from (8).

$$i_{max} = \frac{w}{2EI}\int_0^L x^2\,dx$$

$$= \frac{wL^3}{6EI} \qquad (11)$$

Beam having a load at the middle The results now obtained enable the case of a uniform beam simply supported at its ends to be solved easily by considering the beam as a double cantilever held fixed at the middle of the span and deflected upwards by the reactions ½W at each end (Fig. 201). It is evident that the deflection of C below AB will be equal to the elevation of A and B above the horizontal line through C when the beam is loaded. Hence, using the result obtained in (6) above, we have, by writing ½W for W and ½L for L,

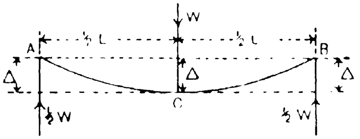


FIG. 201.—Deflection of a simply-supported beam carrying a central load

$$\Delta = \frac{\frac{1}{2}W\,.\,(\frac{1}{2}L)^3}{3EI}$$

$$= \frac{WL^3}{48EI} \qquad (12)$$

The slope at the ends may be obtained similarly from (7).

$$i_{max} = \frac{\frac{1}{2}W\,.\,(\frac{1}{2}L)^2}{2EI}$$

$$= \frac{WL^2}{16EI} \qquad (13)$$

Beam having a uniformly distributed load. This case (Fig. 202 (*a*)) may be regarded also as a double cantilever fixed at the middle of the span. The loading will consist of a downward load ½*w*L on each half span together with a concentrated upward load of $\frac{1}{2}wL$ at each free end. The solution may be derived from the results already obtained by use of the axiom that the resultant deflection and slope of a beam under a combined system of loads will be the algebraic sum of those produced by each load taken separately.

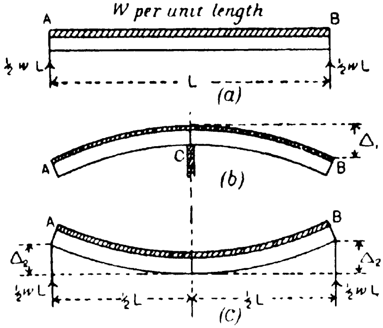


FIG. 202.—Deflection of a simply-supported beam carrying a uniformly distributed load.

In Fig. 202 (*b*) the effect of the distributed load may be examined. Let Δ_1 be the downward deflection of the ends, when we have, by substitution in (10),

$$\Delta_1 = \frac{w(\frac{1}{2}L)^4}{8EI}$$

$$= \frac{wL^4}{128EI}. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Fig. 202 (*c*) shows the effect of the reactions considered alone in producing an upward deflection Δ_2. From (6) we have

$$\Delta_2 = \frac{\frac{1}{2}wL \,.\, (\frac{1}{2}L)^3}{3EI}$$

$$= \frac{wL^4}{48EI}. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

The resultant upward deflection Δ at the supports, and hence the downward deflection at the middle of the span under the proposed loading, will be

$$\Delta = \Delta_2 - \Delta_1$$

$$= \frac{wL^4}{48EI} - \frac{wL^4}{128EI}$$

$$= \frac{5}{384} \cdot \frac{wL^4}{EI} \;\checkmark \ldots\ldots\ldots\ldots (16)$$

Due to the distributed load there will be a downward slope at the supports the value of which, i_1, may be obtained from (11).

$$i_1 = \frac{w(\frac{1}{2}L)^3}{6EI}$$

$$= \frac{wL^3}{48EI} \quad \ldots \quad (17)$$

The upward reactions will produce an upward slope i_2, obtained from (7).

$$i_2 = \frac{\frac{1}{2}wL(\frac{1}{2}L)^2}{2EI}$$

$$= \frac{wL^3}{16EI} \quad \ldots \quad (18)$$

Combining these results, we have for the slope i_{max} at the supports:

$$i_{max} = i_2 - i_1$$

$$= \frac{wL^3}{16EI} - \frac{wL^3}{48EI}$$

$$= \frac{1}{24}\frac{wL^3}{EI} \quad \ldots \quad (19)$$

Graphical solution. The method of obtaining the slope and deflection at the free end of a uniform cantilever, employed on p. 167, may be extended in such a manner as to enable the slope and deflection in more complicated cases to be found graphically.

Referring to Fig. 203 (*a*), the slope i caused by a portion ACDB being elastic, while the remainder of the cantilever is supposed to be rigid, is equal to the angle CPC′, and will be given by

$$i = \frac{CC'}{CP} = \frac{CC'}{m} \text{ radian.} \quad \ldots \quad (1)$$

Now CC′ divided by AC is the strain of AC caused by the stress f; hence,

$$E = \frac{f}{\frac{CC'}{AC}},$$

or

$$CC' = \frac{f}{E} \cdot AC$$

$$= \frac{f \cdot \delta x}{E} \quad \ldots \quad (2)$$

Substituting this value in (1) gives

$$i = \frac{f \cdot \delta x}{m \cdot E} \quad \ldots \quad (3)$$

Again, from the general expression for the strength of a beam, we have

$$M_{AB} = \frac{f}{m} \cdot I_{NA},$$

or

$$\frac{f}{m} = \frac{M_{AB}}{I_{NA}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Inserting this in (3) gives

$$i = \frac{1}{EI_{NA}} \cdot M_{AB}\,\delta x \quad \ldots\ldots\ldots \quad (5)$$

Let the diagram HKL (Fig. 203 (*b*)) be the bending moment diagram for the cantilever. It will be clear that the product $M_{AB} . \delta x$ is the area of the shaded strip of the diagram. Hence we may say that the change of slope i produced by the elastic bending of the portion between AB and CD is given by the area of the strip of the bending-moment diagram lying under BD multiplied by the constant $\frac{1}{EI_{NA}}$. Consider now the whole cantilever to be elastic, then, the slope at E being zero, it follows that the slope at AB will be the sum of the areas of all strips between HL and QR multiplied by $\frac{1}{EI_{NA}}$, or

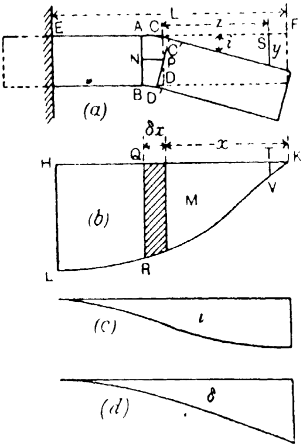


FIG. 203.—Graphical method of deducing slope and deflection

$$\text{Slope at AB} = i_{AB} = \text{area HQRL} \times \frac{1}{EI_{NA}} \quad \ldots\ldots (6)$$

In cases where the bending moment diagram has a simple outline, it may be possible to calculate the required area, otherwise it will be necessary to use a planimeter. If the area has been found in square inches, the result should be corrected by multiplying by the scale in inch-tons per inch used in setting out the ordinates such as QR, and by further multiplying by the scale in inches per inch used in setting out the abscissae such as HQ; E should be taken in tons per square inch, and I_{NA} in inch units.

HK being divided at a convenient number of points, the slope at each point may be found by the above method and a diagram drawn showing the slope at all parts of the cantilever by means of plotting the results and drawing a fair curve through them (Fig. 203 (c)).

Again, referring to Fig. 203 (a), let y be the deflection at a point S, distant z from C, owing to the elasticity of ACDB. Then

$$i = \frac{y}{z},$$

or

$$y = iz.$$

Now, i is given by the area of the shaded strip in the bending moment diagram multiplied by $\frac{1}{EI_{NA}}$; hence,

$$y = \frac{1}{EI_{NA}} \times \text{area of shaded strip} \times z.$$

That is to say, the increment y of the total deflection at S caused by the elasticity of ACDB is given by the moment about S of the shaded strip of the bending-moment diagram multiplied by $\frac{1}{EI_{NA}}$. To obtain the total deflection δ at S, we must therefore evaluate the moment of area of HTVL about T (Fig. 203 (b)) and multiply the result by $\frac{1}{EI_{NA}}$, paying regard to the scales in the manner already noted. Repeating the same operation in order to obtain the deflection at several sections, data will be obtained from which the deflection curve (Fig. 203 (d)) may be drawn.

Some applications of the graphical method. Taking again the case of a uniform cantilever carrying a load W at its free end (p. 167), and referring to Fig. 204, we have

$$M_P = Wx,$$

$$\text{slope at P} = i_P = \text{area CFGD} \times \frac{1}{EI}$$

$$= \tfrac{1}{2}(WL + Wx)(L - x)\frac{1}{EI}$$

$$= \frac{W}{2EI}(L^2 - x^2). \qquad (1)$$

The maximum slope will be at B, and may be obtained by writing

$$x = 0,$$

$$i_B = \frac{WL^2}{2EI} \qquad \ldots\ldots (2)$$

To obtain the deflection at P, we have

$$\delta_P = \text{moment about F of area CFGD} \times \frac{1}{EI}$$

$$= (\text{moment of CFHD} - \text{moment of DGH})\frac{1}{EI}$$

$$= \left\{WL(L-x)\left(\frac{L-x}{2}\right) - (WL - Wx)\tfrac{1}{2}(L-x)\tfrac{1}{3}(L-x)\right\}\frac{1}{EI}$$

$$= \left\{WL\frac{(L-x)^2}{2} - W\frac{(L-x)^3}{6}\right\}\frac{1}{EI}$$

$$= \frac{W}{EI}\left\{L\frac{(L-x)^2}{2} - \frac{(L-x)^3}{6}\right\} \quad \ldots \ldots \quad (3)$$

The maximum deflection will occur at B, and may be obtained by writing $x = 0$.

$$\Delta_B = \frac{W}{EI}\left\{L\frac{L^2}{2} - \frac{L^3}{6}\right\}$$

$$= \frac{WL^3}{3EI} \quad \ldots \ldots \quad (4)$$

The case of a uniform beam simply supported at both ends and carrying a central load may be worked out in a similar manner, and is left as an exercise for the student.

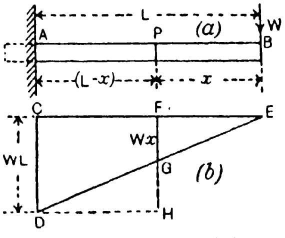


FIG. 204.—Graphical method applied to a cantilever loaded at the free end.

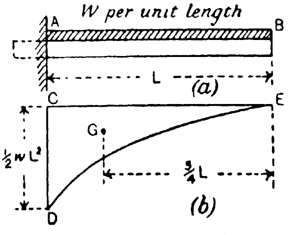


FIG 205 —Graphical method applied to a cantilever uniformly loaded

A uniform cantilever carrying a uniformly distributed load *w* per unit length may be worked out easily so far as its maximum slope and deflection are concerned. The bending-moment diagram is parabolic (Fig 205), and it may be noted that its area is one-third of the area of the circumscribing rectangle, *i.e.* one-third of CD multiplied by CE, and that its centre of area G is distant horizontally three-quarters of CE from E.

Hence,
$$i_B = \text{area CED} \times \frac{1}{EI}$$
$$= \frac{wL^2}{2} \cdot \frac{L}{3} \cdot \frac{1}{EI}$$
$$= \frac{wL^3}{6EI} \quad \ldots\ldots\ldots \quad (5)$$

Also,
$$\Delta_B = \text{moment about E of area CED} \times \frac{1}{EI}$$
$$= \frac{wL^2}{2} \cdot \frac{L}{3} \cdot \frac{3}{4}L \cdot \frac{1}{EI}$$
$$= \frac{wL^4}{8EI} \quad \ldots\ldots\ldots \quad (6)$$

In the case of **a uniform beam supported at both ends, and carrying a uniformly distributed load** (Fig. 206), the bending moment diagram

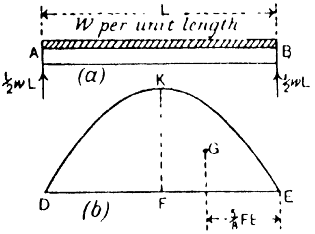


FIG. 206.—Uniformly loaded beam.

is also parabolic. The maximum bending-moment occurs at the middle of the span, and is given by
$$FK = \frac{wL^2}{8}.$$

The slopes at A and B will be equal, and may be found by applying the rule to the area KFE, noting that the slope at the middle will be zero.
$$i_A = i_B = \text{area KFE} \times \frac{1}{EI}$$
$$= \frac{2}{3}\ \text{area of circumscribing rectangle} \times \frac{1}{EI}$$
$$= \frac{2}{3} \cdot \frac{wL^2}{8} \cdot \frac{L}{2} \cdot \frac{1}{EI}$$
$$= \frac{wL^3}{24EI} \quad \ldots\ldots\ldots \quad (7)$$

Noting that the centre of area G of KFE is at a horizontal distance $\frac{5}{8}$FE from E, we have, reckoning the deflection of A or B upwards from the middle,

$$\Delta_A = \Delta_B = \text{moment of area KFE about E} \times \frac{1}{EI}$$

$$= \frac{2}{3} \cdot \frac{wL^2}{8} \cdot \frac{L}{2} \cdot \frac{5}{8} \frac{L}{2} \cdot \frac{1}{EI}$$

$$= \frac{5}{384} \cdot \frac{wL^4}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Encastré beams. We may now examine the case of a uniform beam which is fixed rigidly at both ends by being built into walls or by some other method (Fig. 207 (*a*)). In such cases it may be assumed that the sections at A and B, which are in the plane of the wall before loading, remain in the same plane after loading; hence the slopes at A and B will be zero. In order that this may be the case it is necessary that the means used for fixing the ends should apply restraining bending moments at A and B. We may obtain a fair idea of the conditions by examining the beam shown in Fig. 207 (*b*). Here the bending moments at A and B, applied by the loads W_1W_1 on the overhanging ends, have the effect of keeping vertical the sections at A and B. Hence, in the beam shown in Fig. 207 (*a*), the walls must supply the bending moments at A and B, which in Fig. 207 (*b*) are given by the loads W_1.

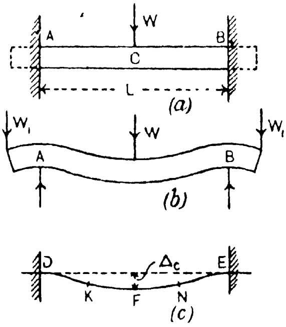


FIG. 207.—Encastré beam loaded at middle

The curve of the bent beam will resemble Fig. 207 (*c*), and will be convex downwards between two points K and N, and convex upwards between D and K and also between E and N. This comes about from the fact that the resultant curve is produced from two component curves, one (Fig. 208 (*a*)) caused by the action of W tending to produce a curve wholly convex downwards, and the other (Fig. 208 (*b*)), caused by the action of the bending moments M_A and M_B (which are obviously equal and are transmitted uniformly throughout the length of the beam), tending to produce a curve which is wholly convex upwards. The resultant bending moment at any section may be obtained by taking the algebraic sum of these moments for that section.

Fig. 209 (*a*) gives the bending-moment diagram for a beam simply supported and carrying a central load W. Its ordinates give the positive bending moments at any section of the beam under consideration due to W alone. Fig. 209 (*b*) shows the uniform negative bending moments due to the fixing of the ends. These diagrams may be combined as shown in Fig. 210 (*a*), when the shaded portions, which show the algebraic sum of the component diagrams, will give the resultant bending moments for the beam.

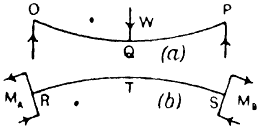


FIG. 208.—Component curves of an encastré beam.

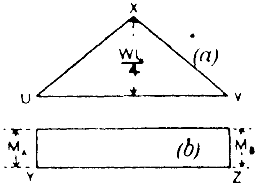


FIG. 209.—Component bending moment diagrams for an encastré beam.

The maximum bending moment due to W alone is represented by *ch* in Fig. 210 (*a*), and is of value $\frac{WL}{4}$. To obtain the values of M_A and M_B, represented by *ad* and *be*, we have the consideration that the slopes at D and E (see Fig. 207 (*c*)) are zero. Hence the areas of the bending-moment diagram *adf* and *fcm* (Fig. 210 (*a*)) must be equal, because the slope at the centre is given by their algebraic sum, and this must be zero for zero slope. For a similar reason the areas *cmg* and *geb* are equal; hence it is easily seen from the figure that the triangular area *acb* must be equal to the rectangular area *adeb*. Thus, *hm* must be one half of *hc*, giving

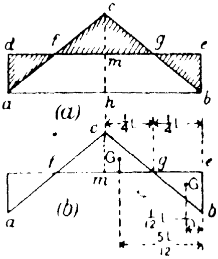


FIG. 210.—Resultant bending moment diagram for an encastré beam.

$$M_A = M_B = \frac{1}{2} \cdot \frac{WL}{4}$$

$$= \frac{WL}{8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

It will also be obvious from Fig. 210 (*a*) that the points *f* and *g*, where the resultant bending moments are of zero value, must lie at one-quarter span.

The deflection upwards of E above F (Fig. 207 (*c*)) may be obtained by taking the algebraic sum of the moments about *e* of the areas *cmg* and *geb* (Fig. 210 (*b*)), and dividing the result by EI. This will give the central deflection Δ of the beam.

$$\Delta = (\text{moment of area } cmg - \text{moment of area } geb)\,\frac{1}{\mathrm{EI}}$$

$$= \left\{\left(\frac{\mathrm{WL}}{8}\cdot\frac{1}{2}\cdot\frac{\mathrm{L}}{4}\cdot\frac{5\mathrm{L}}{12}\right)-\left(\frac{\mathrm{WL}}{8}\cdot\frac{1}{2}\cdot\frac{\mathrm{L}}{4}\cdot\frac{\mathrm{L}}{12}\right)\right\}\frac{1}{\mathrm{EI}}$$

$$= \frac{1}{192}\,\frac{\mathrm{WL}^3}{\mathrm{EI}} \quad \ldots\ldots \quad (2)$$

An encastré beam of uniform section carrying a uniformly distributed load (Fig. 211 (*a*)) may be worked out in a similar manner. The parabolic curve *afcgb* (Fig. 211 (*b*)) represents the bending moment diagram for a beam simply supported at the ends and carrying w per unit length. The maximum bending moment will occur at the middle of the span, and is represented by $ch = \frac{w\mathrm{L}^2}{8}$. The rectangle *adeb* represents the uniform bending moment due to the fixing in the walls. The shaded area gives the resultant bending moment diagram.

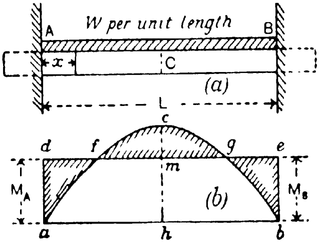


FIG. 211 – An encastré beam uniformly loaded.

Here, as in the last case, there is zero slope at A, C and B; hence the areas *adf*, *fcm*, *cmg* and *geb* are equal, consequently the parabolic area *afcgb* must be equal to the rectangular area *adeb*, giving

$$\frac{2}{3}\cdot\frac{w\mathrm{L}^2}{8}\cdot\mathrm{L} = \mathrm{M_A L};$$

$$\therefore\ \mathrm{M_A} = \mathrm{M_B} = \frac{w\mathrm{L}^2}{12}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

The bending moment at C will be given by

$$\mathrm{M_C} = \frac{w\mathrm{L}^2}{8} - \mathrm{M_A}$$

$$= \frac{w\mathrm{L}^2}{8} - \frac{w\mathrm{L}^2}{12}$$

$$= \frac{w\mathrm{L}^2}{24}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Thus, we see that the bending moment at the walls is double that at the middle of the span.

To obtain the deflection at C, we must find the algebraic sum of the moments about *h* of the areas *cmg* and *gcb*, or, since the result will be the same if the moment of the area *mgbh* be added to each, the calculation may be simplified by taking the algebraic sum of the moments about *h* of the areas *chbg* and *hmcb*. Hence,

$$\Delta_c = (\text{moment of area } chbg - \text{moment of area } hmcb)\,\frac{1}{EI}$$

$$= \left\{\left(\frac{2}{3}\cdot\frac{wL^2}{8}\cdot\frac{L}{2}\cdot\frac{3}{8}\cdot\frac{L}{2}\right) - \left(\frac{wL^2}{12}\cdot\frac{L}{2}\cdot\frac{L}{4}\right)\right\}\frac{1}{EI}$$

$$= \left(\frac{wL^4}{128} - \frac{wL^4}{96}\right)\frac{1}{EI}$$

$$= \frac{1}{384}\,\frac{wL^4}{EI} \qquad (3)$$

The distance of *f* and *g*, the points of zero bending moment, from *d* and *e* respectively will be equal, and may be found by obtaining an expression for the bending moment at a distance *x* from the wall and then equating this to zero. Thus,

$M_x = M_A -$ bending moment at *x* for a beam simply supported

$$= \frac{wL^2}{12} - \left(\frac{wL}{2}\cdot x - \frac{wx^2}{2}\right)$$

$$= \frac{wL^2}{12} - \frac{wLx}{2} + \frac{wx^2}{2}.$$

Equating this to zero gives

$$\frac{x^2}{2} - \frac{Lx}{2} + \frac{L^2}{12} = 0,$$

or

$$x^2 - Lx + \tfrac{1}{6}L^2 = 0,$$

$$x = \frac{L \pm \sqrt{L^2 - \frac{2}{3}L^2}}{2}$$

$$= \frac{L \pm \frac{1}{\sqrt{3}}L}{2}$$

$$= \frac{L \pm 0{\cdot}577L}{2}$$

$$= 0{\cdot}211L \text{ or } 0{\cdot}788L. \quad \ldots\ldots\ldots\ldots(4)$$

Hence the points of zero bending moment lie at 0·211L from each wall.

Points of contraflexure. The two last cases considered provide examples of beams in which the curvature is partly convex downwards and elsewhere convex upwards. The centres of curvature for a portion of the length of the beam lie on the upper side, and for other portions lie on the lower side. Points on a beam where the curvature changes from convex upwards to convex downwards, *i.e.* where the centre of curvature changes from one side of the beam to the other, are called **points of contraflexure.** Curvature which is convex downwards may be called positive, and that which is convex upwards may be considered negative. The curvature changes sign at points of contraflexure, and hence must have zero value at such points.

Considering the equation (p. 166),

$$\text{Curvature} = \frac{1}{R} = \frac{M}{EI},$$

it is evident that, for the curvature to be zero, M must be zero. A point of contraflexure may hence be defined as a point of zero bending moment. Such points occur at quarter span for an encastré beam carrying a single load at the middle of the span (p. 176) and at 0·211 L from the walls in the case of a uniformly distributed load (p. 178). It should be noted that encastré beams differ from beams which are simply supported at both ends in that one or both supports in the latter may suffer sinkage when the load is applied, or by reason of some alteration in the foundation conditions, without thereby affecting the distribution of bending moment along the beam. No such alteration in either of the walls fixing the ends of an encastré beam can occur without affecting the bending moments on the beam. For example, if the encastré beam in Fig. 212 (*a*) should for any reason become loose in the holes in the wall, so that the fixing couples M_A and M_B disappear, the bending-moment diagram will change from that shown in Fig. 210(*a*) to that for a simply supported beam, and the maximum bending moment, and consequently the maximum stress due to bending, will be doubled. In the case of a uniform load (Fig. 212 (*b*)) such an alteration in the wall fixings would produce a change in the maximum bending moment from

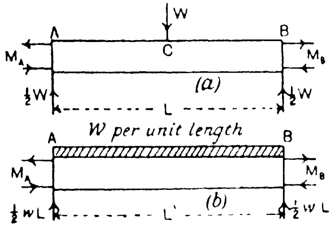


FIG. 212.—Encastré beams.

$-\frac{wL^2}{12}$ to $+\frac{wL^2}{8}$, that is, a numerical increase of 50 per cent It should also be noted that these alterations would be accompanied by very small alterations in the slope and deflection, the inference being that quite a small alteration in the shape or position of the fixing arrangements due to sinkage, or otherwise, will be sufficient to produce a large alteration in the bending moments and stresses

The difficulty may be overcome, if desired, by noting that at points of contraflexure there is zero bending moment, and that the beam may be cut at these points provided that means are provided there for taking up the shear Fig 213 (*a*) shows diagrammatically how this may be effected for an encastre beam carrying a central load W The beam is cut at quarter span, and links CD and EF are used for suspending the middle portion These links will be under pulls of $\frac{1}{2}$W owing to the shearing force. Obviously no moderate changes in the supports can now affect the bending moments in the component parts of the beam A practical method of designing the arrangement is shown in Fig. 213 (*b*), where the central portion is supported on a rocker at E and by a short column at CD. It will be observed that alterations of length, etc, due to expansion on heating, are taken up by this device without inducing stresses on the beam. The artifice of cutting a beam, or an arch, at places where it is desirable that there should be no possibility of any bending moment arising is often resorted to in practice.

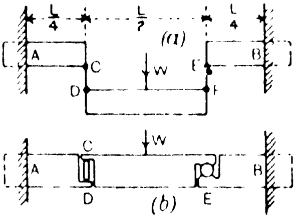


Fig. 213. Beam cut at the points of contraflexure

Propped cantilevers and beams. In Fig 214 (*a*) is shown a **cantilever AB carrying a uniformly distributed load.** The fixing in the wall at A is sufficient alone for the equilibrium of the cantilever, but an additional support or prop has been placed under B. The pressure on this prop depends on the elastic properties of the material of the cantilever and also on the level of the top of the prop. Assuming that the cantilever just touches the top of the prop before application of the load, the reaction of the prop may be calculated as follows.

Supposing the prop to be removed (Fig. 214 (*b*)), the deflection Δ_1

of the cantilever under the action of the distributed load would be given by

$$\Delta_1 = \frac{wL^4}{8EI} \quad \text{(p. 175)}. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now suppose that the distributed load is removed and that the prop is applied and pushed upwards until a deflection at B of the same magnitude as Δ_1 is obtained (Fig. 214 (*c*)). The upward deflection Δ_2 thus produced by the force P exerted by the prop will be

$$\Delta_2 = \frac{PL^3}{3EI} \quad \text{(p. 174)}. \quad (2)$$

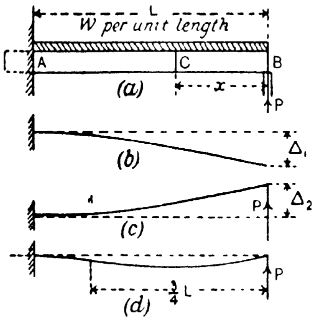


FIG. 214.—Propped cantilever.

If both P and the distributed load be applied simultaneously, the levels will be the same at both A and B (Fig. 214 (*d*)), for Δ_1 and Δ_2 are equal and opposite. Hence,

$$\frac{PL^3}{3EI} = \frac{wL^4}{8EI} = \frac{WL^3}{8EI},$$

where W is the total distributed load. Hence,

$$P = \tfrac{3}{8}W. \quad \ldots \quad \ldots (3)$$

The bending moment at any section C may be calculated now:

$$M_C = Px - \frac{wx^2}{2}$$

$$= \tfrac{3}{8}wLx - \frac{wx^2}{2}. \quad \ldots\ldots (4)$$

Points of contraflexure may be found by equating M_C to zero (p. 180). Thus,

$$\tfrac{3}{8}wLx - \frac{wx^2}{2} = 0.$$

Zero is one value of x satisfying this equation, hence B is a point of contraflexure. To obtain the other point, we have

$$\tfrac{3}{8}wL - \frac{wx}{2} = 0,$$

or

$$x = \tfrac{3}{4}L. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The bending-moment diagram is shown in Fig. 215 (*a*). The bending moment at the wall may be found by writing $x = L$ in (4), giving

$$M_A = \tfrac{3}{8}wL^2 - \frac{wL^2}{2}$$

$$= -\tfrac{1}{8}wL^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

To obtain the bending moment at $\frac{3}{8}$L from B, we have, from (4),

$$M_1 = \tfrac{3}{8}wL \,.\, \tfrac{3}{8}L - \frac{w}{2}(\tfrac{3}{8}L)^2$$
$$= \tfrac{9}{128}wL^2 \quad \ldots\ldots \qquad (7)$$

It will be understood that any vertical displacement of the prop, whether by reason of sinkage of the foundations or by changes in temperature, will alter the bending moment, and hence the stresses throughout the cantilever. The shearing force diagram is given in Fig. 215 (*b*); the values of the shearing force are $+\frac{5}{8}$W at the wall and $-\frac{3}{8}$W at the prop.

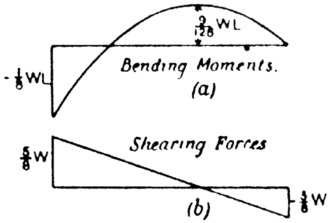


FIG. 215. Bending moment and shearing force diagrams for a propped cantilever.

The case of **a beam resting on three supports** at A, B and C is illustrated in Fig. 216 (*a*). The supports at A and B alone are

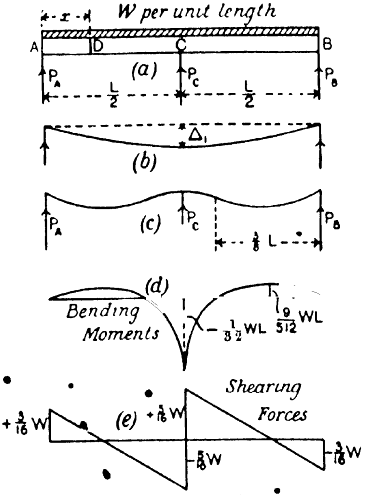


FIG. 216.—A beam resting on three supports.

sufficient for the equilibrium of the beam; hence, in this case also, the reactions, bending moments, and stresses depend on the levels of the supports being preserved.

Suppose the supports to divide the beam into two equal spans and that the supports are all at the same level. If the support at C be removed (Fig. 216 (*b*)), there will be a deflection Δ_1 at C given by

$$\Delta_1 = \frac{5}{384}\frac{wL^4}{EI} \quad \text{(p. 176).} \quad \ldots\ldots(1)$$

Replace the support at C by pushing upwards until the level is restored (Fig. 216 (*c*)). The upward deflection Δ_2 produced by P_C will be given by

$$\Delta_2 = \frac{P_C L^3}{48EI} \quad \text{(p. 169).} \quad \ldots\ldots\ldots\ldots(2)$$

Clearly Δ_1 and Δ_2 are equal. Hence,

$$\frac{P_C L^3}{48EI} = \frac{5}{384}\frac{wL^4}{EI},$$

$$P_C = \tfrac{5}{8}wL$$

$$= \tfrac{5}{8}W, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where W is the total load.

It will be evident that P_A and P_B are equal. Hence,

$$P_A = P_B = \tfrac{3}{16}W. \quad \ldots\ldots\ldots\ldots(4)$$

The bending moment at D may be found from

$$M_D = P_A x - \frac{wx^2}{2}$$

$$= \tfrac{3}{16}wLx - \frac{wx^2}{2}. \qquad (5)$$

Points of contraflexure occur where M_D is zero; to find these, we have

$$\tfrac{3}{16}wLx - \frac{wx^2}{2} = 0.$$

The value zero for x satisfies this equation; hence, A is one point and, from symmetry, B is another point of contraflexure. To obtain others,

$$\tfrac{3}{16}wL - \frac{wx}{2} = 0,$$

$$x = \tfrac{3}{8}L. \quad \ldots\ldots\ldots(6)$$

Points of contraflexure therefore occur at $\tfrac{3}{8}L$ from A and also at an equal distance from B. The complete bending-moment diagram is given in Fig. 216 (*d*) and the shearing-force diagram appears in Fig. 216 (*e*).

The beam here discussed is a simple illustration of continuous beams, *i.e.* beams continuous over several spans and resting on several supports.

Beams of uniform curvature. Considering again the equation

$$\frac{1}{R}=\frac{d^2y}{dx^2}=\frac{M}{EI},$$

it will be remembered that it has been assumed that the moment of inertia is uniform in all the cases considered. This is the case very often in practice; for example, beams of comparatively short span generally consist of a rolled steel beam or of two or more similar beams placed side by side. When we consider larger beams, we find that the section in general is not uniform, but is varied so as to produce more nearly a beam of uniform strength (p. 154). The above equation may be modified so as to include a great number of such cases. Thus,

$$M=\frac{f}{m}I \quad (\text{p. } 146);$$

$$\therefore \frac{1}{R}=\frac{M}{EI}=\frac{1}{EI}\,\frac{f}{m}I$$

$$\frac{1}{R}=\frac{f}{Em} \qquad \ldots.\ (1)$$

A beam of uniform strength is one having uniform maximum stress f. This may be secured by having constant depth and varying the breadth, in which case m will be constant. In equation (1) above, the right-hand side will contain nothing but constants, and therefore $\frac{1}{R}$ will be constant. Such a beam will have constant radius of curvature, and hence will bend into the arc of a circle. In other cases of built-up plate girders having parallel flanges, the breadth is constant, and uniformity in f is secured by adjusting the thickness of the flanges, the number of flange plates becoming greater towards the middle of the span. Assuming that this variation of flange thickness does not alter the depth sensibly, we have a constant value of m, and the girder will have constant curvature.

Constant curvature may also occur in a beam of uniform section. To obtain such a result, M must be constant in the equation

$$\frac{1}{R}=\frac{M}{EI}.$$

This condition may be brought about by the loads and reactions being applied in the form of two equal opposing couples. A carriage axle is a common example (Fig. 217). Here AC = BD; equal loads W, W are applied at A and B, and the wheel reactions P, P at

C and D will be each equal to W. The portion CD of the axle will therefore have uniform bending moment, given by

$$M_{CD} = W \times AC,$$

and hence will bend into the arc of a circle. The curvature in the overhanging portions AC and BD will vary, following the law for the cantilever worked out on p. 167.

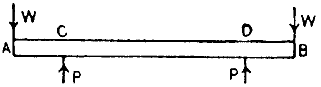


FIG. 217 —A carriage axle.

In Fig. 218 AB is a beam of length L bent into a circular curve ACB. Drawing the diameter EODC perpendicular to the chord AB, and remembering that the deflection will be very small in practice we have, by application of the principle that the products of the segments of two intersecting chords in a circle are equal,

$$ED \times DC = AD \times DB,$$

or, very nearly,

$$2R \times DC = (\tfrac{1}{2}L)^2;$$

hence the deflection DC at the middle will be

$$DC = \frac{L^2}{8R}.$$

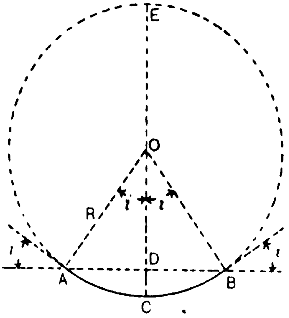


FIG 218 —Beam bent into a circular curve.

Substituting $\frac{M}{EI}$ for $\frac{1}{R}$ in this result gives

$$DC = \frac{ML^2}{8EI}.$$

It will also be evident that the inclination of the tangents at A and B will be equal to the angle AOC. Expressing this in radians so as to obtain the slope at A and B, we have

$$i = \frac{AC}{AO} = \frac{L}{2R}$$

$$= \frac{ML}{2EI}.$$

Relation of stress and deflection. In all the cases of deflection which have been considered, it will be noted that the expression for the maximum deflection has the form

$$\Delta = c\frac{WL^3}{EI},$$

where c is a numerical coefficient, the value of which depends on the circumstances of the case. Hence we may write,

$$\Delta \propto \frac{WL^3}{EI}. \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Taking the general equation for the strength of a beam (p. 146),

$$M = \frac{f}{m}I, \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

it will be noted that M is always proportional to WL, and that m is always proportional to d, the overall depth of the beam. Hence, from (2),

$$WL \propto \frac{f}{d}I.$$

Substitution of this in (1) gives

$$\Delta \propto WL\frac{L^2}{EI}$$

$$\propto \frac{f}{d}I\frac{L^2}{EI}$$

$$\propto \frac{fL^2}{dE}. \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Hence, in beams constructed of the same material, for which E will be constant, we may state that the maximum deflection will be directly proportional to the square of the length and inversely proportional to the depth when the beams are carrying loads which produce the same maximum value of f.

EXAMPLE. A steel bar of rectangular section is supported at its ends and carries a central load. The ratio of maximum deflection to span is not to exceed $\frac{1}{500}$; the maximum stress is not to exceed 5 tons per square inch. Find the ratio of span to depth if $E = 13,500$ tons per square inch.

$$\Delta = \frac{WL^3}{48EI} \text{ (p. 169),} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

$$M = \frac{f}{m}I,$$

or

$$\frac{WL}{4} = \frac{f}{\frac{1}{2}d}I = \frac{2fI}{d};$$

$$\therefore \quad WL = \frac{8fI}{d} \qquad \text{........(2)}$$

Hence, from (1),

$$\Delta = \frac{8fI}{d} \cdot \frac{L^2}{48EI}$$

$$= \frac{fL^2}{6dE}, \qquad \text{........(3)}$$

or

$$\frac{L}{d} = \frac{6E}{f} \cdot \frac{\Delta}{L} \qquad \text{.......(4)}$$

$$= \frac{6 \times 13500}{5} \times \frac{1}{500}$$

$$= \underline{32 \cdot 4}$$

EXERCISES ON CHAPTER VIII

1. A bar of steel of square section, 2 inches edge, is used as a cantilever, projecting 24 inches beyond the support, and has a load of 400 lb. at its free end. Find the values of the radius of curvature for sections at 3 inches intervals throughout the length. Plot R and the length of the cantilever. E = 13,500 tons per square inch.

2. Find the slope and deflection at the free end of the cantilever given in Question 1.

3. A beam of **I** section, 8 inches deep, 4·5 inches wide, metal 0·5 inch thick, is simply supported on a span of 10 feet and carries a central load of 1·5 tons. Calculate the maximum deflection and also the slope at the ends. What will be the radius of curvature at the middle of the span? Take E = 13,500 tons per square inch.

4. Answer Question 3, supposing that the beam carries only a uniformly distributed load of 2 tons.

5. An encastré beam of **I** section has its ends fixed into walls 12 feet apart. The depth is 12 inches, and I is 50 in inch units. If the stress is limited to 5 tons per square inch, what central load would be safe? Draw the diagrams of bending moment and shearing force.

6. Answer Question 5, supposing that the load is to be distributed uniformly.

7. Calculate the deflections at the middle of the span for the beams given in Questions 5 and 6.

8. A cantilever projects 8 feet from a wall and carries a load of 1·5 tons at 4 feet from the wall and another load of 0·75 ton at the free end. Draw the diagrams of bending moment, slope and deflection, in each case giving the scale of the diagram. State the values of the slope and deflection at the free end. Take I = 350 in inch units and E = 13,500 tons per square inch.

9. Calculate the uniform bending moment which must be applied to a bar of steel of 0·25 inch in diameter in order to make it bend into the arc of a circle of 20 feet radius. E = 30,000,000 lb. per square inch. If the bar is 5 feet in length, what will be the deflection at its centre?

10. A girder is 40 feet span by 4 feet deep, and rests on its supports. The uniformly distributed load produces a maximum stress due to bending of 5 tons per square inch. Find the deflection at the middle of the span. E = 13,500 tons per square inch.

11. Supposing the girder given in Question 10 to have uniform flange stress of 5 tons per square inch, what will be its radius of curvature? Calculate the deflection at the centre.

12. A cantilever of uniform section is built securely into a wall, and its outer end just touches a prop when there is no load. The cantilever is 8 feet long, and carries a uniformly distributed load of 1000 lb. per foot length. Find the reaction of the prop, and draw the diagrams of bending moment and shearing force; give the calculations required for these.

13. A beam 40 feet in length rests on three supports A, C and B at the same level; the supports divide the beam into two equal spans. If there is a uniformly distributed load of 1·5 tons per foot length, find the reactions of the supports, and draw the diagrams of bending moments and shearing force, showing the necessary calculations.

14. A piece of flat steel has to be bent round a drum 5 feet in diameter; what is the maximum thickness which the strip can be made so that there shall be no permanent deformation when it is removed from the drum? The steel has an elastic limit of 14 tons per square inch. E = 14,000 tons per square inch. (I. C. E.)

15. Three rolled steel joists 6 inches deep are placed side by side spanning an opening of 10 feet; the moment of inertia of the two outer joists is 20 and that of the inner one 44 inch units. A central load of 5 tons is so placed as to deflect each of the three joists equally; state the amount of the load carried by each joist and the maximum unit stress (*i.e.* stress in tons per square inch) in the centre joist only. (I. C. E.)

16. A beam is firmly built into a wall at one end, and rests freely at its other end on a vertical column whose centre line is distant 8 feet from the wall. The beam supports a wall, whose weight added to that of the beam itself is equivalent to a uniformly distributed load of 3200 lb. per foot run of the beam. Find (*a*) the total load supported by the column; (*b*) the bending moment and shear force at the section of the beam adjoining the wall; (*c*) the position of the point of zero bending moment. Sketch complete bending moment and shear diagrams. (B. E.)

17. A rectangular timber beam, supported at the ends, is of uniform section from end to end, and it carries a uniformly distributed load. If the working intensity of stress in the wood is not to exceed 2000 lb. per square inch, and if the modulus of elasticity of the wood is 1,700,000 lb. per square inch, determine the ratio of the depth of cross-section of beam to span of beam in order that the deflection may not exceed $\frac{1}{500}$th part of the span. (B. E.)

18. A horizontal beam, span 25 feet, is fixed at the ends. It carries a central load of 5 tons, and loads of 2 tons each at 5 feet from the ends. Determine the maximum bending moment, the bending moment at the centre of the span and the position of the points of contraflexure; sketch also a diagram of shear force. (L. U.)

19. A floor, carrying a uniformly distributed load of 2 cwt. per square foot over a span of 20 feet, is proposed to be carried by *either* (*a*) I joists, 10 inches deep; area, 12·35 square inches; I (maximum), 212 inch-units; pitch, 4 feet. Or, (*b*) I joists, 12 inches deep; area, 15·9 square inches; I (maximum), 375 inch-units; pitch, 6 feet. Compare these two propositions by finding the ratio of strengths, deflections and total weights of girders. Find the maximum skin stress in case (*a*). (L.U.)

20. A uniform beam, 30 feet long, fixed at the ends, has a load of 20 tons spread uniformly along it. It has also two loads of 3 tons, each hung from points which are 10 feet from the ends. What is the bending moment everywhere, and what is its greatest value? (B.E.)

CHAPTER IX.

WORKING LOADS. BEAMS AND GIRDERS.

Dead and live loads. The loads to which any structure is subjected may be divided into **dead** and **live loads**. The dead loads include the weights of all the permanent parts of the structure; the live loads may consist of travelling weights and other forces, such as wind pressure, which may occur periodically. Dead loads produce stress of constant magnitude in the parts of the structure; the live loads produce fluctuating stresses; hence each part of the structure may be called upon to withstand stresses which fluctuate between maximum and minimum values.

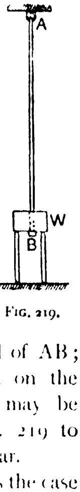


FIG. 219.

A load may be applied to a bar in three different ways: (*a*) in **gradual application**, the load on the bar is at first zero and the magnitude of the load is increased uniformly and slowly until the bar is carrying the whole load; (*b*) **sudden application** may be realised by reference to Fig. 219, in which the load W is supported by short rods so that it is just touching the collar at the lower end of AB; if the rods be knocked out, the load will suddenly rest on the collar; (*c*) **impulsive application** may be obtained by allowing W in Fig. 219 to drop from a height on to the collar.

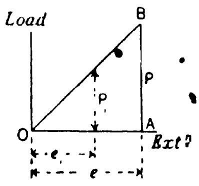


FIG. 220.—Diagram for a gradually applied load.

Resilience. Fig. 220 illustrates the case of gradual application of pull to a bar. The bar extends by an amount proportional to the load, up to the elastic limit, and at any instance the resistance of the bar is equal to the pull applied. When the load applied is P_1, the extension of the bar is e_1 and its resistance is equal to P_1. It will be evident from the figure, that the average value of the load is $\frac{1}{2}P$, and as this force acts through a distance e, the work done will be given by the product of these quantities (p. 325).

$$\text{Work done in stretching the bar} = \tfrac{1}{2}Pe. \quad \ldots\ldots\ldots\ldots \quad (1)$$

As the resistance of the bar is at all times equal to the pull, it follows that the energy stored in the bar will be equal to $\tfrac{1}{2}Pe$.

Let a = the sectional area of the bar in square inches.

P = the final pull in tons.

$f = \dfrac{P}{a}$ = the stress produced by P in tons per square inch.

L = the original length of the bar in inches.

e = the extension produced by P in inches.

E = Young's modulus in tons per square inch.

Then
$$E = \frac{P}{a} \cdot \frac{L}{e},$$

or
$$e = \frac{PL}{aE} = f\frac{L}{E}.$$

Also,
$$P = af.$$

Substituting these values in (1) gives:

$$\text{Energy stored in the bar} = \tfrac{1}{2}af \cdot f\frac{L}{E}$$

$$= aL\frac{f^2}{2E} \text{ inch-tons.} \quad \ldots \quad \ldots \quad (2)$$

This quantity is called the **resilience** of the bar. The **resilience of the material** is stated usually as the energy which can be stored in a cubic inch when stressed up to the elastic limit. This may be obtained from (2) by taking f to be the elastic limit stress and noting that aL is the volume of the bar in cubic inches. Hence

$$\text{Resilience} = \frac{f^2}{2E} \text{ inch-tons per cubic inch.}$$

Load suddenly applied. If the load be applied suddenly as in Fig. 219, and if the bar extends by an amount e, gravity is doing work on W throughout this extension. Hence,

$$\text{Work done on } W = We \text{ inch-tons.}$$

This work may be represented by the rectangular diagram OKLM (Fig. 221), in which OK represents W and OM represents e. The resistance offered by the bar during the extension still follows the same law as before, *i.e.* at first the resistance is zero and it gradually increases, being proportional to the extension up to the elastic limit.

This may be represented by the triangular diagram OMQ. At PN the resistance of the bar and the weight of the load are equal, but extension does not stop here, since more work has been done by gravity than can be stored in the rod. Extension will go on until the work done by gravity has been stored entirely in the rod, *i.e.* until the area OKLM is equal to the area OMQ. This will occur evidently when OM is equal to twice OP, or when MQ is twice OK. Now, had W been applied gradually, the stretch would have been OP, hence the sudden application of W has produced a stretch double of this amount. Also PN would have been the final resistance of the bar had W been applied gradually, hence the sudden application has produced a resistance of twice this magnitude, and therefore also a stress equal to double of that which would have occurred with gradual application.

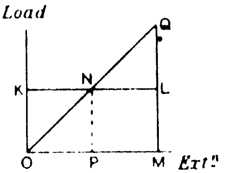


Fig. 218. Diagram for a load applied suddenly.

The conditions are not attained easily in practice, but the effects of live loads in producing stress are often taken account of by estimating what the stress would be had the load been applied gradually and then taking double this stress as that which the part will be called upon to carry.

Impulsive application of a load. In an impulsive application, let W be dropped from a height H inches (Fig. 219). Then

$$\text{Total work done by gravity} = W(H+e) \text{ inch-tons.}$$

Extension will go on until the whole of this work is stored in the bar. From equation (2) (p. 192), we have

$$\text{Energy stored in the bar} = aL\frac{f^2}{2E}.$$

Hence,

$$aL\frac{f^2}{2E} = W(H+e),$$

or

$$f = \sqrt{\frac{2EW(H+e)}{aL}} \quad \dots\dots\dots(3)$$

If e is small compared with H, as is the case generally, then

$$f = \sqrt{\frac{2EWH}{aL}} \quad \dots\dots\dots(4)$$

Working stresses. The stresses which may occur in any part of a structure are estimated by first calculating the stress produced by the dead load. Separate calculations are then made in order to determine the stress produced by the live loads. The stress in the part under consideration will then fluctuate between known limits, and it remains to determine what ought to be the safe stress permitted in the part. The determination may be based on the known breaking strength of the material by taking the working stress as a fraction of the ultimate strength. The reciprocal of this fraction is called a **factor of safety,** and its value depends on the kind of material and the nature of the loads. Thus, for wrought iron and steel, the factor of safety may be 3 for a dead load, 5 for a stress which does not change from pull to push, 8 for a stress which alternates from a certain pull to an equal push and 12 for parts subjected to shock. Somewhat higher factors may be taken for cast iron and for timber, as these materials are less trustworthy.

It may be noted here that a load which a piece of material may carry for an indefinite time, if applied steadily, will ultimately cause fracture if it is applied and removed many times. The effect is more marked if the load be alternated, *i.e.* applied first as a pull and then as a push, in the manner in which the piston rod of a steam engine is loaded. The experiments of Bauschinger, Wohler, Stanton and others, on the effects of repeatedly applied and alternating loads show that the strength to resist an indefinite number of repetitions depends on the range of stress rather than on the actual values of the maximum and minimum stresses.

The following rule has been deduced by Unwin* from the results of Wohler's experiments, and applies to cases of varying stresses

Let f_s = the breaking strength of the material in tons per square inch under a load applied gradually.

f_1 = the breaking strength of the same material in tons per square inch when subjected to a variable load which fluctuates from f_1 to f_2 and is repeated an indefinite number of times. Let this be of the same kind (push or pull) as f_s.

f_2 = the lower limit, in tons per square inch, to which the material is subjected, + if f_2 is of the same kind as f_1 and f_s, − if f_2 is of the opposite kind.

$r = f_1 - f_2$ = the range of stress.

Then **Unwin's formula** is

$$f_1 = \frac{r}{2} + \sqrt{f_s^2 - nrf_s} \quad \dots\dots\dots\dots\dots (1)$$

**Machine Design*, Part I. Prof. W. C. Unwin. (Longmans, 1909.)

n has the value 1·5 for wrought iron and mild steel. From equation (1), we have

$$\left(f_1 - \frac{r}{2}\right)^2 = f_s^2 - nrf_s,$$

or

$$f_s^2 - nrf_s - \left(f_1 - \frac{r}{2}\right)^2 = 0,$$

$$\therefore f_s = \frac{nr + \sqrt{n^2r^2 + 4\left(f_1 - \frac{r}{2}\right)^2}}{2}, \quad \ldots (2)$$

in which the negative sign has been disregarded. Equation (2) gives a dead load stress f_s which would produce, when applied steadily to the member, the same effect as the actual fluctuating stresses.

If each side of (2) be multiplied by the sectional area of the bar, the stresses in the equation become total forces on the bar. Using capital letters to represent the total forces corresponding to the stresses f_s, r and f_1, we have

$$\text{Equivalent steady load } F_s = \frac{nR + \sqrt{n^2R^2 + 4\left(F_1 - \frac{R}{2}\right)^2}}{2}. \quad (3)$$

The **Launhardt-Weyrauch formula** also takes account of stress variation. Let F_1 and F_2 tons be the maximum and minimum forces to which the bar may be subjected, and let f_s tons per square inch be the breaking strength of the material under a gradually applied stress. Then

$$\text{Breaking stress} = \tfrac{2}{3}f_s\left(1 + \frac{1}{2}\frac{F_2}{F_1}\right) \text{ tons per square inch.} \quad \ldots.(4)$$

Applying a factor of safety of 3 to this, we have

$$\text{Working stress} = \tfrac{2}{9}f_s\left(1 + \frac{1}{2}\frac{F_2}{F_1}\right) \text{ tons per square inch.} \quad \ldots (5)$$

EXAMPLE. A certain bar in a structure carries a pull of 80 tons due to the dead load; the live load produces forces in the same bar varying from 20 tons pull to 40 tons push. Find the working stress and the cross-sectional area of the bar. The breaking strength of the material under a gradually applied pull is 30 tons per square inch.

First Method. By doubling the live load pull and adding the result to the dead load pull, we have

$$\text{Equivalent dead load} = 80 + (2 \times 20) = 120 \text{ tons.}$$

Taking 9 tons per square inch as the working stress, we have

$$\text{Sectional area of bar} = \frac{120}{9} = \underline{13.3} \text{ sq. inches.}$$

Second Method. By Unwin's formula (3),

$$R = 60 \text{ tons.}$$

$$F_1 = 80 + 20 = 100 \text{ tons}$$

$$n = 1\cdot5.$$

$$\text{Equivalent dead load} = \frac{(1\cdot5 \times 60) + \sqrt{(\frac{9}{4} \times 3600) + 4(100 - 30)^2}}{2}$$

$$= 128 \text{ tons.}$$

Again using 9 tons per square inch as the working stress, we have

$$\text{Sectional area of bar} = \frac{128}{9} = \underline{14\cdot2} \text{ sq. inches.}$$

Third Method. By the Launhardt-Weyrauch formula (5), we have

$$f_s = 30 \text{ tons per square inch.}$$

$$F_1 = 80 + 20 = 100 \text{ tons pull.}$$

$$F_2 = 80 - 40 = 40 \text{ tons pull.}$$

$$\text{Working stress} = \tfrac{2}{9} \times 30(1 + \tfrac{40}{200})$$

$$= 8 \text{ tons per square inch.}$$

$$\text{Cross-sectional area} = \frac{F_1}{8} = \frac{100}{8}$$

$$= \underline{12\cdot5} \text{ square inches.}$$

Wind pressure. If wind pressure be treated as a live load, then 30 lb. per square foot of vertical surface may be assumed to be the maximum. If treated as a dead load, then pressures up to 55 lb. per square foot of vertical surface may be taken. Stanton's experiments at the National Physical Laboratory give, for small plates,

$$p = 0\cdot0027V^2 \text{ lb. per square foot,}$$

or, for large plates,

$$p = 0\cdot0032V^2 \text{ lb. per square foot,}$$

where V is the velocity of the wind in miles per hour. Hutton's formula may be used in calculating the normal pressure on inclined surfaces.

Let p = the pressure in lb. per square foot on a surface perpendicular to the direction of the wind.

p_n = the normal pressure in lb. per square foot on a surface inclined at an angle θ to the direction of the wind.

Then Hutton's formula gives

$$p = p_n \cdot \sin\theta^{(1\cdot84\cos\theta - 1)}$$

$$= ap_n,$$

where a is a coefficient depending on the value of θ. Values of a

corresponding to different values of θ have been plotted in Fig. 222, and the value of a appropriate to any given surface may be taken from the curve.

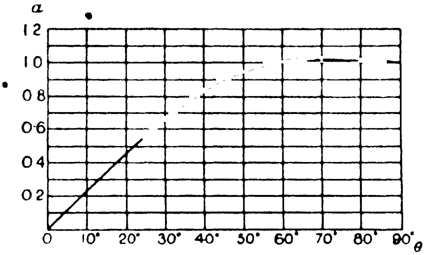


FIG. 222.—Values of a in Hutton's formula.

Travelling load. In Fig. 223 (a) AB is a beam simply supported at A and B and carrying a load W. The effects of W alone will be considered, any other loads being disregarded. If W remains fixed in position, the reactions P and Q as well as the bending moment and shearing force at any section, such as D, have definite values. These values will alter if W travels along the beam, and it then becomes necessary to determine what position W must occupy when a given section is subjected to the greatest bending moment it will be called upon to resist, as well as the value of this bending moment. The same questions must also be considered in relation to the shearing force at any given section.

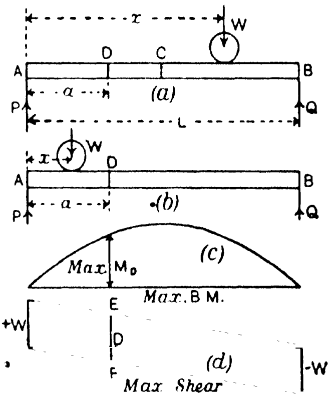


FIG. 223.—Beam carrying a single rolling load.

Let x = the distance of W from A.

a = the distance of the given section D from A.

L = the span of the beam, all in the same units.

Then, taking moments about B, we have

$$PL = W(L - x),$$

or $$P = \left(\frac{L - x}{L}\right)W = \left(1 - \frac{x}{L}\right)W. \quad \text{......... .. ....} \quad (1).$$

Let W be on the right-hand side of D as shown in Fig. 223 (*a*). Then the bending moment at D will be

$$\begin{aligned} M_D &= Pa \\ &= \left(1 - \frac{x}{L}\right)Wa. \quad \text{....................} \quad \text{...}(2) \end{aligned}$$

Hence, as x diminishes, *i.e.* as W travels towards the left and so approaches the section D, the bending moment at D increases.

Now let W be on the left-hand side of D as shown in Fig. 223 (*b*). Writing down the bending moment at D, we have

$$\begin{aligned} M_D &= Pa - W(a - x) \\ &= \left(1 - \frac{x}{L}\right)Wa - W(a - x) \\ &= Wx\left(1 - \frac{a}{L}\right). \quad \text{... .. ..} \quad (3) \end{aligned}$$

This result indicates that as x diminishes, *i.e.* as W, still travelling towards the left, recedes from the section D, the bending moment at D is becoming smaller. Therefore, the greatest bending moment which the section D will be called upon to resist will occur when W is immediately over the section. The value of this bending moment may be obtained by writing $x = a$ in either (2) or (3) above, giving

$$\begin{aligned} \text{Maximum bending moment } M_D &= \left(1 - \frac{a}{L}\right)Wa \\ &= W\left(a - \frac{a^2}{L}\right). \quad \text{......... .} \quad (4) \end{aligned}$$

If *a* be varied so as to obtain the maximum bending moments for other sections of the beam, and if the results of calculation from equation (4) be plotted, a parabolic curve will be obtained (Fig. 223 (*c*)), the ordinates of which will show the maximum bending moment for all sections of the beam. The centre section C is called upon to resist a bending moment $\frac{WL}{4}$. It will, of course, be understood that the values shown by the ordinates in Fig. 223 (*c*) are not attained simultaneously. The diagram must be interpreted as indicating that the bending moment at any section, say D, is zero when W is off the beam, and increases gradually as W travels towards the section; the maximum value M_D is attained when W reaches the section.

The shearing force at D, when W occupies any position on the beam lying on the right of D, will be positive and equal to P; hence, from (1),

$$S_D = P = \left(1 - \frac{x}{L}\right)W. \quad \ldots\ldots \quad (5)$$

This result shows that the shearing force increases as x diminishes, *i.e.* as W approaches the section D from the right.

Taking W in a position on the left of D, the shearing force will be negative, and will be given by

$$S_D = P - W \quad \left(1 - \frac{x}{L}\right)W - W$$

$$= -\frac{x}{L}W. \quad \ldots\ldots\ldots \quad (6)$$

We infer from this result that the negative shearing force at D diminishes as x becomes smaller, *i.e.* as W recedes from the section and approaches the left-hand support.

The inferences from this discussion are that the shearing force at any section attains a maximum positive value when W lies close to the right-hand side of the section, that it becomes zero as W crosses the section, and attains a maximum negative value when W lies close to the left-hand side of the section. To obtain the values of these shearing forces, write $x = a$ in equations (5) and (6), giving

$$\text{Maximum positive shearing force } S_D = \left(1 - \frac{a}{L}\right)W. \quad (7)$$

$$\text{Maximum negative shearing force } S_D = -\frac{a}{L}W. \ \ldots \quad \ldots \ (8)$$

Varying a so as to obtain values of the shearing forces for other sections and plotting the values so found (Fig. 223 (*d*)), we obtain two sloping straight lines. This diagram must be interpreted as follows: the shearing force at any section is zero when W is off the beam; as W travels along the beam from right to left, the shearing force at any section D is positive and increases gradually, until the maximum value DE is attained when W is on the point of arriving at the section. As W crosses the section, the shearing force becomes negative and attains the maximum value DF when W reaches the other side of the section.

Uniform travelling load. In Fig. 224 (*a*) is illustrated the case of a beam AB simply supported at A and B and subjected to a uniform load *w* per unit length. Taking the load of sufficient length to cover the whole span, it will be evident that the maximum bending moment at any section will occur when the beam is loaded fully, *i.e.* when the whole span is covered by the load. The bending-moment diagram will therefore be parabolic (Fig. 224 (*b*)) and of maximum height $\frac{wL^2}{8}$.

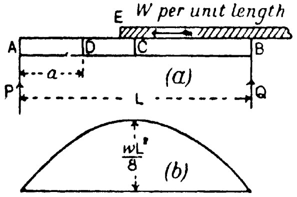


FIG. 224.—Beam carrying a uniform travelling load; maximum bending-moment diagram.

In Fig. 224 (*a*) the nose E of the load is advancing towards a given section D. The shearing force at D is positive and is equal to P; hence it increases as E approaches D. When E has crossed to the other side of D (Fig. 225 (*a*)), the shearing force is P diminished by the portion of the load lying between D and E. This shearing force will be less than that existing at D when the nose E is vertically over D, for P will then have a certain value, and this value will be increased in Fig. 225 (*a*) by a fraction only of the portion of the load lying between D and E; as the whole of the latter must be deducted from P in calculating the shearing force in Fig. 225 (*a*), it follows that the positive shearing force at D in Fig. 225 (*a*) is diminishing. Hence the maximum positive shearing force at any section occurs when the whole of the part of the beam lying to the right of the section is covered by the load, the end of the load being vertically over the section. In the same manner it may be shown that the maximum negative shearing force at any section occurs when the part of the beam lying on the left of the section is covered by the load (Fig. 225 (*b*)).

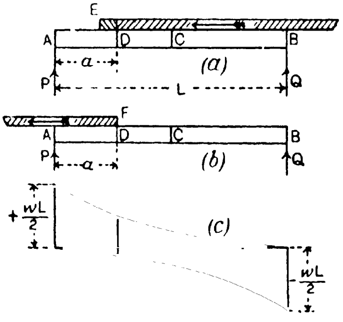


FIG. 225.—Beam carrying a uniform travelling load; maximum shearing-force diagram.

To obtain the values of these shearing forces, first let the load cover DB (Fig. 225 (*a*)) and take moments about B,

$$PL = w(L-a)\frac{(L-a)}{2}$$

$$= \tfrac{1}{2}w(L-a)^2.$$

$$\therefore \text{ maximum positive shearing force } S_D = P = \frac{w}{2L}(L-a)^2. \quad . \quad (9)$$

Now let the load cover AD (Fig. 225 (*b*)) and take moments about A;

$$QL = wa\frac{a}{2} = \tfrac{1}{2}wa^2,$$

or

$$Q = \frac{wa^2}{2L}.$$

$$\text{Hence,}\quad \text{Maximum negative shearing force } S_D = Q = \frac{wa^2}{2L}. \quad \ldots \quad (10)$$

Variation of a in (9) and (10) so as to obtain values for other sections will evidently produce two parabolic curves when plotted (Fig. 225 (*c*)). The interpretation of this diagram is similar to that of Fig. 223 (*d*)). The end ordinates are of magnitudes $\pm\tfrac{1}{2}wL$.

Combined dead and travelling loads. If, in addition to the travelling or live loads, the dead loads be considered, diagrams of bending moments and shearing forces may be drawn separately for the latter. Combined diagrams of bending moments and shearing forces may then be constructed by adding algebraically the corresponding ordinates of the diagrams. This has been done in Fig. 226 for a uniformly distributed dead load and a single rolling load.

In Fig. 226 (*b*), ACB is the bending-moment diagram for the dead load and ADB is that for the live load; AEB is the combined diagram. In Fig. 226 (*c*), FGKH is the shearing-force diagram for the dead load; FGL and FGM are the shear diagrams for the live load; FGKRN is the shear diagram for the com-

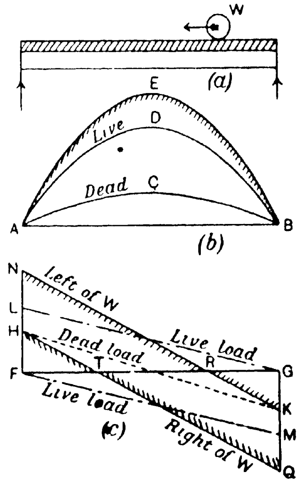


FIG. 226.—Diagrams for a beam carrying a dead load and a single rolling load.

bined loads, and shows the shearing force on any section lying close to the left of the live load; GFHTQ is a similar diagram for sections lying close to the right of the live load. The construction consists in making FN = FH + FL and joining NK; also make GQ = GK + GM and join QH. Zero shearing force occurs at T, and at R. Sections lying between F and T are subjected to positive shear only, those lying between R and G have negative shearing force only; sections lying between T and R have to resist both kinds of shearing force.

Maximum bending moments for a non-uniform travelling load. In designing bridge girders, it is necessary sometimes to consider the

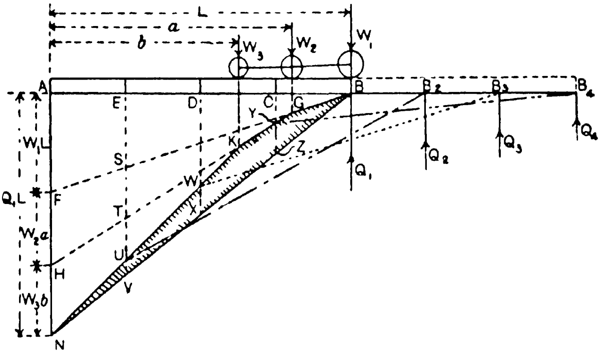


FIG. 227.—Bending-moment diagrams for a system of rolling loads.

effects of a non-uniform travelling load. Fig. 227 illustrates a convenient method; bending-moment diagrams for the girder when the load is occupying several given positions are obtained first; from these diagrams the maximum bending moment at any section is determined.

AB is the girder resting on supports at A and B; sections at E, D and C divide the girder into four equal bays. Three loads W_1, W_2 and W_3 at fixed distances apart have been chosen, but it will be understood that the method applies to any number of loads.

First let W_1 be vertically over B, and let the distances of W_2 and W_3 from A be a and b respectively.

Take moments about A and set these off along AN, which is drawn at right angles to AB.

Moment of $W_1 = W_1L$, represented by AF.
„ $W_2 = W_2a$, „ FH.
„ $W_3 = W_3b$, „ HN.

Let Q_1 be the reaction at B, then the sum of the above moments is equal to the moment of Q_1 about A; hence the moment of Q_1 about A is represented by AN.

Join BF and produce the line of W_2 to cut BF in G. Join GH and produce the line of W_3 to cut GH in K. Join KN and also BN. Draw the vertical ESTUV.

From the laws of proportion applied to similar triangles, the following statements may be made:

Moment of Q_1 about E = EV.
" W_1 " E = ES.
" W_2 " E = ST.
" W_3 " E = TU.

Now, by taking moments about E, we have

$$\begin{aligned}\text{Bending moment at E} &= \text{moment of } Q_1 - \text{moment of } W_1 \\ &\quad - \text{moment of } W_2 - \text{moment of } W_3 \\ &= EV - ES - ST - TU \\ &= UV.\end{aligned}$$

In the same way $M_D = WX$ and $M_C = YZ$. Hence the bending-moment diagram for the given position of the loads is the shaded diagram in Fig. 227.

To obtain the bending-moment diagram when W_1 is vertically over the section C, instead of moving the loads, leave them in their original position and shift the girder towards the right until C is vertically under W_1. The end B will then be at B_2 and A will coincide with the original position of E. The reaction Q_2 at B_2 will be obtained by taking moments about E, giving

$$\begin{aligned}Q_2 \times B_2E &= \text{moment of } W_1 + \text{moment of } W_2 + \text{moment of } W_3 \\ &= ES + ST + TU \\ &= EU.\end{aligned}$$

Join UB_2, when it will follow, by similar reasoning to that already employed, that the bending-moment diagram for W_1 over C is

$B_2BGKWUB_2$.

Similarly, when W_1 is at D, the diagram of bending moments will be $B_3B_2BGKWB_3$; also when W_1 is at E, the bending moment diagram will be $B_4B_3B_2BGYB_4$.

Measure these diagrams so as to obtain from each the bending moments at E, D and C. If these are tabulated, there will be no difficulty in obtaining the maximum value of the bending moment at each section by inspection of the table.

Position of load W_1.	Bending moments at sections.		
	C	D	E
B			
C			
D			
E			
A	0	0	0

It should be noted that the loads may run on to the bridge girder from either end, and that either W_1 or W_3 may lead. The effect of this on sections lying equidistant from the middle of the span, such as E and C, may be taken account of fully by choosing the maximum of the tabular values for C and E as being the bending moment to which both E and C may be subjected depending on which way the load runs on to the bridge.

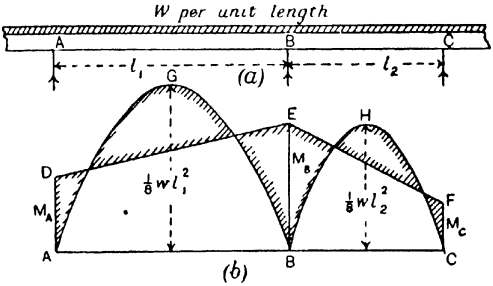


FIG 228 — Portion of a continuous beam.

Continuous beams. Let ABC (Fig. 228) be a portion of a beam which is continuous over several supports; three of the supports are situated at A, B and C respectively, the spans being l_1 and l_2 respectively. For simplicity, the load is taken as w per unit length, uniformly distributed throughout.

There will be bending moments at each support owing to the beam being one continuous piece; let these be M_A, M_B and M_C respectively. Erect perpendiculars AD, BE and CF to represent these bending moments (Fig. 228 (*b*)) and join DE and EF. Draw also the parabolic bending-moment diagrams AGB and BHC for the two segments AB and BC taken as cut at A, B and C, and simply resting on the supports. Then, as has been shown for an encastré beam (p. 176), the difference between the bending-moment diagrams, shown shaded, will be the bending-moment diagram for the portion ABC of the continuous beam.

It is evident that the solution will depend on the determination of M_A, M_B and M_C. To determine these, we have the principle that if all the supports are at the same level, then the deflections at A, B and C must be zero, whatever may be the changes in deflection occurring in the spans. Hence, taking moments of area about A, the moment of ADEB must equal that of AGB, also taking moments of area about C, the moment of BEFC must equal that of BHC.

Taking moments about A, and remembering that the parabolic area is two-thirds that of the circumscribing rectangle, we have

$$\frac{2}{3}\cdot\frac{wl_1^2}{8}\cdot l_1\cdot\frac{l_1}{2} = M_A l_1\frac{l_1}{2} + (M_B - M_A)\frac{l_1}{2}\cdot\frac{2}{3}l_1$$

It will be noted that the right hand side has been obtained by splitting ADEB into a rectangle of height AD and base AB, and a triangle of height (BE - AD) and base equal to AB. The equation is reduced as follows:

$$\frac{wl_1^3}{24} = \frac{1}{2}M_A l_1 + \frac{1}{3}(M_B - M_A)l_1$$

$$= \frac{1}{6}M_A l_1 + \frac{1}{3}M_B l_1. \qquad \ldots\ldots\ldots\ldots\ (1)$$

Taking moments about C in the same manner, we have

$$\frac{2}{3}\cdot\frac{wl_2^2}{8}\cdot l_2\cdot\frac{l_2}{2} = M_C l_2\frac{l_2}{2} + (M_B - M_C)\frac{l_2}{2}\cdot\frac{2}{3}l_2,$$

$$\frac{wl_2^3}{24} = \frac{1}{2}M_C l_2 + \frac{1}{3}(M_B - M_C)l_2$$

$$= \frac{1}{6}M_C l_2 + \frac{1}{3}M_B l_2. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Add (1) and (2):

$$\frac{w}{24}(l_1^3 + l_2^3) = \frac{1}{6}M_A l_1 + \frac{1}{3}M_B(l_1 + l_2) + \frac{1}{6}M_C l_2,$$

or

$$\frac{w}{4}(l_1^3 + l_2^3) = M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2. \ \ldots\ldots\ldots\ldots\ldots\ (3)$$

Should the spans be carrying uniformly distributed loads of different values, let w_1 and w_2 be the loads per unit length on AB and BC respectively. Then (1) and (2) will become

$$\frac{w_1 l_1^3}{24} = \frac{1}{6}M_A l_1 + \frac{1}{3}M_B l_1,$$

$$\frac{w_2 l_2^3}{24} = \frac{1}{6}M_C l_2 + \frac{1}{3}M_B l_2.$$

Adding these and reducing as before gives

$$\frac{w_1 l_1^3}{4}+\frac{w_2 l_2^3}{4}=M_A l_1+2M_B(l_1+l_2)+M_C l_2. \quad \ldots\ldots\ldots(4)$$

Equations (3) and (4) are cases of **Clapeyron's theorem of three moments.** By use of these, an equation may be written down for any three successive supports of a continuous beam. If there are n supports, there will be $(n-2)$ equations; other two equations may also be written from the data supplied for the ends of the beam. Thus, if the beam simply rests on the support at each end, the bending moments at these supports will be zero, and the other equations will be sufficient in order to obtain the complete solution.

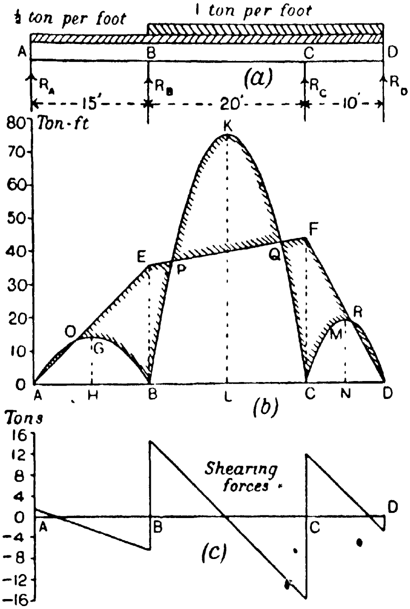


FIG. 229.—A continuous beam having three spans.

EXAMPLE I. A continuous beam rests on four supports on the same level and carries loads as shown in Fig. 229. Find the bending moments at the supports.

Equation (4) applied to A, B and C gives

$$\left(\frac{0\cdot5\times15^3}{4}\right)+\left(\frac{1\cdot5\times20^3}{4}\right)=15M_A+2M_B(15+20)+20M_C.$$

Also, $M_A = 0$;

$$\therefore\ 422 + 3000 = 70M_B + 20M_C \ \dots\dots\dots\dots\dots (1)$$

Equation (4) applied to B, C and D gives

$$\left(\frac{1\cdot5 \times 20^3}{4}\right) + \left(\frac{1\cdot5 \times 10^3}{4}\right) = 20M_B + 2M_C(20 + 10) + 10M_D.$$

Also, $M_D = 0$;

$$\therefore\ 3000 + 375 = 20M_B + 60M_C \ \dots\dots\dots\dots\dots (2)$$

From (1) and (2),

$$\begin{aligned} 210M_B + 60M_C &= 10{,}266, \\ 20M_B + 60M_C &= 3375\,; \\ 190M_B &= 6891, \\ M_B &= \underline{36\cdot3} \text{ ton-feet.} \end{aligned}$$

From (2),

$$\begin{aligned} (20 \times 36\cdot3) + 60M_C &= 3375, \\ 60M_C &= 2649, \\ M_C &= \underline{44\cdot15} \text{ ton-feet.} \end{aligned}$$

EXAMPLE 2. Find the reactions of the supports of the beam given in Example 1.

To find R_A, write down an expression for the bending moment at B, obtained by calculating the moments about B of the forces acting on AB.

$$\begin{aligned} (0\cdot5 \times 15 \times \tfrac{15}{2}) - (R_A \times 15) &= M_B = 36\cdot3, \\ 56\cdot25 - 15R_A &= 36\cdot3, \\ R_A &= \underline{1\cdot33} \text{ tons.} \end{aligned}$$

In the same way, R_B may be found by writing down an expression for M_C, taking moments about C of all the forces acting on ABC.

$$\begin{aligned} (0\cdot5 \times 35 \times \tfrac{35}{2}) + (1 \times 20 \times \tfrac{20}{2}) - (R_A \times 35) - (R_B \times 20) &= M_C = 44\cdot15, \\ 306\cdot25 + 200 - (1\cdot33 \times 35) - 20R_B &= 44\cdot15, \\ 20R_B &= 415\cdot5, \\ R_B &= \underline{20\cdot77} \text{ tons.} \end{aligned}$$

To find R_C, take moments about D of all the forces acting on the beam.

$$\begin{aligned} (0\cdot5 \times 45 \times \tfrac{45}{2}) + (1 \times 30 \times \tfrac{30}{2}) - (R_A \times 45) - (R_B \times 30) - (R_C \times 10) &= M_D = 0, \\ 506\cdot25 + 450 - (45 \times 1\cdot33) - (30 \times 20\cdot77) - 10R_C &= 0, \\ 10R_C &= 273\cdot25, \\ R_C &= \underline{27\cdot32} \text{ tons.} \end{aligned}$$

Also, $R_A + R_B + R_C + R_D =$ the total load on the beam,

$$\begin{aligned} R_D &= 52\cdot5 - (1\cdot33 + 20\cdot77 + 27\cdot32) \\ &= \underline{3\cdot08} \text{ tons.} \end{aligned}$$

In order to check the accuracy, calculate R_D by taking moments about C of the forces acting on CD.

$$\begin{aligned} (1\cdot5 \times 10 \times \tfrac{10}{2}) - (R_D \times 10) &= M_C = 44\cdot15, \\ 10R_D &= 30\cdot85, \\ R_D &= \underline{3\cdot08} \text{ tons.} \end{aligned}$$

EXAMPLE 3. Draw the diagrams of bending moment and shearing force.

$$\text{S on the right of A} = +R_A = +\underline{1\cdot33}\text{ tons}$$

$$\begin{aligned}\text{S on the left of B} &= R_A - (0\cdot5 \times 15)\\ &= 1\cdot33 - 7\cdot5 = -\underline{6\cdot17}\text{ tons.}\end{aligned}$$

$$\begin{aligned}\text{S on the right of B} &= R_A + R_B - (0\cdot5 \times 15)\\ &= 1\cdot33 + 20\cdot77 - 7\cdot5 = +\underline{14\cdot6}\text{ tons.}\end{aligned}$$

$$\begin{aligned}\text{S on the left of C} &= R_A + R_B - (0\cdot5 \times 35) - (1 \times 20)\\ &= 1\cdot33 + 20\cdot77 - 17\cdot5 - 20\\ &= \underline{15\cdot4}\text{ tons.}\end{aligned}$$

$$\begin{aligned}\text{S on the right of C} &= R_A + R_B + R_C - 17\cdot5 - 20\\ &= 1\cdot33 + 20\cdot77 + 27\cdot32 - 37\cdot5\\ &= +\underline{11\cdot92}\text{ tons.}\end{aligned}$$

$$\text{S on the left of D} = -R_D = -\underline{3\cdot08}\text{ tons}$$

The shearing force varies uniformly between the supports; the complete shearing-force diagram is given in Fig. 229 (*c*).

The following quantities, together with the bending moments at the supports, are required for the bending-moment diagram. They are obtained by calculating the bending moments at the middle of each span, assuming that the beam is cut at B and C.

$$\begin{aligned}\text{Bending moment at the centre of AB} &= \frac{w_1 l_1^2}{8} = \frac{0\cdot5 \times 15 \times 15}{8}\\ &= \underline{14\cdot06}\text{ ton-feet.}\end{aligned}$$

$$\begin{aligned}\text{Bending moment at the centre of BC} &= \frac{w_2 l_2^2}{8} = \frac{1\cdot5 \times 20 \times 20}{8}\\ &= \underline{75}\text{ ton-feet.}\end{aligned}$$

$$\begin{aligned}\text{Bending moment at the centre of CD} &= \frac{w_2 l_3^2}{8} = \frac{1\cdot5 \times 10 \times 10}{8}\\ &= \underline{18\cdot75}\text{ ton-feet.}\end{aligned}$$

The bending-moment diagram is given in Fig. 229 (*b*), and is drawn by making BE and CF equal to M_B and M_C respectively and joining AE, EF and FD. GH, KL and MN are then set up from the centres of AB, BC and CD, and are made equal to 14·06, 75 and 18·75 ton-feet respectively. The curves AGB, BKC and CMD are parabolic. The difference of these diagrams, shown shaded, is the bending-moment diagram for the beam. Points of contraflexure (p. 180) occur at O, P, Q and R, as the bending moments are zero there.

Plate girders. Plate girders are used instead of rolled **I** sections when the dimensions of the girder become large. Such girders consist of top and bottom flange plates (Fig. 231) and a web plate secured to the flanges by riveted angles. The flange plates, as may be observed in Fig. 230, increase in number towards the middle of the span, where the bending moment is large. The web plate is generally of uniform thickness in girders of comparatively small span; in very large girders, in which the web is built of several plates placed end to end, the plates near the supports may be made thicker than those at the middle, thus making allowance for the larger shearing forces near the supports. In calculating the dimensions of the parts, it is customary to assume that the flanges supply the whole of the resistance to bending and that the web supplies the whole of the resistance to shearing. The web is liable to buckling, and requires to be stiffened at intervals. For this purpose vertical stiffeners are riveted to the web plate at intervals as shown in Fig. 230, these are of closer pitch near the supports, and may be constructed of angles as in Fig. 231, or may be of T section.

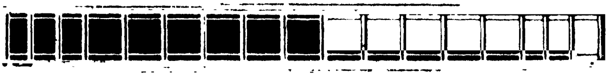


Fig. 230.—Side elevation of a plate girder.

The method of finding the principal dimensions may be understood by study of the following example.

EXAMPLE. A plate girder of 30 feet span with parallel flanges has to carry a uniformly distributed dead load of 2 tons per foot length, including the weight of the girder. Find the principal dimensions.

Taking the depth as $\frac{1}{12}$th of the span gives a depth of 2·5 feet. The breadths of the flanges may be $\frac{1}{35}$th of the span, giving 10·5 inches for this dimension.

The total load will be 60 tons. The maximum bending moment will be

$$M_{max} = \frac{WL}{8} = \frac{60 \times 30}{8} = 225 \text{ ton-feet.}$$

Taking working stresses of 7 tons per square inch pull and 6 tons per square inch push, the sectional areas of the flanges at the centre of the span may be found. Let these be A_t and A_c square inches for the bottom and top flanges respectively. The moment of resistance of the section to bending will be $7A_t \times$ the depth of the girder, or $6A_c \times$ the depth, according as the bottom or the top flange is considered. Equating these to the bending moment at the centre of the span gives

$$7A_t \times 2\tfrac{1}{2} = 225,$$

$$A_t = \frac{225}{17\cdot5} = \underline{12\cdot85} \text{ square inches.}$$

$$6A_c \times 2\tfrac{1}{2} = 225,$$

$$A_c = \frac{225}{15} = \underline{15} \text{ square inches.}$$

It will be noted, from inspection of the section given in Fig. 231, that two rivet holes occur in each flange. In the case of the flange under push, it may be assumed that the rivets fill the holes perfectly and that no compensation is necessary. In the case of the flange under pull the sectional area of the two rivet holes must be deducted from the total sectional area of the flange plates. The rivets in the present example may be taken as $\frac{3}{4}''$ in diameter; it is not customary to exceed this dimension to any extent on account of the difficulty of closing larger rivets by hand, as has sometimes to be done during erection. The sectional area of the horizontal limbs of the angles used for securing the flange plates to the web plate may be included in the flange area. Angles $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$ are used in the present case.

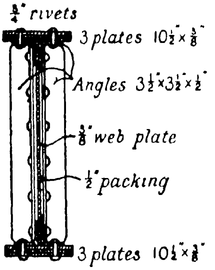

$\frac{3}{4}''$ rivets
3 plates $10\frac{1}{2}'' \times \frac{3}{8}''$
Angles $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$
$\frac{3}{8}''$ web plate
$\frac{1}{2}''$ packing
3 plates $10\frac{1}{2}'' \times \frac{3}{8}''$

FIG. 231.—Section of a plate girder

Taking the bottom flange first, in which the rivet hole allowance must be made, we have

$$\text{Net area of the horizontal limbs of two angles} = 2(3\tfrac{1}{2} - \tfrac{3}{4})\tfrac{1}{2}$$
$$= \underline{2{\cdot}75} \text{ square inches.}$$

Using plates $\frac{3}{8}''$ thick,

$$\text{Net area of one plate } 10{\cdot}5'' \times \tfrac{3}{8}'' = (10{\cdot}5 - 1{\cdot}5)\tfrac{3}{8}$$
$$= \underline{3{\cdot}375} \text{ square inches.}$$

If three such plates are used,

$$\text{Net area} = 3 \times 3{\cdot}375$$
$$= \underline{10{\cdot}125} \text{ square inches.}$$

Adding this to the area provided by the angles, we have

$$\text{Sectional area supplied in bottom flange} = 2{\cdot}75 + 10{\cdot}125$$
$$= \underline{12{\cdot}875} \text{ square inches.}$$

This is slightly in excess of the area actually required, viz. 12·85 square inches, and may thus be adopted with safety.

Considering now the top flange, which is under push, and using the same dimensions of angles and also the same thickness of plates, we have

$$\text{Area of the horizontal limbs of two angles} = 2 \times 3\tfrac{1}{2} \times \tfrac{1}{2}$$
$$= \underline{3{\cdot}5} \text{ square inches.}$$
$$\text{Area of one plate, } 10{\cdot}5'' \times \tfrac{3}{8}'' = \underline{3{\cdot}94} \text{ square inches.}$$
$$\text{Area of three plates} = \underline{11{\cdot}84} \text{ square inches.}$$
$$\text{Total flange area} = 3{\cdot}5 + 11{\cdot}84$$
$$= \underline{15{\cdot}34} \text{ square inches.}$$

The area actually required is 15 square inches; hence the assumed dimensions may be adopted.

The method of finding the lengths of the flange plates may be understood by reference to Fig. 232. The bending-moment diagram for the girder is drawn on a base AB, and is also redrawn inverted. The moment of resistance of the angle limbs is calculated, and also the moment of resistance of each plate separately, making allowance for rivet holes in the cases of those under pull. These are set off vertically from AB and horizontal lines ruled. The angles and the plates adjacent to the web must run the whole length of the girder. The other plates may stop at

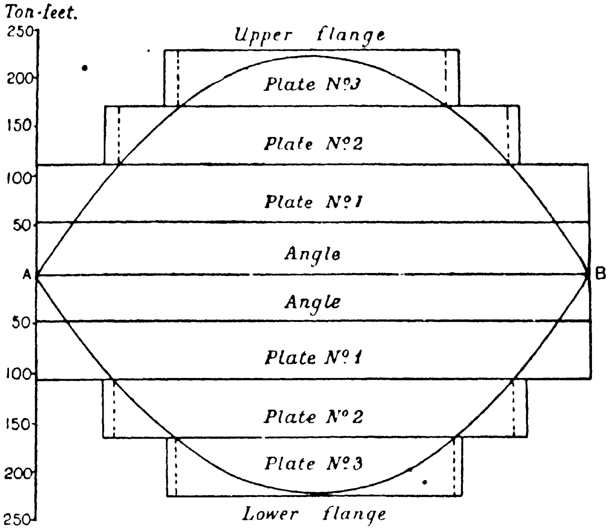


FIG. 232.—Construction for obtaining the flange-plate lengths in a plate girder.

the points where their moment-of-resistance lines cut the bending-moment diagram, but are made a little longer in order that the riveting at the ends of the plates may be carried out properly.

The thickness of the web plate may be found on the assumption that the shearing force is distributed uniformly over the section of the web. Assuming a shearing stress of 6 tons per square inch and taking a section close up to either support where the shearing force is a maximum and attains the value of 30 tons, the area required will be

$$\text{Sectional area of web} = \frac{30}{6} = 5 \text{ square inches.}$$

For a plate 30 inches deep this would give a thickness of $\frac{5}{30} = \frac{1}{6}$ inch. To guard against the effects of rusting, no plate should be less than $\frac{3}{8}$ inch thick ; further, buckling has to be considered ; hence the web may be taken as $\frac{3}{8}$ inch thick

The stiffeners should have a pitch not exceeding the depth of the girder, and the pitch may be halved near the supports.

To find the pitch of the rivets connecting the flanges to the web, taking a section near the end of the girder, the shearing force is 30 tons, and as the girder is 2·5 feet deep this will be equivalent to an average shearing force of $30 \div 2{\cdot}5 = 12$ tons per foot Now the shearing force per foot of vertical section must be equal to the shearing force per foot of horizontal section (p. 126); hence the resistance which must be provided by the rivets will be 12 tons per horizontal foot.

Taking rivets ¾ inch in diameter, a shearing stress of 6 tons per square inch and a bearing stress of 10 tons per square inch, we have

$$\text{Bearing resistance of a } \tfrac{3}{4} \text{ inch rivet in a } \tfrac{3}{8} \text{ inch plate} = \tfrac{3}{4} \times \tfrac{3}{8} \times 10$$
$$= 2{\cdot}81 \text{ tons.}$$

$$\text{Shearing resistance, under double shear} = 1\tfrac{3}{4} \times \frac{\pi d^2}{4} \times 6$$
$$= 4{\cdot}64 \text{ tons.}$$

Hence the bearing resistance must be taken.

$$\text{Number of rivets per foot} = \frac{12}{2{\cdot}81} = 4;$$
$$\text{pitch} = 3 \text{ inches.}$$

As the shearing force diminishes for sections taken nearer to the centre of the span, the pitch may be increased towards the centre. It is, however, undesirable that the pitch should change too frequently. To find the section at which the pitch may be changed to 6 inches is equivalent to finding the section at which the shearing force is half the maximum, viz. 15 tons. This will occur evidently at quarter span; hence the middle 15 feet of the girder may have a rivet pitch of 6 inches

Parallel braced bridge girder. Fig. 233 shows in outline a bridge constructed of two Pratt girders A and B, one on each side of the bridge; the roadway is supported by cross girders C which are attached to the main girders at the lower panel points *a*, *b*, *d*, *f*, etc., and transmit the road loads W_1, W_2, W_3, etc., to the girders at these points. The main girders each consist of two parallel booms, connected by inclined web bracings and vertical bars. The forces in the various parts of the girders are found generally by calculation in the following manner.

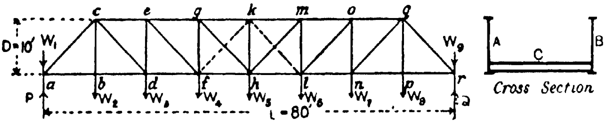


FIG. 233.—Bridge having two Pratt girders.

Forces in the top boom. Consider the bar *ce* (Fig. 233); if this bar were dropped out, the portion *acd* would rotate about *d*. Take moments about *d* of all the forces acting on *acd* which is shown separately in Fig. 234. T_{ce} is the force in *ce*. W_3 has zero moment.

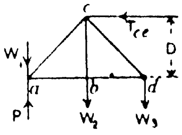


FIG. 234.

$$(T_{ce} \times D) + (W_1 \times ad) + (W_2 \times bd) = P \times ad,$$

or $$T_{ce} \times D = (P \times ad) - (W_1 \times ad) - (W_2 \times bd).$$

The right hand side of this expresses the bending moment at *d*; writing this as M_d, the equation gives

$$T_{ce} = \frac{M_d}{D}. \quad \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In the same way, $$T_{eg} = \frac{M_f}{D}.$$

And $$T_{gk} = \frac{M_h}{D}.$$

There is no necessity for calculating the forces in the bars on the other side of *k*, as, with symmetrical loading, it is evident that the forces will repeat themselves.

Forces in the bottom boom. It is evident that, as there are only horizontal and vertical forces at the joint *b* (Fig. 233), the force in *ab* will be equal to that in *bd*. If the bar *bd* be dropped out, then the portion *abc* will rotate about *c*. Taking moments about *c* of all the forces acting on *abc* (Fig. 235), we have

$$(T_{bd} \times D) + (W_1 \times ab) = P \times ab,$$
$$T_{bd} \times D = (P \times ab) - (W_1 \times ab)$$
$$= M_b;$$
$$\therefore\ T_{bd} = \frac{M_b}{D}. \quad \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

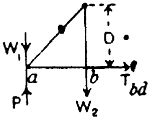


FIG. 235.

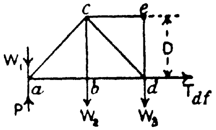


FIG. 236.

If *df* be dropped out (Fig. 233), *adec* will rotate about *e* (Fig. 236). Hence,

$$(T_{df} \times D) + (W_1 \times ad) + (W_2 \times bd) = P \times ad,$$
$$T_{df} \times D = (P \times ad) - (W_1 \times ad) - (W_2 \times bd)$$
$$= M_d;$$

$$\therefore\ T_{df} = \frac{M_d}{D}. \qquad (3)$$

In the same way, $$T_{fh} = \frac{M_f}{D}.$$

Forces in the inclined braces. Considering the bar ac (Fig. 233), evidently the horizontal component of the force in it will be balanced by the force in abd. Let θ be the angle of inclination of the brace to the horizontal (Fig. 237). Then

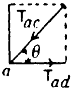


FIG. 237.

$$T_{ad} = T_{ac}\cos\theta,$$

or $$T_{ac} = T_{ad}\sec\theta. \qquad (4)$$

In the same way, the horizontal component of the force in cd is balanced by the forces in bd and df (Fig. 238). Hence,

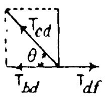


FIG. 238.

$$T_{df} - T_{bd} = T_{cd}\cos\theta,$$

or $$T_{cd} = (T_{df} - T_{bd})\sec\theta \qquad (5)$$

$$= \left(\frac{M_d - M_b}{D}\right)\sec\theta. \qquad (6)$$

In the same manner, $$T_{ef} = (T_{fh} - T_{df})\sec\theta$$

$$= \left(\frac{M_f - M_d}{D}\right)\sec\theta.$$

The force in gh requires special treatment. If the forces in gh and hm be both resolved vertically, the sum will be equal to W_5 (Fig. 239). Hence,

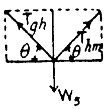


FIG. 239.

$$2T_{gh}\sin\theta = W_5,$$

$$T_{gh} = \frac{W_5}{2}.\operatorname{cosec}\theta. \qquad (7)$$

Forces in the vertical bars. The only force possible in bc is the load W_2 applied at its lower end (Fig. 233). There can be no force in hk, as there is no load at its upper end. Consider the bar de; the force in this bar is balanced by the vertical component of the force in the inclined brace ef which is connected to its top. Hence (Fig. 240),

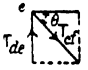


FIG. 240.

$$T_{de} = T_{ef}\sin\theta. \qquad (8)$$

In the same way, $$T_{fg} = T_{gh}\sin\theta.$$

The forces in the various members due to the dead load may be found also by graphical methods. Fig. 241 shows the force diagram for the girder under discussion.

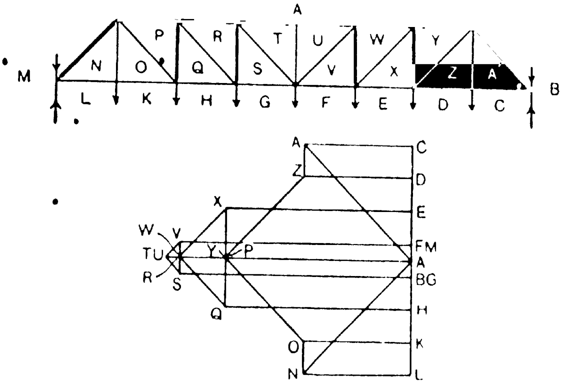


Fig. 241. Graphical solution of a Pratt girder carrying a dead load.

Live load forces. Suppose that a uniform live load, of length sufficient to cover the entire span, may run from either end on to the girder shown in Fig 233. It is evident that maximum bending moment will occur at all sections when the span is covered wholly by the live load, hence maximum forces will then occur in all the members of both top and bottom booms. If both live and dead loads are uniform, producing a ratio of live to dead load per foot length of girder equal to n, then the bending moments at any section, produced by these loads, will also have the ratio n, and the force in any boom member due to the live load will be n times the force in the same member due to the dead load.

In finding the maximum live-load forces in the inclined bars of the web, it may be taken that the shearing force in any panel is balanced by the vertical component of the force in the inclined bar belonging to that panel. Maximum force in any inclined bar will therefore occur when maximum shearing force exists in the panel to which the bar belongs. The following simple practical rule gives results sufficiently accurate. Assume that maximum pull in the inclined bars *cd*, *ef*, *gh* (Fig 233) occurs when each panel point situated on the right of the bar is carrying a load W, W being the live load per panel, also that the maximum push in the inclined bars *hm*, *lo*, *nq*, occurs when each panel point situated on the right of the bar is carrying a load equal to W. Under these conditions, the shearing force in the panel will be equal to the left-hand reaction P, P being calculated from the loads applied to the selected panel points. The force in the bar may then be found from the product P cosec θ. The maximum forces in the end bars *ac* and *qr* (Fig. 233) may be found by resolving vertically and horizontally the forces at *a* and *r* when the span is wholly covered by the live load.

It will be noted that corresponding bars on each side of the middle of the span, such as *cd* and *nq*, undergo reversal from pull to push, owing to the condition that the live load may run on to the girder from either end of the bridge. In general it will be found, when the dead-load forces are combined with the live-load forces, that the inclined bars near the ends of the girder have forces fluctuating between maximum and minimum pulls, and that a few only near the middle of the span undergo actual reversal from pull to push. It is customary to design the inclined bars in such girders to withstand pull only, and to counterbrace those panels in which the inclined bars suffer reversal from pull to push as shown by the results of the calculations indicated above. Counterbracing is shown by dotted lines in the two centre panels of the girder shown in Fig 233. It is assumed that the counterbraces *fk* and *kl* take as pulls the forces which would otherwise have to be carried as pushes by *gh* and *hm*.

Having found the maximum forces which may occur in the inclined bars due to the live load, the forces in the verticals may be found by considering the upper panel points *c*, *e*, *g*, etc. (Fig 233). The force in any vertical bar will be equal to the vertical component of the force in the inclined bar which is connected to the same upper panel point.

Bridge girder of varying depth. The principles underlying the solution of a bridge girder of varying depth may be understood by

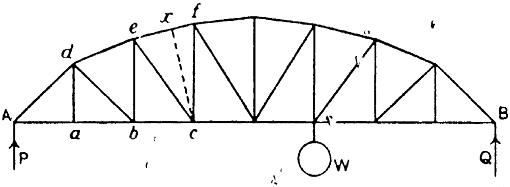


FIG. 242.—Bridge girder of varying depth.

reference to Fig. 242. A single load W is alone considered, and P and Q are calculated first.

To find the force in the member *bc* belonging to the bottom boom, it will be noticed that, if the bar be removed, *Abed* will rotate about *e*. Taking moments about *e*, we have

$$\text{Force in } bc \times be = P \times Ab = M_b;$$

$$\therefore \text{ force in } bc = \frac{M_b}{be}. \qquad (1)$$

To find the force in the member *ef* of the top boom, the rotation point would be *c* if the bar were dropped out. Taking moments about *c*, first drawing *cx* perpendicular to *ef*, we have

$$\text{Force in } ef \times cx = P \times Ac = M_c;$$

$$\therefore \text{ force in } ef = \frac{M_c}{cx}. \qquad (2)$$

To find the force in *ec*, reference is made to Fig 243, showing *Abed* together with all the forces acting on it. T_1, T_2 and T are the forces in *ef*, *bc* and *ec* respectively. T_1 and T_2 intersect, when produced, at *z*, and hence have no moment about *z*. Draw *zm* perpendicular to the line of T and take moments about *z*.

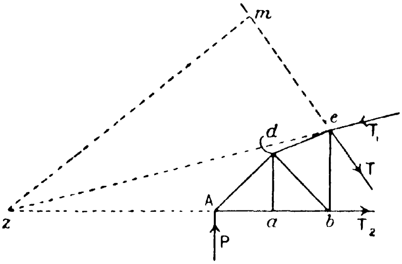


FIG. 243.—Construction for finding the force in a diagonal brace.

$$P \times Az = T \times zm,$$

$$T = P \times \frac{Az}{zm} \qquad (3)$$

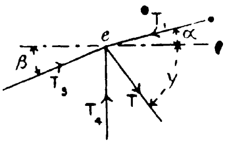


FIG. 244.—Construction for finding the force in a vertical member.

To find the force in *be*, resolve horizontally and vertically all the forces acting at the top joint *e* (Fig. 244). For balance of the vertical components, we have

$$T \sin\gamma + T_1 \sin\alpha = T_4 + T_3 \sin\beta;$$

$$\therefore \ T_4 = T \sin\gamma + T_1 \sin\alpha - T_3 \sin\beta. \qquad \ldots(4)$$

Double Warren girder. As an example of another method of solution, consider the double Warren girder shown in Fig. 245 (*a*). The girder may be taken as made up of two component girders shown separately in Figs. 245 (*b*) and (*c*), each carrying the loads which hang, in the complete girder, from panel points belonging to the component girder. Each component girder should be solved separately. The force in any member of the complete girder may then be found by adding algebraically the forces in the corresponding bars of the component girders.

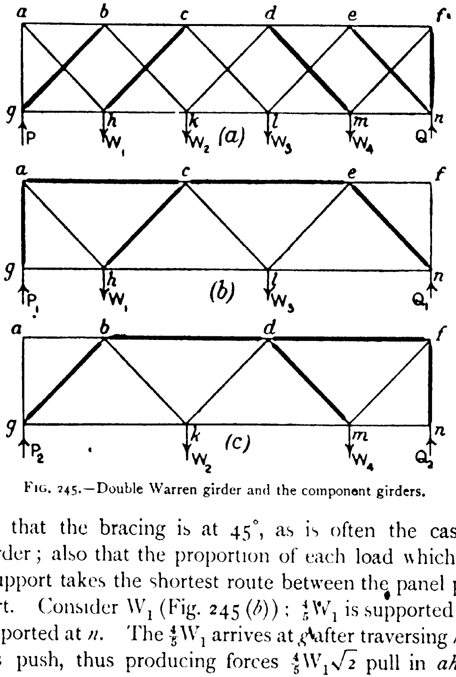


FIG. 245.—Double Warren girder and the component girders.

Assume that the bracing is at 45°, as is often the case in this type of girder; also that the proportion of each load which is borne by each support takes the shortest route between the panel point and the support. Consider W_1 (Fig. 245 (*b*)); $\frac{4}{5}W_1$ is supported at *g*, and $\frac{1}{5}W_1$ is supported at *n*. The $\frac{4}{5}W_1$ arrives at *g* after traversing *ha* as pull and *ag* as push, thus producing forces $\frac{4}{5}W_1\sqrt{2}$ pull in *ah* and W_1 push in *ag*. The $\frac{1}{5}W_1$ arrives at *n* after traversing *hc* as pull, *cl* as push, *le* as pull and *en* as push, and produces forces equal to $\frac{1}{5}W_1\sqrt{2}$ in each of these bars.

In the same manner, $\frac{3}{5}W_3$ arrives at *n* by producing $\frac{3}{5}W_3\sqrt{2}$ pull in *le* and $\frac{3}{5}W_3\sqrt{2}$ push in *en*; also $\frac{2}{5}W_3$ arrives at *g* by producing $\frac{2}{5}W_3\sqrt{2}$ pull in *lc*, $\frac{2}{5}W_3\sqrt{2}$ push in *ck*, $\frac{2}{5}W_3\sqrt{2}$ pull in *ha* and $\frac{2}{5}W_3$ push in *ag*.

The total forces in these bars in Fig. 245 (*b*) may be found now by adding algebraically the results calculated for each. The forces in the boom members are best found by calculation from the bending moments in the manner described on p. 213 It will be noted that there are no forces in *gh*, *cf* and *fn*.

The solution of the other component girder (Fig 245 (*c*)) is obtained in a similar manner. The force in any member such as *bc* in Fig. 245 (*a*) will be found by adding algebraically the forces in *ac* (Fig. 245 (*b*)) and *bd* (Fig. 245 (*c*)).

In girders of the double Warren type containing a large number of panels and uniformly loaded, the assumption may be made that the inclined bars in any panel share equally the shearing force in that panel. This assumption should not be made if the number of panels is small, as it then leads to absurd results.

If vertical bars *bh*, *ck*, etc., be added to the girder shown in Fig. 245 (*a*), it may be assumed that each vertical bar transfers one half the load applied at the lower panel point to the upper panel point, and the solution may then be obtained in the same manner as before, with the vertical bars left out.

Reinforced concrete beams. In Fig. 246 (*a*) is shown the section of a concrete beam having steel reinforcement bars near the bottom edge.

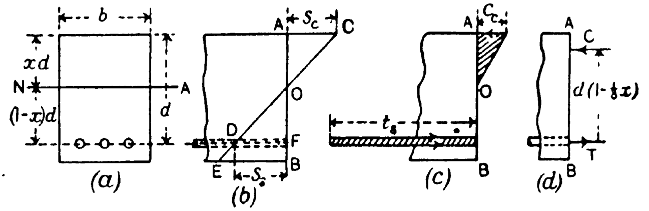


FIG. 246.—Reinforced concrete beam.

In making calculations regarding the strength of beams of this type, it may be assumed, as has been done for metal beams, that there is pure bending, that there is no resultant pull or push along the length of the beam and that cross sections which were plane in the unloaded beam remain plane when the beam is loaded. It follows from the last assumption, that the strains of longitudinal filaments will be proportional to the distance from the neutral layer (p. 143). Hence the strains all over any section AB (Fig. 246 (*b*)) may be represented in the side elevation of the beam by a sloping straight line CE which passes through the neutral axis at O, giving the two strain diagrams AOC and BOE, the horizontal breadths of which show the strain at any point.

Let s_c be the maximum strain in the concrete under compression, represented by AC (Fig. 246 (*b*)), and let s_s, represented by DF, be the strain in the steel. Let d be the depth of the beam measured from the top to the centre of the reinforcement bars, and let xd be the distance of the neutral axis from the top. Then

$$\frac{s_c}{s_s} = \frac{xd}{(1-x)d} = \frac{x}{1-x}. \qquad \text{.....} \quad \text{.................}(1)$$

Let c_c be the push stress in the concrete corresponding to the strain s_c and connected with it by

$$\text{E}_c = \frac{c_c}{s_c},$$

where E_c is Young's modulus for the concrete. Also let t_s be the pull stress in the steel corresponding to the strain s_s, the connection being

$$\text{E}_s = \frac{t_s}{s_s},$$

where E_s is Young's modulus for the steel. Then

$$\frac{\text{E}_s}{\text{E}_c} = \frac{t_s}{s_s} \cdot \frac{s_c}{c_c} = \frac{t_s}{c_c} \cdot \frac{s_c}{s_s}$$

$$= \frac{t_s}{c_c} \cdot \frac{x}{(1-x)}, \text{ (from 1).} \qquad \text{..............}(2)$$

The ratio of $\frac{\text{E}_s}{\text{E}_c}$, denoted by m, is rather variable owing to the nature of concrete; the average value of 15 is taken in practice; hence the above result may be written

$$m = \frac{t_s}{c_c} \cdot \frac{x}{(1-x)} = 15. \qquad \text{.................} \quad \text{...}(3)$$

It is customary to allow safe stresses of 600 lb. per square inch push in the concrete and 16,000 lb. per square inch pull in the steel. Suppose that the section is so proportioned as to secure that these values occur simultaneously on a certain load being applied. Then, from (3),

$$\frac{16{,}000}{600} \cdot \frac{x}{1-x} = 15,$$

whence

$$x = \tfrac{9}{25} = 0{\cdot}36. \qquad \text{.....................}(4)$$

A section so designed is referred to generally as an **economic section.**

In estimating the strength of the section to resist bending, it is usual to disregard the stresses in that portion of the concrete lying below the neutral axis, and hence under pull stress. It follows that the stress diagram for the section will resemble that shown shaded in Fig. 246 (*c*), in which push stress on the concrete is proportional to the distance from the neutral axis, c_c being the maximum value, and the stress t_s on the steel is assumed to be distributed uniformly over the steel. These stresses will give rise to equal resultant forces C and T on the concrete and steel respectively (Fig. 246 (*d*)), equal because there is no resultant force along the length of the beam.

Let p be the ratio of the area of the steel to the rectangular area bd (Fig. 246 (*a*)), and let A_s be the total sectional area of the steel bars in square inches. Then

$$A_s = pbd, \qquad (5)$$

and

$$T = t_s pbd. \qquad (6)$$

Also, Area of the concrete under push $= bxd$ (Fig. 246 (*a*)).

Average push stress in the concrete $= \frac{1}{2}c_c$.

$$\text{Total push in the concrete} = C = \tfrac{1}{2}c_c bxd. \quad \ldots\ldots\ldots \quad (7)$$

Also

$$T = C;$$

$$\therefore\ t_s pbd = \tfrac{1}{2}c_c bxd,$$

or

$$t_s p = \tfrac{1}{2}c_c x,$$

$$p = \frac{1}{2}\frac{c_c}{t_s}x \quad \ldots\ldots\ldots\ldots \quad (8)$$

$$= \frac{1}{2}\frac{x^2}{m(1-x)} \text{ (from (3))}. \quad \ldots\ldots \quad (9)$$

If the beam is of the economic section, then x is 0·36 from (4), and (9) becomes

$$p = 0{\cdot}00675$$

$$= 0{\cdot}675 \text{ per cent.} \quad \ldots \quad (10)$$

To obtain the moment of resistance to bending, we must calculate the moment of the couple formed by T and C. C acts at a distance $\frac{1}{3}xd$ from the top; hence the distance between C and T is

$$(d - \tfrac{1}{3}xd) = d(1 - \tfrac{1}{3}x).$$

Hence, $\qquad \text{Moment of resistance} = Cd(1 - \tfrac{1}{3}x), \quad \ldots \quad (11)$

or $\qquad$ „ $\qquad$ „ $\qquad = Td(1 - \tfrac{1}{3}x). \quad \ldots \quad (12)$

From (7) and (11), we have

$$\text{Moment of resistance} = \tfrac{1}{2}c_c bxd^2(1 - \tfrac{1}{3}x), \quad \ldots\ldots \quad (13)$$

or, from (6) and (12),

$$\text{Moment of resistance} = t_s pbd^2(1 - \tfrac{1}{3}x). \quad \ldots\ldots\ldots \quad (14)$$

Reinforced concrete **T** beams are much used. There are two cases, one in which the neutral axis falls below the slab (Fig. 247 (*a*)) and the other in which the neutral axis falls within the slab (Fig. 248 (*a*)). As the concrete under pull is neglected, the stress diagram for the latter (Fig. 248 (*b*)) is identical with that for a beam of rectangular section (Fig. 246 (*c*)); hence all the results already found apply to this case. In the former case (Fig. 247 (*a*)), it is customary to disregard the shaded area, representing a small portion of the concrete under push. The stress diagram will then take the form shown in Fig. 247 (*b*), and the equations become somewhat altered.

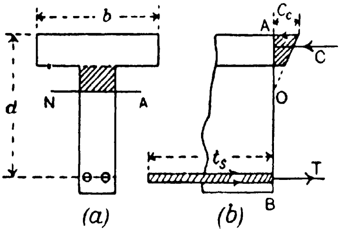


FIG. 247.—Reinforced concrete T beam, NA below the slab.

It should be noted that reinforced concrete buildings are practically monolithic; columns, beams and floors are so constructed as to form one piece. Hence all such beams must be regarded as fixed at the ends. It has been shown already (p. 177) that in such beams the bending moment reverses in sense near the walls; hence the top sides of the beams near the walls will be under pull, and some of the reinforcement bars should be brought diagonally upwards and run near the top of the section over the supports.

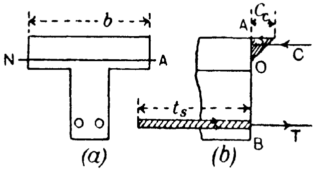


FIG. 248.—Reinforced concrete T beam, NA within the slab.

EXAMPLE 1. A reinforced concrete beam is 9 inches wide, and is to have a moment of resistance of 200,000 lb.-inches. The stresses of 600 lb. per square inch on the concrete and 16,000 lb. per square inch on the steel are to be attained simultaneously. Ratio of elastic moduli = 15. Find the depth of the beam and also the sectional area of steel required and the position of the neutral axis.

From (10), $p = 0{\cdot}675$ per cent.

From (4), $x = 0{\cdot}36$.

From (13), $M = \frac{1}{2} c_c b x d^2 (1 - \frac{1}{3} x)$,

or
$$d=\sqrt{\frac{2M}{xbc_c(1-\frac{1}{3}x)}}$$
$$=\sqrt{\frac{2\times200{,}000}{0\cdot36\times9\times600(1-0\cdot12)}}$$
$$=\underline{15\cdot3}\text{ inches.}$$

Distance from the top to the neutral axis
$$=xd=0\cdot36\times15\cdot3$$
$$=\underline{5\cdot51}\text{ inches.}$$

Sectional area of steel $=A_s=pbd$
$$=0\cdot00675\times9\times15\cdot3$$
$$=\underline{0\cdot93}\text{ square inch.}$$

EXAMPLE 2. A reinforced concrete beam 9 inches wide by 18 inches deep has three steel reinforcement bars, each 0·75 inch in diameter. Find the position of the neutral axis and the moment of resistance. Neither of the stresses of 600 lb. per square inch for the concrete and 16,000 lb. per square inch for the steel may be exceeded. Take the ratio of E_s to $E_c=15$.

Sectional area of steel $=A_s=3\times\frac{\pi d^2}{4}=\frac{3\times22\times9}{4\times7\times16}$
$$=1\cdot33\text{ square inches.}$$

From (3),
$$\frac{t_s}{c_c}=\frac{15(1-x)}{x}.\qquad\text{.................................(1)}$$

Also,
$$T=C;$$
$$\therefore\ 1\cdot33\times t_s=\tfrac{1}{2}c_cbdx=\tfrac{1}{2}\times9\times18\times c_cx;$$
$$\frac{t_s}{c_c}=\frac{81x}{1\cdot33}.\qquad\text{.......................................(2)}$$

Equating (1) and (2) above, we have
$$\frac{15(1-x)}{x}=\frac{81x}{1\cdot33},$$
$$(15-15x)1\cdot33=81x^2,$$
$$19\cdot95-19\cdot95x=81x^2,$$
$$81x^2+19\cdot95x-19\cdot95=0;$$

whence
$$x=0\cdot387.$$

Distance of the neutral axis from the top $=xd=0\cdot387\times18$
$$=\underline{6\cdot96}\text{ inches.}$$

Again, from (1),
$$\frac{t_s}{c_c}=\frac{15(1-x)}{x}=\frac{15\times0\cdot613}{0\cdot387},$$

or
$$t_s=23\cdot8c_c.$$

Suppose t_s be taken as 16,000. Then
$$c_c=\frac{16{,}000}{23\cdot8}=673\text{ lb. per square inch,}$$

a value inadmissible by the data. Take, therefore, c_c as 600 lb. per square inch, giving

$$t_s = 600 \times 23{\cdot}8$$
$$= \underline{14{,}280} \text{ lb. per square inch,}$$

and $$c_c = \underline{600} \text{ lb. per square inch.}$$

From (14),

$$\text{Moment of resistance} = t_s p b d^2(1 - \tfrac{1}{3}x)$$
$$= 14{,}280 A_s d(1 - \tfrac{1}{3}x)$$
$$= 14{,}280 \times 1{\cdot}33 \times 18\left(1 - \frac{0{\cdot}387}{3}\right)$$
$$= \underline{298{,}000} \text{ lb.-inches.}$$

EXAMPLES ON CHAPTER IX.

1. A steel bar is 20 feet long and has a sectional area of 4 square inches. Find the work done while a pull of 24 tons is applied gradually. Take $E = 13{,}500$ tons per square inch. Find also the energy stored in a cubic inch of the bar.

2. Suppose, in Question 1, that the load is applied suddenly, and calculate the maximum stress produced. What will be the momentary extension of the bar?

3. It is found that a steady load of 400 lb. resting at the middle of a beam produces a deflection at the centre of 0·01 inch. What central deflection would be produced by a load of 100 lb. dropped on to the middle of the beam from a height of 16 inches?

4. A certain steel bar in a girder carries a constant pull of 20 tons owing to the dead load. The live load produces in the same bar forces which range from 60 tons pull to 10 tons push. Find the working stress and the sectional area of the bar. Take ultimate tensile strength = 30 tons per square inch.

5. A single load of 10 tons rolls along a girder of 30 feet span. Draw curves showing the maximum bending moments and shearing forces at every section. State the scales.

6. Answer Question 5 for a uniformly distributed travelling load of 1·5 tons per foot length which may cover the whole span.

7. Supposing that the girder in Question 5 is uniform in section and weighs 8 tons. Draw the diagrams of M and S for the dead load. Then combine these diagrams with those already drawn for the single rolling load in order to show the effects of combined live and dead loads.

8. A girder of 40 feet span is traversed by three concentrated loads of 6 tons each at 7 feet centres, followed at an interval of 6 feet by a uniformly distributed load of 0·5 ton per foot. Find graphically the maximum bending moments at sections of the girder taken at 5 feet intervals. The load may run on to the girder from either end.

9. A continuous beam of length 50 feet rests on four supports on the same level. The left-hand span is 20 feet and the others are 15 feet each. The left-hand span carries a uniform load of 2 tons per foot, the other spans carry uniform loads of 1 ton per foot. Find the bending moments at the supports.

10. In Question 9, find the reactions of the supports.

11. In Question 9, draw diagrams of bending moments and shearing forces for the complete beam. State the scales.

12. A plate girder 24 feet span, 2 feet deep, flanges 10 inches wide, carries a uniformly distributed load of 45 tons. The angle sections are 3·5 × 3·5 × 0·5 in inches. Take stresses as follows: pull, 7 tons per square inch; push, 6 tons per square inch; shearing, 6 tons per square inch; bearing, 10 tons per square inch. Find the sectional area of each flange; state the number and thickness of plates required for each flange at the middle of the span. What thickness of web plate would be suitable? If the rivets are 0·75 inch in diameter, what will be the pitch of those near the ends of the girder?

13. In Question 12, find the length of each plate in (*a*) the top flange, (*b*) the bottom flange.

14. A Pratt girder (Fig. 241) 48 feet span has 6 equal bays of 8 feet each. The bracing bars make angles of 45° with the horizontal. There is a uniform dead load of 1 ton per foot length. Find the forces in the horizontal top and bottom bars of the two central bays; also those in the two inclined bars nearest to one support. Find also the force in the vertical bar second from one support.

15. In Question 14 a uniform live load of 1·25 tons per foot travels along the girder. Find the maximum forces it will produce in the same bars. The load is long enough to cover the whole girder.

16. A model reinforced concrete beam 3·5 inches wide by 4·25 inches deep from the top to the centre of the reinforcement has to be made so that stresses of 600 and 16,000 lb. per square inch will occur in the concrete and in the steel respectively. Taking the ratio of the elastic moduli as 15, find the percentage of reinforcement required, the sectional area of the steel, the position of the neutral axis and the moment of resistance of the section.

17. A reinforced concrete beam of rectangular section 12 inches wide by 18 inches deep has three steel reinforcement bars each 1·25 inches in diameter. Find the position of the neutral axis and the moment of resistance. Stresses of 600 and 16,000 lb. per square inch respectively for the concrete and steel must not be exceeded. Take the ratio of the elastic moduli as 15.

18. Experiments upon some wrought-iron bars showed that a permanent set was taken when the bars were strained to a degree greater than that produced by a stress of 20,000 lb. per square inch, but not when strained to a less degree. At that point the average strain was 0·0006 foot per foot of length; what was the resilience of this quality of iron in foot-pounds per square inch section per foot of length? (I.C.E.)

19. An iron bar 10 feet long having $E = 14,000$ tons per square inch and a limit of elasticity $= 14$ tons per square inch is subjected to shocks of a total value of 224 foot-pounds. The bar is not to have any permanent set produced in it, this being guaranteed by the adoption of a factor of safety of 2. Find the required sectional area of the bar. (I.C.E.)

20. A vertical steel rod, 10 feet long, the cross section of which is 1 square inch, is fixed at its upper end and has a collar at its lower end. An annular weight of 300 lb. is allowed to fall through a height of 3 inches upon this collar. Determine the maximum intensity of stress produced in the steel rod if Young's modulus is 12,500 tons per square inch. (B.E.)

21. Two bars, A and B, of circular section and the same material, are each 16 inches long. A is 1 inch in diameter for 4 inches of its length and 2 inches in diameter for the remainder; B is 1 inch in diameter for 12 inches of its length and 2 inches in diameter for the remainder. A receives an axial blow which produces a maximum stress in it of 10 tons per square inch. Calculate the maximum stress produced by the same blow on B. How much more energy can B absorb in this way than A without exceeding a given stress within the elastic limit of the material? (L.U.)

22. A double Warren girder (Fig. 245) is 50 feet span and 10 feet deep and has five equal bays of 10 feet each. It is supported at the ends and carries a load of 12 tons at each of the four lower panel points (48 tons in all). Find the forces in the members. State the assumptions made. (L.U.)

CHAPTER X.

COLUMNS. ARCHES

Ties and struts. Those portions of a structure which are intended to be under pull are called **ties**; parts under push are called **struts**, or **columns**. Columns are usually vertical pieces intended to carry weights. There is an essential difference which modifies greatly the method of calculating the strengths of ties and struts; a loaded tie exhibits no tendency to bend if it is straight originally, and will tend to become straight if originally curved. A strut, if originally curved, will have its curvature increased by application of the load, and, if straight at first, may very easily be under such conditions of loading as will produce bending; want of uniformity in the elastic properties of the material may produce a similar effect.

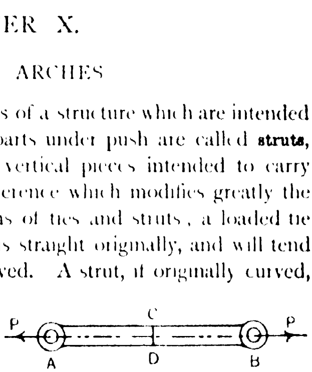


FIG. 249. A straight tie.

A straight tie AB is shown in Fig. 249, loaded with pulls P, P, applied in the axis of the bar. It is evident there is no tendency to bend the tie, and any cross section CD, at 90° to the axis of the bar, will have a uniformly distributed pull stress. A bent tie bar AB is shown in Fig. 250 (*a*). The nature of the stresses on CD may be understood by considering the equilibrium of one half of the bar (Fig. 250 (*b*)). It will be observed that there is a bending couple of clockwise moment Pd; this is balanced by the moment of resistance at the section CD, the latter being represented by the forces Q, Q. It is apparent that the bending couple Pd is endeavouring to straighten the bar.

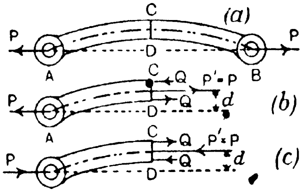


FIG. 250.—Bent ties and struts.

Had a strut of similar shape been chosen, the forces acting on one half of it would be as shown in Fig. 250 (*c*). Here the couple P*d* is anti-clockwise, and tends to produce further bending.

In each of these cases there will be two kinds of stresses on the section CD: (*a*) a stress of uniform distribution due to the axial force P'; (*b*) a stress due to the bending couple, varying from a maximum push stress at one edge to a maximum pull stress at the opposite edge. An initially straight strut which has been allowed to bend under the load will have a similar stress distribution. It may be taken that the effects of bending may be disregarded in axially loaded straight ties, but must be taken account of in all struts.

Euler's formula for long columns. This formula may be deduced by considering the bending of a long flexible column of uniform cross section and carrying a load applied axially. If such a column is perfectly straight to begin with, and there are no inequalities in the elastic properties of the material, the application of an axial load will not tend to bend the column. On increasing the load, a certain critical load is reached, the magnitude of which depends on the method of fixing the ends of the column; under this load the material of the column becomes elastically unstable. This condition is evidenced by the column refusing to spring back if slightly deflected from the vertical, while it does so readily for loads lower than the critical load. The slightest increase in the load beyond the critical value will cause a small deflection imparted to the column to increase without limit, and the column collapses. It will be evident from what has been said regarding the conditions to be realised, that it is not possible to obtain a column of such ideal material and construction as will show perfect agreement under test with Euler's result. But the formula is of service in enabling other more practical formulae to be devised.

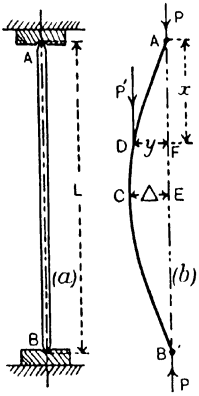


FIG. 251.—Euler's theory of long struts.

Considering a long column AB (Fig. 251 (*a*)) of uniform cross section and length L. Let both ends be rounded, or pivoted, in such a manner that, if bending does occur, the column will assume a curve resembling a bow (Fig. 251 (*b*)). The effect of a load P applied at A in producing stresses at any section D will be understood by shifting P from A to D as shown by P′; the section at D will evidently be under an axial load $P' = P$ producing uniform stress, together with a couple of moment Py. The couple gives a bending moment, and the effect of this alone is considered in the following; the stress at D caused by the axial load P′ is disregarded, as it is small compared with that produced by the bending moment in the case of a long column. The maximum bending moment will be found at the middle section C, at which the maximum value of y, viz. Δ (Fig. 251 (*b*)), occurs, and will be given by

$$\text{Maximum bending moment} = M_C = P\Delta. \quad . \quad . \quad (1)$$

Taking the equation for the curvature of a beam (p. 166), we have for the curvature at D,

$$\frac{1}{R_D} = \frac{M_D}{EI}$$

$$= \frac{P}{EI} y. \quad . \quad (2)$$

Since $\frac{P}{EI}$ will be constant for a given load on a given column, we may write

$$\frac{1}{R_D} \propto y.$$

It may be shown readily that a curve, plotted so that its ordinates y (Fig. 252) are the sines of the angles α represented by its

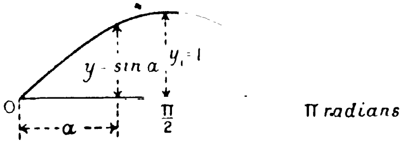


FIG. 252.—Curve of sines.

abscissae, possesses the same property, viz. the curvature at any point is directly proportional to the ordinate $y = \sin \alpha$. It may thus be inferred that the curve of the bent column is a curve of sines to some scale. The scales of x and y may be stated by taking the origin at A (Fig. 251 (*b*)), when $AB = L$ will represent π radians, $AE = \frac{1}{2}L$ will represent $\frac{1}{2}\pi$ radians; also $CE = \Delta$ will represent $\sin \frac{\pi}{2} = 1$, and y will represent the sine of an angle AF, which will have a value $\frac{x}{L}\pi$. Hence,

$$y : \Delta = \sin \frac{x}{L}\pi : \sin \frac{\pi}{2},$$

$$y = \Delta \sin \frac{\pi}{L} x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

It can be shown that, if a curve be given by an equation showing the relation of y and x, the curvature at any point may be obtained by finding the second differential coefficient, viz. $\frac{d^2y}{dx^2}$, provided that the curvature is not too great. Application of this process to equation (4) will lead to a result which may be equated to that of equation (2) above. Thus,

$$y = \Delta \sin \frac{\pi}{L} x,$$

$$\frac{dy}{dx} = \Delta \frac{\pi}{L} \cdot \cos \frac{\pi}{L} x,$$

$$\frac{d^2y}{dx^2} = -\Delta \frac{\pi^2}{L^2} \sin \frac{\pi}{L} x$$

$$= -\frac{\pi^2}{L^2} y. \quad \text{(from 4)} \quad \dots\dots\dots\dots\dots \quad (5)$$

This gives the curvature at D (Fig. 251 (*b*)), viz. $\frac{1}{R_D}$. The negative sign may be disregarded, as its only significance has reference to the position of the centre of curvature. Equating (2) and (5), we have

$$\frac{P}{EI} y = \frac{\pi^2}{L^2} y.$$

It is important to note that y cancels, giving

$$\frac{P}{EI} = \frac{\pi^2}{L^2},$$

$$P = \frac{\pi^2 EI}{L^2}. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6)$$

This is Euler's formula for a long column having both ends rounded. The meaning to be attached to the deflection y disappearing from the final result is that no deflection will occur until a certain load P given by (6) is applied. When the load attains this value, any small deflection will increase indefinitely with the consequent collapse of the column.

A more general way of writing Euler's formula is

$$P = \frac{\pi^2 EI}{l^2}, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (7)$$

where l is a function of the length L of the column. The value of l depends on the method of fixing the ends, a point which we now proceed to examine.

Effect of fixing the ends of columns. In the case of the column discussed in finding Euler's formula, the ends were taken as rounded and the column bent as a whole. There was no bending moment at the ends, and these may be looked upon as points of contraflexure (p. 180). Fixing the ends will produce a stiffer and consequently stronger column. This may be taken account of in the formula by writing, instead of L, the length of the column, the distance *l* between the points of contraflexure in the actual curve of the bent column. Some cases are noted below; reference is made to Fig. 253.

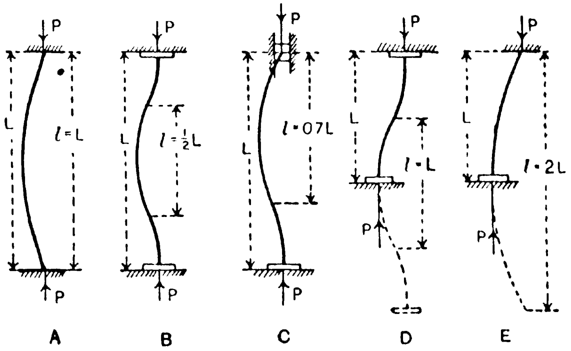


Fig. 253.—Various methods of end fixing in columns.

Case A. Both ends rounded. This is the case examined above; $l=L$.

Case B. Both ends fixed and so controlled that the forces P, P remain in the same vertical line. Here $l=\frac{1}{2}L$.

Case C. One end (the lower) fixed; the other end guided so that the forces P, P remain in the same vertical, but the column is otherwise free at this end to take up any direction. In this case $l=0{\cdot}7L$.

Case D. Both ends are fixed so that the directions at the ends of the curve of the column remain vertical, but one end is free to move horizontally relative to the other end, so that the forces P, P are not in the same vertical when the column bends. Only one point of contraflexure will occur in the column itself; the position of the second point may be seen by producing the curve of the column downwards (shown dotted in Fig. 253 D). In this case $l=L$.

CASE E. One end fixed, the other end perfectly free. In this case, the free end is a point of contraflexure. The second point may be obtained by producing the curve of the column downwards. Here $l = 2L$.

Using Euler's formula, it will be noticed that, as l has to be squared, the effect of fixing both ends of the column as in Case B will be to give the column four times the strength of the same column having both ends rounded.

Curve illustrating Euler's formula. Euler's formula may be modified by writing

$$I = Ak^2,$$

where A is the sectional area of the column and k is the least radius of gyration of the section, *i.e.* k is taken with reference to that axis containing the centre of the area of the section for which I has the minimum possible value. It is evident that the column will bend in a plane perpendicular to this axis. Two instances are given in Fig. 254 (*a*) and (*b*); in each of these OX is the axis perpendicular to the plane of bending, and k should be taken with respect to OX. Inserting this expression for I in equation (7), (p. 230), we have

$$P = \frac{\pi^2 EAk^2}{l^2} = \pi^2 EA\left(\frac{k}{l}\right)^2.$$

Let $l = nL$, where n is a coefficient depending on the method of fixing the ends. Then

$$P = \frac{\pi^2 EA}{n^2}\left(\frac{k}{L}\right)^2,$$

or

$$\frac{P}{A} = \frac{\pi^2 E}{n^2}\cdot\left(\frac{k}{L}\right)^2.$$

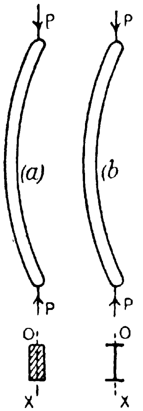


FIG. 254.—Plane of bending in columns.

The left-hand side of this expresses the collapsing load p per unit of sectional area. Hence,

$$p = \frac{\pi^2 E}{n^2}\left(\frac{k}{L}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This will be in tons per square inch, provided the following units are employed:

E = Young's modulus, in tons per square inch.

k = the least radius of gyration, in inch units.

L = the length of the column, in inches.

In columns of a given material and having a stated method of fixing the ends, the quantity $\frac{\pi^2 E}{n^2}$ will be constant, hence p may be calculated for different ratios of L to k, and a curve may be plotted from the results. Fig. 255 gives such a curve for mild steel struts having both ends hinged. In this case $n = 1$ and E has been taken as 13,500 tons per square inch.

It will be noticed from Fig. 255 that, for small ratios of L to k, the collapsing stress obtained is absurd. Only when the ratio is large is a reasonable value obtained. This leads to the conclusion that agreement of Euler's formula with practical results of tests should be looked for only in the case of struts which are very long as compared with the cross-sectional dimensions.

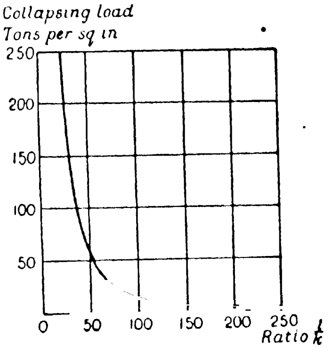


FIG. 255.—Euler's curve for mild steel struts, both ends hinged.

Ewing's composite formula. Sir J. A. Ewing has suggested a composite formula made up of the crushing strength of a very short block together with the elastic instability load of a very long column, both composed of the same material as the actual column.

Let f_c = the crushing strength of a short block, in tons per square inch.

P_1 = the crushing load, in tons, applied axially.

A = the area over which P_1 is distributed, in square inches.

Then $$P_1 = f_c A. \qquad (1)$$

Let P_2 = the elastic instability load of a long column, in tons; the column having the same sectional area A as the short block.

Then $$P_2 = \frac{\pi^2 EI}{l^2}. \qquad (2)$$

Combining these in accordance with Ewing's method gives the following formula for P, the collapsing load of the ordinary practical column when loaded axially:
$$P = \frac{f_c A}{1 + f_c A \frac{l^2}{\pi^2 EI}}. \qquad (3)$$

This formula has the advantage of being continuous, and does not give an absurd result for a column of any practical length. If l is very short, the second term in the denominator becomes very small, and may be disregarded. The formula then reduces to the expression (1) for the crushing load of a short block. If l is very long, the formula reduces to Euler's formula by neglecting the unimportant terms.

Rankine's formula for columns. This formula is the one in most frequent practical use. It is practically the same as Ewing's, although in a slightly altered form. Thus,

$$\mathrm{P} = \frac{f_c \mathrm{A}}{1 + f_c \mathrm{A} \dfrac{l^2}{\pi^2 \mathrm{EI}}}$$

$$= \frac{f_c \mathrm{A}}{1 + \dfrac{f_c \mathrm{A}}{\pi^2 \mathrm{E}} \cdot \dfrac{l^2}{\mathrm{I}}}.$$

Now, I may be written as $\mathrm{I} = \mathrm{A}k^2$,

where A is the sectional area and k is the least radius of gyration. Hence,

$$\mathrm{P} = \frac{f_c \mathrm{A}}{1 + \dfrac{f_c \mathrm{A}}{\pi^2 \mathrm{EA}} \cdot \dfrac{l^2}{k^2}}$$

$$= \frac{f_c \mathrm{A}}{1 + \dfrac{f_c}{\pi^2 \mathrm{E}} \cdot \dfrac{l^2}{k^2}}.$$

It is apparent that $\dfrac{f_c}{\pi^2 \mathrm{E}}$ will be constant for a given material, and may be written c, the value of which is to be determined by experiments on the collapsing strength of columns. Hence,

$$\mathrm{P} = \frac{f_c \mathrm{A}}{1 + c\dfrac{l^2}{k^2}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This is Rankine's form of the formula, and gives the total collapsing load on the column. The collapsing load per square unit of sectional area will be given by

$$p = \frac{\mathrm{P}}{\mathrm{A}} = \frac{f_c}{1 + c\dfrac{l^2}{k^2}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This will be in tons per square inch, provided f_c is in tons per square inch and l and k are in inch units. It is assumed that the column is of uniform section and is loaded axially. The least value of k should be chosen as in the case of the Euler formula, and the value of l is the same as in the cases given on p. 231 for various methods of fixing the ends. Values of f_c and c are given in the table below; a suitable factor of safety should be applied in order to obtain the safe load:

COEFFICIENTS IN RANKINE'S FORMULA.

Material	f_c for collapsing load		c.*
	Lb. per sq. inch.	Tons per sq. inch.	
Cast iron - - -	80,000	36	$\frac{1}{1600}$
Hard steel - - -	70,000	31·2	$\frac{1}{5000}$
Mild steel - - -	48,000	21·4	$\frac{1}{7500}$
Wrought iron - -	36,000	16	$\frac{1}{9000}$
Timber (varies greatly)	7,200	3·2	$\frac{1}{750}$

It will be evident on inspection of the Rankine formula that allowance is made both for direct crushing and for bending. Owing to the radius of gyration entering into the formula, due regard has been paid to the distribution of the material in the section, *i.e.* the shape as well as the area of the section has been taken into account.

It is useful to plot curves from equation (2) showing the collapsing load per square inch of sectional area for different ratios of L to k, varying the material and the method of fixing the ends. Such a curve for mild steel, both ends hinged, is shown in Fig. 256, the corresponding Euler curve being given on the same diagram.

Another useful set of curves is given in Fig. 257; here the safe loads per square inch of sectional area for mild-steel, wrought-iron and cast-iron columns have been plotted for different ratios of l/k. The factor of safety employed is 5. The curves indicate that mild steel may always carry a higher stress than wrought iron; also that, at the ratio of $l/k = 40$ approximately, the safe stresses on mild steel and cast iron are equal; hence equal columns of cast iron and mild steel having this ratio of l/k would carry equal safe loads. Wrought iron and cast iron have equal safe stresses at a ratio of l/k of 65 approximately. In designing a column to carry a given load, cast iron is the material calling for the smallest sectional area for ratios of L to

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* Note that l should be taken from the cases shown in Fig. 253. For values of k^2, see p. 151.

k under 40, and mild steel demands the smallest sectional area for ratios of L to k above 40. Wrought iron would require a smaller sectional area than cast iron for ratios of L to k over 65.

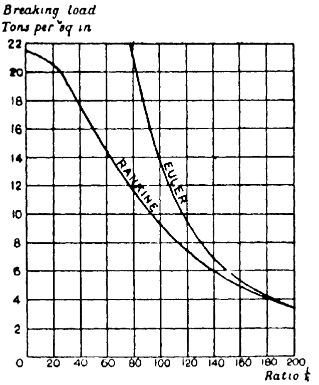


FIG. 256.—Rankine and Euler curves for mild steel columns, both ends hinged.

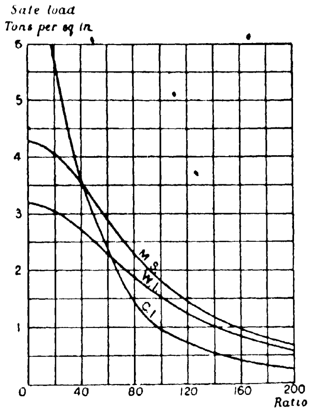


FIG. 257.—Rankine curves for columns of different materials, ends hinged.

Gordon's formula. The formula bearing Gordon's name, and formerly in common use, is

$$p = \frac{f_c}{1 + a\left(\frac{l}{d}\right)^2},$$

where f_c and a are experimental coefficients and d is the least transverse dimension of the section. The formula is objectionable, from the fact that no allowance is made for the distribution of the material over the section. For example, referring to Fig. 258, in which are shown an I section and a box section of equal areas, and also having equal over-all dimensions, Gordon's formula would have the same value of d for both sections, viz. B, and would give the same value of collapsing load for both. Rankine's formula would give fairer consideration to the box section, which is obviously the stiffer and stronger, from the fact that the radius of gyration of the box section with reference to

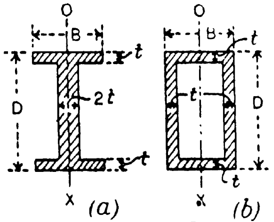


FIG. 258.

OX is greater than that for the **I** section, and hence would give a greater load for the box section.

Secondary flexure in columns. Professor Lilly has pointed out that columns constructed of thin plates are liable to fail by secondary flexure, *i.e.* the column may not fail by bending as a whole, but by the material buckling over a short length. Lilly has made many experiments in support of his views, and has proposed a formula in which account is taken of the ratio of the thickness of the plate to the radius of gyration. Further experimental work is required in order to settle the values of the experimental factors involved.

Recent tests made at the University of Illinois on built-up columns indicate that the stress distribution may be very erratic; especially in the neighbourhood of riveted joints. It is admitted that our knowledge of the strength of columns, struts and compression members generally is far from being complete. At present, most designers rely on the Rankine formula coupled with a liberal factor of safety.

Effect of a non-axial load. In Fig. 259 (*a*) is shown a column the axis of which is AB, *i.e.* AB passes through the centres of area of all horizontal sections of the column. A load P is applied at C at a distance a from the axis. P may be moved from C to A provided a couple of moment Pa is applied. We have now an axial load $P' = P$ together with a couple Pa which will give a uniform bending moment at all horizontal sections of the column. Let A be the area of the section, then P will produce a uniformly distributed push stress p_1 given by

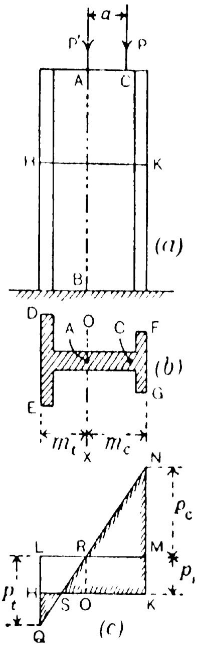


FIG. 259 – Column carrying a non-axial load

$$p_1 = \frac{P}{A} \quad \ldots\ldots\ldots\ldots \quad (1)$$

The bending moment Pa will give a stress distribution similiar to that of a beam under pure bending, which will vary from a pull stress p_t at the edge DE (Fig. 259 (*b*)), to a push stress p_c at the edge FG. Let m_t and m_c be the distance of these edges respectively from OX.

Then, using the equation (p. 146)

$$M = \frac{f}{m} I,$$

we have $$Pa = \frac{p_t}{m_t} I = \frac{p_c}{m_c} I.$$

Whence $$p_t = \frac{Pam_t}{I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and $$p_c = \frac{Pam_c}{I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The stress due to the bending moment will vary uniformly between these values, being zero at OX. The stress figure for the direct stress combined with the bending stress may be drawn as shown in Fig. 259 (*c*), where HKML shows the uniformly distributed stress p_1, and MNRQL shows the varying stress due to the bending moment. The resultant stress figure is shaded, and shows that the maximum push stress occurs at the edge FG (Fig. 259 (*b*)), and is given by

$$\text{Maximum push stress} = p_1 + p_c. \quad \ldots\ldots (4)$$

In the case shown there will be no stress at S (Fig. 259 (*c*)); the portion SK will be under push stress, and SH will be under pull stress, the maximum value of the latter occurring at the edge DE (Fig. 259 (*b*)) and given by

$$\text{Maximum pull stress} = p_t - p_1. \quad \ldots\ldots (5)$$

The presence of pull stress in a metal column is permissible, but is objectionable in a column of stone, brick, or other construction in which the jointing of the blocks of material is not considered to be trustworthy under pull. The extreme limit of stress distribution in such cases is taken usually to be zero stress at one edge and increasing gradually to a maximum push stress at the opposite edge.

Taking a rectangular section (Fig. 260 (*a*)) of dimensions *b* and *d*, the values of m_c and m_t will be equal; hence p_c and p_t will also be equal, and the stress figure (Fig. 260 (*b*)) shows that the condition of no pull stress is

$$p_c - p_1 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Also, $$p_1 = \frac{P}{A} = \frac{P}{bd}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and $$p_c = \frac{Pam_c}{I} = Pa\frac{d}{2} \div \frac{bd^3}{12}$$

$$= \frac{6Pa}{bd^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Hence,
$$\frac{6Pa}{bd^2}-\frac{P}{bd}=0,$$

or
$$a=\frac{d}{6}. \qquad \text{(9)}$$

It therefore follows that P may be applied at a distance not exceeding $\frac{1}{6}d$ from the centre of the section in a direction parallel to d. Similarly P may be applied at a distance not exceeding $\frac{1}{6}b$ in a direction parallel to b. We may thus state that P may be applied within the middle third of OX and OY (Fig. 260 (a)) without giving rise to pull stress.

In the same way it may be shown, for a column of solid circular cross section of radius r, that the load may be applied anywhere inside a circle of radius $0{\cdot}25r$, having its centre on the axis of the column, without the production of pull stress.

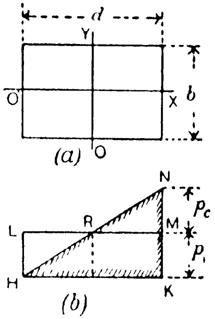


FIG. 260.—Rectangular section carrying a non axial load.

EXAMPLE 1. A wrought-iron stanchion of square section 2 inches × 2 inches is 8 feet high. Both ends are fixed. Find the safe axial load, using a factor of safety of 5.

Here
$$l=\frac{1}{2}L=48 \text{ inches.}$$
$$I=Ak^2=\frac{s^4}{12};$$
$$\therefore\ k^2=\frac{s^2}{12}=\frac{4}{12}=\frac{1}{3} \text{ inch units.}$$
$$f_c=16 \text{ tons per square inch.}$$
$$c=\frac{1}{9000}.$$
$$P=\frac{16\times2\times2}{1+\left(\frac{1}{9000}\times48\times48\times3\right)}=\frac{64}{1+0{\cdot}768}$$
$$=36{\cdot}2 \text{ tons.}$$
$$\text{Safe load}=\frac{36{\cdot}2}{5}=\underline{7{\cdot}24} \text{ tons.}$$

EXAMPLE 2. A cast-iron column, of circular solid cross section 6 inches diameter is bolted down firmly at its lower end and is perfectly free at the top. If the length is 15 feet, what axial load would cause rupture?

Here $l=2L=360$ inches. $f_c=36$ tons per square inch.

$k^2=\frac{r^2}{4}=\frac{9}{4}$ inch units. $c=\frac{1}{1600}$.

$$A=\frac{\pi d^2}{4}=\frac{22}{7}\times\frac{36}{4}$$

$$=\frac{198}{7}\text{ square inches.}$$

$$P=\frac{36\times\frac{198}{7}}{1+\left(\frac{1}{1600}\times 360\times 360\times\frac{4}{9}\right)}$$

$$=\frac{36\times 198}{37\times 7}$$

$$=\underline{27\cdot 5}\text{ tons.}$$

EXAMPLE 3. Taking a factor of safety of 5, find the diameter of a solid mild-steel strut 6 feet long to carry safely a load of 3 tons. Both ends are rounded.

Let $d=$ diameter of strut in inches.

Then $k^2=\frac{r^2}{4}=\frac{d^2}{16}$ inch units.

$l=L=72$ inches. $f_c=21\cdot 4$ tons per square inch.

$A=\frac{\pi d^2}{4}$. $c=\frac{1}{7500}$.

Also, Collapsing load $=P=3\times 5=15$ tons.

$$15=\frac{21\cdot 4\times\frac{\pi d^2}{4}}{1+\left(\frac{1}{7500}\times 72\times 72\times\frac{16}{d^2}\right)}$$

$$=\frac{16\cdot 81d^2}{1+\frac{11\cdot 06}{d^2}},$$

or $$15+\frac{165\cdot 9}{d^2}=16\cdot 81d^2,$$

$$16\cdot 81d^4-15d^2-165\cdot 9=0;$$

whence $d=\underline{1\cdot 9}$ inches.

Straight-line formula. Very fair approximation to the strength of a strut may be obtained by use of a straight-line formula, *i.e.* one for which the graph is a straight line, and the calculations required in designing a strut to fulfil given conditions become much simpler. The usual form of such formulae is

$$f_c=f\left(1-c\frac{L}{k}\right),$$

where f_c is the safe stress per square inch of sectional area of the column, f is the safe stress for a short block of the same material and c is a coefficient depending on the material, and ranging in value from 0·005 for mild steel and wrought iron to 0·008 for cast iron and timber.

Arches. In Fig. 261 (*a*) is shown a number of loads W_1, W_2, etc., supported by an arrangement of links ABCDE, forming part of a link polygon. The construction necessary to determine the directions of the links is given in Fig. 261 (*b*) and has been explained on p. 67.

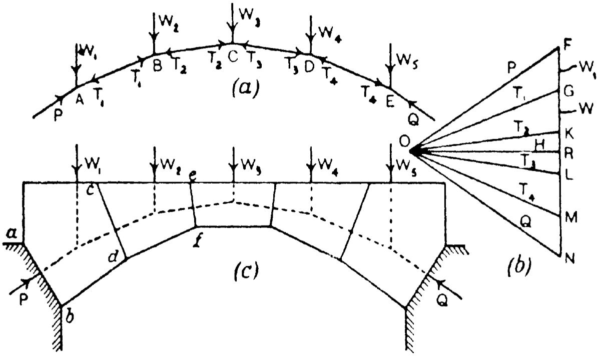


FIG. 261 — Principle of the arch.

The thrusts in the links are T_1, T_2, etc., and may be scaled from the lines radiating from O. OF = P and NO = Q give the forces required to maintain the links in position. Instead of links we might have employed blocks (Fig. 261 (*c*)), drawing the joints *ab*, *cd*, *ef*, etc., perpendicular to the lines of P, T_1 and T_2 respectively. The arrangement now gives an **arch** such as might be constructed in masonry or brickwork. The original link polygon is called the **line of resistance** of the arch; the forces acting at the joints of the blocks will have the same values P, T_1, T_2, etc., as in the link polygon.

The best arrangement would be produced by having the line of resistance passing through the centre of each joint and perpendicular to the joint. Such would give a uniform distribution of stress over the joints, and there would be no tendency for any block to slide on its neighbours. Generally, it is not possible to secure these conditions, but it is usual to endeavour to satisfy the following conditions:

(1) The line of resistance is arranged to come within the middle third of each joint; this secures that there will be no tendency for the joints to open out either at the top or the bottom (p. 239).

(2) The stress on the joint produced by the forces P, T_1, T_2, etc., is limited to a value which can be carried safely by the material.

(3) The line of resistance should not be inclined to the normal to the joint at an angle greater than the limiting angle of resistance (see p. 363); this secures that there shall be no slip, independently of any binding effect owing to the mortar.

Condition (2) above may be understood more clearly by reference to Fig. 262, in which are shown two of the blocks in equilibrium under the action of W_1, W_2, P and T_2. T_2 may be split into components T_1 and S, normal and tangential respectively to the section *ef*. If T_1 acts at the centre of the joint, a uniformly distributed normal stress will be produced. Otherwise, as explained for a column on p. 237, a varying normal stress will act on the section and may be represented by the stress figure *efhg*. The maximum stress p_1 is limited to a safe value depending on the material of the blocks.

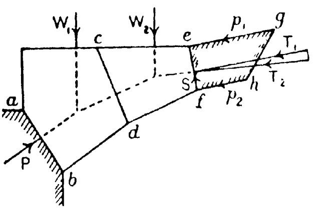


FIG. 262.—Stress at an arch joint.

Reference to Fig. 261 (*b*) will show that the horizontal component of any of the forces P_1, Q, T_1, T_2, etc., is given by OR = H. H is called the horizontal thrust of the arch, and is constant throughout a given arch carrying given vertical loads.

It will be understood that the link polygon ABCDE (Fig. 261 (*a*)) may have a greater or smaller vertical height depending on the position chosen for the pole O in Fig. 261 (*b*). The effect of this on the arch will be to give it a greater rise if O is nearer FN in Fig. 261 (*b*); H will be diminished thereby. Hence, an arch of given span and carrying given loads will have the horizontal thrust diminished by increasing the rise.

Metal arches. From what has been said regarding the line of resistance falling within the middle third of the joints, it will be clear that the bending moment at any section of a masonry arch is limited to a small quantity only. The rule is unnecessary in the case of metal arches, as these are capable of withstanding large bending moments.

Metal arches are of three principal types: (*a*) arches continuous from abutment to abutment, and firmly anchored to the abutments or springings; (*b*) arches continuous throughout their length, but hinged at the abutments by means of pin joints; (*c*) arches having pin joints at the abutments, and also a pin joint at the crown. These types are shown in outline in Fig. 263 (*a*), (*b*) and (*c*).

In the types (*a*) and (*b*) difficulties arise in the solution by reason of the inability of the arch to change its shape freely in order to accommodate changes in dimensions due to elastic strains of the metal, or to changes in temperature In type (*a*), both the span and the directions of the tangents to the arch at the abutments are unaltered when the arch is under strain. In type (*b*) the directions of the tangents at the abutments may alter, but the span remains constant. In type (*c*) the arch may rise freely at the crown to accommodate any strains of the metal; hence this type is not liable to being self-stressed, nor can changes in temperature produce any stresses in the metal. Type (*c*) alone is considered here.

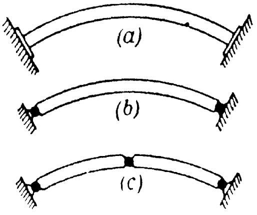


FIG. 263.—Types of metal arches.

Three-pin arch. In Fig. 264 (*a*) is shown an arch having pins A and B at the abutments, or springings, and one at the crown C. A single load W is being supported and all other weights are disregarded meanwhile. Let T_A and T_B be the abutment reactions. Acting on the arch are three external forces only, viz. W, T_A and T_B, and these are in equilibrium; hence their lines must meet at a point. Further, there will be two forces only acting on the portion

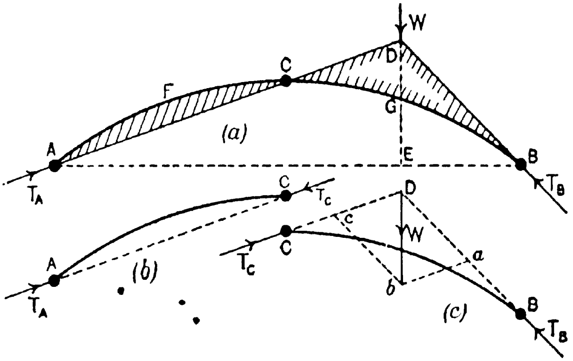


FIG. 264.—Three-pin arch.

AC, viz. T_A and a reaction T_C at C coming from the right-hand portion of the arch; these forces are in equilibrium, and must therefore act in the same straight line AC (Fig. 264 (*b*)). It follows that the line of T_A in Fig. 264 (*a*) is AC, and production of AC to cut the line of W in D will give the point where T_B must also intersect W; therefore T_B acts in the line BD. The equilibrium of the right-hand portion CB is indicated in Fig. 264 (*c*). T_C (now reversed in sense), T_B and W intersect at D and are in equilibrium. Both T_A and T_B may be found from the parallelogram of forces D*abc*.

It may be noted that W (Fig. 264 (*a*)) might be supported by means of straight rods, or links, AD and BD jointed at A, D and B, and that these rods would be under thrust only. ADB is usually termed the **linear arch.** Again, if W were supported by a beam simply resting on supports at A and B, then ADB would be the bending-moment diagram for the beam to a scale in which the bending moment at E is represented by DE.

The bending moment at any section of the arch may be found in the following manner. Let AB (Fig. 265) be a transverse section of an arch, let OX be the centre line of the arch, *i.e.* the line containing the centres of area of all transverse sections, and let OX intersect AB at C. Draw DC vertically to meet the linear arch at D. The thrust T in the linear arch at D will act in the direction of the tangent DE to the linear arch at D, and may be transferred to C as shown by $T'=T$, provided a couple of moment $T \times CE$ be applied, CE being perpendicular to DE. The moment of this couple is the bending moment at AB; the normal thrust and shearing forces at AB may be obtained by resolving T′ into components respectively normal and tangential to AB.

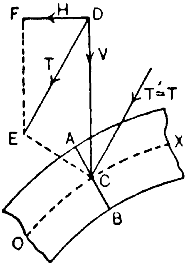


FIG. 265 —Bending moment, thrust and shear at a section of an arch.

A convenient manner of expressing the bending moment may be obtained: Resolve T at D into horizontal and vertical components H and V by means of the triangle of forces DFE (Fig. 265). The triangles EFD and DEC are similar; hence

$$\frac{T}{H}=\frac{DE}{FD}=\frac{DC}{CE};$$

$$\therefore\ T \times CE = H \times DC.$$

Now, since the linear arch is also the link polygon for the given loads, H is constant for any point in the arch (p. 242). Hence the intercepts DC (measured to the same scale as that used in drawing the arch), when multiplied by the constant horizontal thrust H, will give the bending moment at AB. It will be noted that DC is the vertical intercept between the arch centre line OX and the linear arch; hence the area between these will represent the bending-moment diagram for the arch.

The bending-moment diagram for the arch in Fig. 264 (*a*) is shaded. H may be found by first obtaining T_A or T_B and then taking the horizontal component. It will be noted that the diagram for AC falls below the arch centre line AFC; the inference is that this portion of the arch is under negative bending. Reference to Fig. 264 (*b*) will render this point more clear; the forces T_A and T_C tend to increase the curvature of AC. The bending-moment diagram for CGB falls above the centre line, CGB is under positive bending and will have its curvature diminished on application of the load.

The following directions will be of service in dealing with more complicated loading; reference is made to Fig. 266, in which ACB is the arch centre line.

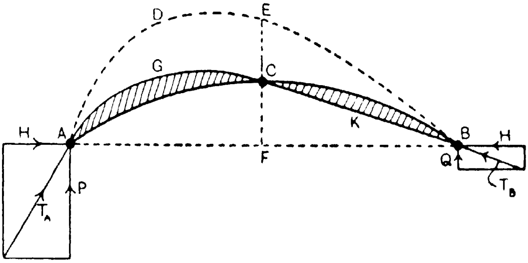


FIG. 266.—Bending moments and reactions for a three-pin arch.

Consider a simply supported beam having the same span as the arch and carrying the same loads. Draw the bending-moment diagram ADEB for this beam, using any convenient scale. The arch has zero bending moment at A, C and B; hence the linear arch may be obtained by redrawing ADEB so that it passes through A, C and B. To do this, reduce all the ordinates of ADEB in the ratio of CF to EF, giving the linear arch AGCKB. The shaded area will be the bending-moment diagram for the arch. To obtain its scale, CF represents M_F, the bending moment at F for the simply supported beam; hence the scale of the shaded area is found by equating CF to this bending moment.

The horizontal thrust H may be found from

$$M_F = H \times CF,$$

or

$$H = \frac{M_F}{CF},$$

CF being measured to the scale used in drawing the arch. T_A and T_B may be found by compounding with H the reactions P and Q for the simply supported beam at A and B respectively.

Suspension bridges. A simple type of suspension bridge is shown in Fig. 267, in which the roadway FG is supported by means of two chains AB, one on each side of the bridge. The chains pass over rollers or sliding pieces on the tops of towers at A and B and are anchored securely at D and E. Suspending bars connected to the chain support the weight of the roadway.

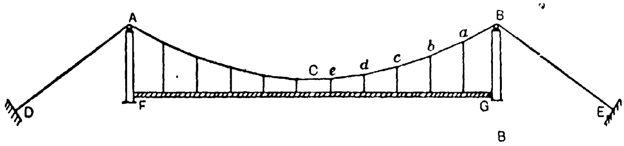


FIG. 267—Suspension bridge.

Assuming that the weight of the roadway is distributed uniformly and that the weight of the chain is small by comparison, also that the roadway is fairly flexible, the tensions at the points B and C may be found as shown in Fig. 268. The portion of the chain hanging between B and C will support one quarter of the whole weight of the bridge, and this may be concentrated at its centre of gravity. The horizontal pull H at C passes through the line of $\frac{1}{4}W$ at G_0; T_0, the pull at B, must pass through the same point. The triangle of forces *abc* (Fig. 268 (*b*)) will then give the values of H and T_0.

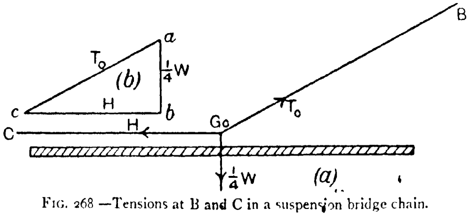


FIG. 268—Tensions at B and C in a suspension bridge chain.

Let *w* (Fig. 269 (*a*)) be the load communicated by each suspender to the chain. T_0 and H in this figure are the pulls at B and C respectively. To obtain the directions of the chain throughout, the pull in *ab*, together with H, supports the load carried by the four suspenders passing through *b*, *c*, *d* and *e*. The resultant of the four loads will intersect H at their centre G_1, and the pull in *ab* must pass through the same point, thus determining the direction of *ab*.

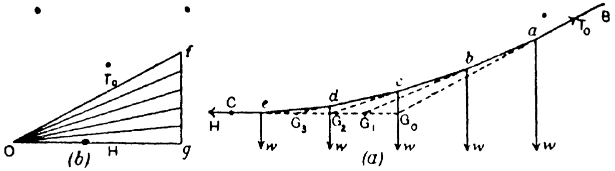


FIG. 269.—Shape of a suspension bridge chain.

bc will pass through G_2, the point in which the resultant force in the suspenders *c*, *d* and *e* intersects the line of H. Similarly *cd* passes through G_3 and *de* completes the half-chain. If a curve were drawn to touch the lines *ab*, *bc*, etc., its shape would be parabolic, owing to the geometrical property involved in the above construction.

It will be evident that *abcde* is a link polygon capable of supporting the given loads. The pull in any link may be found from the force diagram (Fig. 269 (*b*)).

The effect of a load passing along the bridge may be observed by inspection of Fig. 270. As both chain and roadway are flexible, the

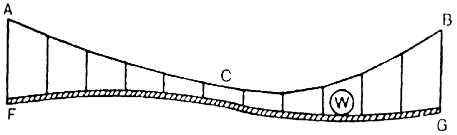


FIG. 270.—Effect of a load on a suspension bridge.

chain alters in shape as shown. To avoid this undesirable effect, the roadway may be stiffened by the insertion of stiffening girders. The best type of such girders consists of two on each side of the roadway (Fig. 271 (*a*)), connected at the middle C by a hinge and also having hinges at the piers of the bridge, D and F. Girders of this type are free to rise or fall at the middle of the span and thus avoid any complications of stress which would result from any alteration in the length of the chain owing to changes of temperature or stretching.

To understand the effect of a live load W on the chain in Fig. 271 (*a*), it should be noted that the chain will alter its curve to a very small extent only, owing to the action of the stiffening girders; any alteration will be due to the elastic strains. Supposing the chain to be parabolic initially, and to remain parabolic, it follows that the effect of W on the chain must be the same as would be produced by equal pulls in all the suspenders, this being the condition under which alone will the chain assume a parabolic curve. Hence, if there be N suspending rods, the pull in each will be $\frac{W}{N}$. The forces acting on the left-hand stiffening girder will be as shown in Fig. 271 (*b*);

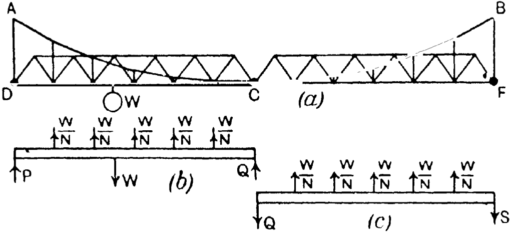


FIG. 271.—Stiffening girders for a suspension bridge

those acting on the right-hand girder are indicated in Fig. 271 (*c*). It will be noted that a reaction P from the left-hand abutment together with another Q communicated through the pin at C from the right-hand girder are required for the equilibrium of the left-hand girder. The right-hand girder requires holding down against the pulls of the suspending rods; hence the reactions Q and S act downwards. Knowing the loads, these reactions can be calculated, and the diagrams of bending moments and shearing forces for the girders may be drawn by application of methods already described.

The length of parabolic chain required for a suspension bridge may be calculated approximately from the following formula:

Let L = the half length of the chain, in feet.
S = the span, in feet.
D = the dip, in feet.

Then $$L = \frac{S}{2} + \frac{4}{3}\frac{D^2}{S}.$$

EXERCISES ON CHAPTER X.

1. Calculate the elastic instability load by Euler's formula for a bar of mild steel 10 feet long and 0·5 inch in diameter, fixed at both ends. Take $E = 13{,}500$ tons per square inch.

2. A mild-steel tube 1·1 inches in external diameter and 1·0 inch internal diameter and 8 feet long is used as a strut, having both ends hinged. What would be the collapsing load by Euler's formula? $E = 13{,}500$ tons per square inch.

3. A series of struts, having both ends rounded, have ratios of L to k of 40, 60, 80, etc., up to 200. Calculate the collapsing loads per square inch of sectional area, using Euler's formula, and plot these loads with the ratios of L to k. $E = 13,000$ tons per square inch.

4. Answer Question 3 if both ends are fixed.

5. Find the breaking load of the strut given in Question 1 by application of Rankine's formula. Take the coefficients from the table on p. 235.

6. A solid mild-steel strut 2 inches diameter is 6 feet high. Use Rankine's formula and find the safe axial load if both ends are rounded. Factor of safety = 5.

7. A wrought-iron tube is 4 inches in external diameter, and is made of metal 0·25 inch thick. It is used as a column 8 feet high, and has both ends fixed. Find the breaking load by use of Rankine's formula.

8. A rolled **I** section of mild steel, flanges 5 inches wide, depth 9 inches, metal 0·6 inch thick, is used as a strut 10 feet long, having one end fixed and the other end perfectly free. Find the safe axial load by Rankine's formula, taking a factor of safety of 6.

9. A solid strut of mild steel is 1·5 inches in diameter and has both ends fixed. Find the length for which the breaking loads by Rankine and by Euler will be equal. Take $E = 13,500$ tons per square inch.

10. The column given in Question 8 carries a load of one ton at the centre of area of one flange. Calculate the maximum and minimum stresses, and draw a stress diagram for a horizontal cross section of the column.

11. Take the tube given in Question 7 and calculate at what distance from the axis a load may be applied without thereby producing tensile stress.

12. A semicircular arch of 4 feet radius, hinged at the crown and springings, carries a uniform load of 500 lb. per horizontal foot. Draw the bending-moment diagram. State from the diagram the maximum bending moment.

13. The centre line of a three-pinned arch is a circular arc; the horizontal distance from springing to springing is 150 feet and the rise is 15 feet. There is a uniformly distributed load of 0·5 ton per horizontal foot together with concentrated loads of 10, 15 and 5 tons at horizontal distances from one springing of 20, 40 and 60 feet respectively. Draw the bending-moment diagram and state its scale; find the horizontal thrust and the reactions at the springings.

14. A suspension bridge is 100 feet span and the chains have a dip of 12 feet. Suppose the uniform load on one chain to be 500 lb. per horizontal foot, and find the maximum and minimum pulls in the chain.

15. Find the length of chain required for the bridge in Question 14. Suppose the chain were to stretch 0·25 inch, what will be the change in the dip?

16. A hollow cast-iron column, 12 inches in external diameter, 10 inches in internal diameter and 8 feet long, is subjected to a direct compressive load of 40 tons. A bracket bolted to the side of the column supports the end of a girder, which transmits to the bracket a load of 5 tons. The line of action of this load may be assumed to be 12 inches from the axis of the column. Find the maximum and minimum stresses in a cross section of the column due to these loads. (B.E.)

17. A horizontal link of rectangular section 4 inches deep and 2 inches thick is subjected to tension, the load being P tons. The line of action of the load is in the central plane of the thickness and 2·25 inches from the bottom face of the link. (*a*) Find the load P if the greatest tensile stress in the straight part of the link is 6 tons per square inch. (*b*) If the tensile stress on a cross section of the link varies uniformly from 6 tons per square inch at the top to 2 tons per square inch at the bottom, find P and the position of its line of action. (L.U.)

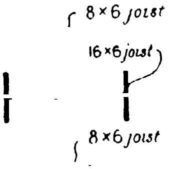


FIG. 272.

18. A hollow cylindrical steel strut has to be designed for the following conditions: Length 6 feet, axial load 12 tons, ratio of internal to external diameter 0·8, factor of safety 10. Determine the necessary external diameter of the strut and the thickness of the metal if the ends of the strut are firmly built in. Use the Rankine formula, taking $f=21$ tons per square inch, and a for rounded ends $=\frac{1}{7500}$. (L.U.)

19. Find the radii of gyration of a column consisting of three steel rolled joists, riveted together as shown in the sketch (Fig. 272), their properties being

	Area, sq. in	I_x, inch units	I_y, inch units	Thickness of web, inch.
16 inch × 6 inch joist.	18·22	726·0	27·0	0 55
8 inch × 6 inch joist.	10·29	110·6	17·9	Not needed here.

What would be the working load of such a column 24 feet long and with fixed ends, using the following straight-line formula:

$$f_c=\left(14560-56\frac{l}{r}\right)\text{lb},$$

where f_c is the working stress in lbs. per square inch ; l is the length of the column in inches ; r is the least radius of gyration in inches. (I.C.E.)

CHAPTER XI.

SHAFTS. SPRINGS.

Twisting moment on a shaft. A shaft is a piece used for the transmission by rotation of motion and power. A moment tending to rotate the shaft is communicated at one place and is transmitted, by stresses in the material of the shaft, to the desired place. Considering a shaft AB (Fig. 273 (*a*)), having one end A fixed rigidly, and having an arm BC mounted at the other end. The effect of a force P applied at C may be examined by applying equal and opposite forces P′ and P″, each equal and parallel to P, at B so as to act through the axis of the shaft. These forces equilibrate and consequently do not interfere with P. The system now consists of a couple formed by the forces P and P″, the sole tendency of which will be to rotate the shaft about its axis, together with a force P′, the tendency of which will be to bend the shaft. The shaft as a whole would be equilibrated by the application of forces (not shown in the figure) at the rigid connection at A.

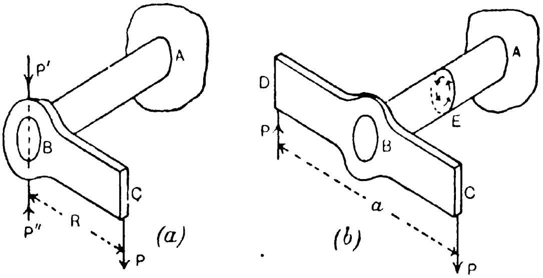


FIG. 273.—Twisting moments on shafts.

A shaft is said to be under **pure twist** when there is no tendency to bend it, nor to produce push or pull in the direction of its axis. The shaft in Fig. 273 (*a*) would have been under pure twist had the couple formed by P and P″ been applied alone. One method of securing this result is shown in Fig 273 (*b*), in which a double arm CBD is used and two forces P, P, forming a couple of moment P*a*, are applied at its ends The moment of the couple is called the **twisting moment**, or **torque**, and is written T generally. Neglecting the weight of the shaft, the equilibrium of the whole as in Fig. 273 (*b*) requires the application at A of a couple having a moment equal and contrary to that of T. The condition to be fulfilled in order that a shaft may be under pure twist is that it must be equilibrated by two equal opposing couples acting in planes perpendicular to the axis of the shaft.

Shearing stresses produced by torque. Consider the shaft to be cut at any cross section E in Fig. 273 (*b*), the section being perpendicular to the axis of the shaft. To equilibrate the outer portion of the shaft under the action of the applied couple P, P requires an equal contrary couple at the section E, and acting in the plane of the section. Such a couple can be brought about in the uncut shaft only by the existence of shearing stresses distributed in some manner over the section. The nature of the distribution may be understood by considering the straining of the shaft under the action of the couples. Experiment justifies the assumptions that, in a **round shaft**, sections such as that at E remain plane, *i.e.* unwarped, when the couples are applied, and that any radius of such a section changes its direction but remains straight; it is assumed in this that the elastic limit is not exceeded.

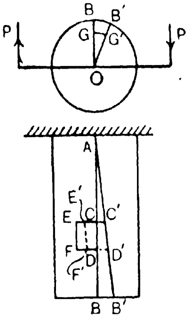


FIG. 274.—Torsional strains.

In Fig. 274, AB is a line drawn on the surface of the shaft parallel to the axis before straining. As A is fixed rigidly, it will remain unaltered in position, but the other end will rotate under the straining; the result is that AB will change position to AB′. Any small rectangle such as CDFE drawn on the surface of the shaft will change its position and shape as shown at C′D′F′E′. The angle through which CD has rotated in order to assume the new position C′D′ is clearly equal to that through which AB has turned. This angle, BAB′, equal to θ, is therefore the shear strain at all parts of the surface of the shaft. Had we been able to draw a rectangle inside the material at a radius OG, its circumferential movement and change of shape evidently would have been proportional to its radius. Thus, on the outer end cross section, G would move to G′ and B to B′, and we have

$$GG' : BB' = OG : OB.$$

We may therefore state that, since the shear strain at any point on a cross section of the shaft is proportional to the radius, the shear stress at that point will also be proportional to the radius, provided the elastic limit is not exceeded. It will also be evident that the shear stress at any point on a cross section will have a direction perpendicular to the radius of the point.

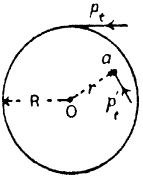


FIG. 275

Moment of resistance to torsion. In Fig. 275 is shown a cross section of a shaft under pure twist. Consider a small area a.

Let R = radius of shaft, in inches;
r = radius of a, in inches;
p_t = intensity of stress at outer skin, in lb. or tons per square inch;
p_t' = intensity of stress on a.

Then

$$p_t' : p_t = r : R,$$

$$p_t' = \frac{r}{R} p_t;$$

and

$$\text{Force on } a = p_t'a = \frac{r}{R} \cdot p_t \cdot a.$$

$$\text{Moment of force on } a \text{ about } O = \frac{r}{R} \cdot p_t ar = \frac{p_t}{R} \cdot ar^2.$$

Taking the sum of such moments all over the section, we have

$$\text{Total moment} = \frac{p_t}{R} \Sigma_0^R ar^2 = \frac{p_t}{R} I_{OZ} = \frac{p_t}{R} \cdot \frac{\pi R^4}{2}$$

$$= \frac{p_t \cdot \pi R^3}{2} \text{ lb.- or ton-inches.}$$

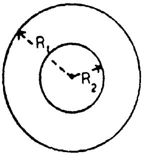


FIG. 276.

This expression is called the **moment of resistance to torsion** for a solid round shaft. The case of a hollow round shaft of external radius R_1 and internal radius R_2 (Fig. 276) would be worked out similarly, with the substitution of limits R_2 and R_1 for o and R in the integration. Thus,

$$\text{Total moment} = \frac{p_t}{R_1} \Sigma_{R_2}^{R_1} ar^2 = \frac{p_t}{R_1}\left(\frac{\pi R_1^4 - \pi R_2^4}{2}\right)$$

$$= \frac{p_t}{R_1} \cdot \pi(R_1^2 - R_2^2)\left(\frac{R_1^2 + R_2^2}{2}\right) \text{ lb.- or ton-inches.}$$

Any question regarding the safe strength of a round shaft under pure twist may be solved by equating the given torque to the proper expression for the moment of resistance to torsion. It will be clear that a hollow shaft will have a greater strength than a solid one of the same weight. Apart from the practical consideration that the boring of an axial hole may lead to the detection of otherwise unsuspected flaws in the material, there are the considerations that the intensity of shear stress, as well as the arm for taking moments, are small near the axis in a solid shaft, so that material near the axis is being employed unprofitably.

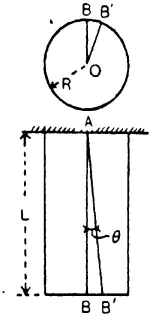


FIG. 277.—Angle of twist of a shaft.

Torsional rigidity of a round shaft. It is often of importance to estimate the angle through which one end of a shaft will twist relatively to the other end—briefly the **angle of torsion.** Referring to Fig. 277, one end of the shaft being fixed rigidly, the other end rotates through a small angle BOB′, denoted by α, on application of a torque T, and the shear strain is given by the angle BAB′, equal to θ radians. Taking the expression for the modulus of rigidity (p. 109), viz.

$$C=\frac{p_t}{\theta}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

we may substitute for p_t and θ as follows:

$$T=\frac{p_t \pi R^3}{2};$$

$$\therefore\ p_t=\frac{2T}{\pi R^3}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Again, $$\frac{BB'}{AB}=\theta, \text{ in radians};$$

or $$BB'=AB\,.\,\theta.$$

Also, $$\frac{BB'}{BO}=\alpha, \text{ in radians};$$

or $$BB'=BO\,.\,\alpha;$$

$$\therefore\ BO\,.\,\alpha=AB\,.\,\theta;$$

$$\therefore\ \theta=\frac{BO}{AB}\,.\,\alpha=\frac{R}{L}\,.\,\alpha, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where R and L are the radius and length of the shaft in inches. Substituting the values found in (2) and (3) in (1) gives

$$C = \frac{\dfrac{2T}{\pi R^3}}{\dfrac{R}{L} \cdot \alpha};$$

$$\therefore \alpha = \frac{2TL}{\pi R^4 C}, \text{ radians} \qquad . \qquad (4)$$

The result for a hollow round shaft of external radius R_1 and internal R_2 may be found in a similar manner, using the expressions

$$T = \frac{p_t \pi (R_1^4 - R_2^4)}{2R_1};$$

$$\theta = \frac{R_1}{L}\alpha.$$

The final result is

$$\alpha = \frac{2TL}{\pi(R_1^4 - R_2^4)C}, \text{ radians} \; . \; . \; . \; . \; . \; . \; . \qquad (5)$$

By substitution from (3) in (1) an expression may be obtained suitable for cases where the maximum shear stress is given. Thus,

$$C = \frac{p_t L}{\alpha R},$$

or

$$\alpha = \frac{p_t L}{CR}, \text{ radians.} \; . \; . \; . \; . \; . \qquad (6)$$

This expression is applicable to both solid and hollow shafts by taking R as the external radius.

The torsional rigidity, or stiffness, of a shaft may be measured by the reciprocal of α.

Comparison of hollow with solid shafts. The relative strengths of two shafts may be estimated by comparing the torques, which may be applied without exceeding a given intensity of stress. Let two shafts, one hollow, the other solid, have the same external radius R. Let the internal radius of the hollow shaft be nR, where n is a numerical coefficient. Let p_t be the maximum intensity of shearing stress in each case. Then, for the solid shaft,

$$T_s = \frac{p_t \pi R^3}{2}; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and for the hollow shaft,

$$T_h = \frac{p_t \pi (R^4 - n^4 R^4)}{2R}$$

$$= \frac{p_t \pi R^3 (1 - n^4)}{2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence,
$$\frac{T_h}{T_s} = 1 - n^4. \qquad \ldots\ldots (3)$$

For example, if the internal radius of the hollow shaft is one-third of the external radius, $n = \frac{1}{3}$, and

$$\frac{T_h}{T_s} = 1 - \frac{1}{81}$$
$$= \frac{80}{81}.$$

Comparison may also be made of the strength of a hollow with a solid shaft having the same cross-sectional area, *i.e.* having the same weight per unit length. We have, for the solid shaft,

$$T_s = \frac{p_t \pi R^3}{2}; \qquad \ldots\ldots (4)$$

and for the hollow shaft, $T_h = \frac{p_t \pi (R_1^4 - R_2^4)}{2R_1}$.

Putting $R_2 = nR_1$, gives

$$T_h = \frac{p_t \pi R_1^4(1 - n^4)}{2R_1}$$
$$= \frac{p_t \pi R_1^3(1 - n^4)}{2} \qquad \ldots\ldots (5)$$

Also, as the cross-sectional areas are equal,

$$\pi R^2 = \pi(R_1^2 - R_2^2);$$
$$\therefore\ R^2 = R_1^2(1 - n^2); \qquad \ldots\ldots (6)$$
$$\therefore\ \frac{R_1^2}{R^2} = \frac{1}{1 - n^2}.$$

Hence,
$$\frac{T_h}{T_s} = \frac{\frac{p_t \pi R_1^3(1 - n^4)}{2}}{\frac{p_t \pi R^3}{2}} = \frac{R_1^3}{R^3}(1 - n^4)$$
$$= \frac{1 - n^4}{(1 - n^2)\sqrt{1 - n^2}}$$
$$= \frac{1 + n^2}{\sqrt{1 - n^2}}. \qquad \ldots\ldots (7)$$

For example, if $n = \frac{1}{3}$, $\frac{T_h}{T_s} = \frac{1 + \frac{1}{9}}{\sqrt{1 - \frac{1}{9}}}$
$$= \underline{1{\cdot}18}.$$

Thin tube under torsion. In the case of a thin tube under torsion it may be assumed that the shearing stress is distributed uniformly.

Let p_t = stress intensity, lb. per sq inch,
R = mean radius of tube, inches,
t = thickness of the tube walls, inch.

Then Cross-sectional area $= 2\pi Rt$,
Total shearing force $= 2\pi Rtp_t$,
Moment of this force $= 2\pi Rtp_tR$,
Moment of resistance to torsion $= 2\pi R^2tp_t$, lb-inches.

Horse-power transmitted by shafting. Formulae connecting the horse-power (see p 326) with the dimensions of the shaft and its speed are based on the average torque transmitted, and should therefore be used with caution. The maximum torque, on which will depend the maximum intensity of shearing stress, may exceed the average torque considerably, leading to a result for the diameter of the shaft which may be much too small.

Considering a solid round shaft under pure twist, let

T = torque transmitted, lb. inches;
R = radius of shaft, inches;
p_t = maximum shear stress, lb per sq. inch;
N = revolutions per min.

Then Work per revolution $= \dfrac{T}{12} \cdot 2\pi$ (p. 339)

$$= \frac{\pi T}{6} \text{ foot lb.}$$

Now $$T = p_t \frac{\pi R^3}{2};$$

$$\therefore \text{ work per revolution} = \frac{p_t \pi^2 R^3}{12}.$$

$$\text{Work per minute} = \frac{p_t \pi^2 R^3 N}{12};$$

$$\therefore \text{ H.P.} = \frac{p_t \pi^2 R^3 N}{12 \times 33{,}000}$$

$$= \frac{p_t R^3 N}{40{,}081} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

EXAMPLE. Find the horse-power which may be transmitted by a shaft 2 inches in diameter at 180 revolutions per minute. The maximum shearing stress is 10,000 lb. per square inch.

$$\text{H.P.} = \frac{p_t R^3 N}{40{,}081} = \frac{10{,}000 \times 1 \times 180}{40{,}081}$$

$$= 45 \text{ nearly.}$$

Equation (1) above may be altered so as to give the diameter o shaft required for a given power. Thus,

$$R^3 = \frac{40{,}081 \times \text{H.P.}}{p_t N},$$

or

$$D^3 = \frac{8 \times 40{,}081 \times \text{H.P.}}{p_t N},$$

$$D = \text{a coefficient} \times \sqrt[3]{\frac{\text{H.P.}}{N}}.$$

A common value of the coefficient is about 3·3 for steel shafts.

Principal stresses for pure torque. In Fig. 278 is shown a shaft

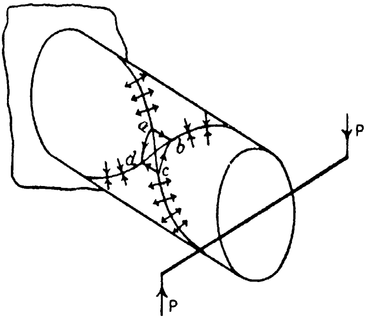


FIG. 278.—Principal stresses for pure torque.

under pure torque. A small square *abcd*, having its edges *ab* and *cd* parallel to the axis of the shaft, has been sketched on the surface. Each edge of the square will be subjected to shearing stresses of magnitude p_t; hence the diagonals *ac* and *bd* have purely normal pull and push stresses respectively, the magnitude being also p_t (p. 128). These diagonals are therefore principal axes of stress, and the stresses on them are the principal stresses for the case of the shaft being under pure torque. If the diagonals be produced round the shaft surface, it is evident that they will form helices having an inclination of 45° to the shaft axis.

A shaft made of material weak under pull, and strong under both shear and push, would fracture along the helix of which *ac* forms a part. This fact may be illustrated by means of a stick of blackboard chalk; on applying opposite couples by the fingers to the ends of the chalk, the fracture will be found to follow very closely a helix of 45° inclination. Pure bending applied to the chalk will cause it to fracture across a section at 90° to the axis; it is therefore evident that bending and torque simultaneously applied will cause fracture to take place on some section intermediate between 45° and 90°. A shaft made of cast iron would behave in a similar manner to the stick of chalk, as the stress properties are similar.

Shafts made of ductile material, such as mild steel, behave in a different way. Fracture under pure torque takes place across a section at 90° to the axis, as the strength under pull and also under push is higher than that under shear. It may be shown that materials loaded in a complex manner have sections mutually perpendicular on which the stress is purely normal, *i.e.* the stresses are principal stresses. There is also a particular section which has a shearing stress greater than that on any other section. There is strong evidence for believing that brittle materials break down when the principal stress of tension reaches a certain value depending on the material; many ductile materials break down, or yield, when the maximum shearing stress reaches a certain value.

More general case of principal stresses. Let AB and BC be two sections of a body intersecting at 90° at B (Fig. 279 (*a*)). Let

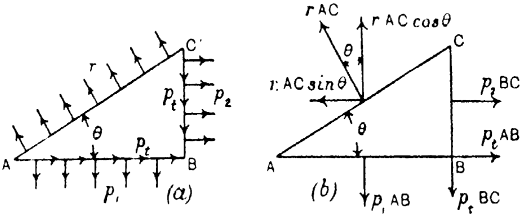


FIG. 279.—Principal stresses and axes.

AB and BC have normal stresses p_1 and p_2 respectively, and let each be subjected to equal shearing stresses p_t. Let AC represent a third section of the body, cutting AB at an angle θ, and let the stress r on AC be purely normal. The wedge ABC will be in equilibrium under the action of these stresses, and it is required to determine from this condition the values of θ and of r. AC and r will then be a principal axis of stress and a principal stress respectively. For simplicity, let the thickness of the wedge from front to back be unity.

Due to the given stresses p_1, p_2 and p_t, the faces AB and BC will have resultant forces acting as shown in Fig. 279 (*b*). The forces

$p_1 . AB$ and $p_2 . BC$ will act at the centres of AB and BC respectively. The forces $p_t . AB$ and $p_t . BC$ will act along AB and BC respectively. Due to r, a normal force $r . AC$ will act at the centre of AC and will have an inclination to the vertical equal to θ. Hence its vertical and horizontal components will be $r . AC . \cos\theta$ and $r . AC . \sin\theta$ respectively. For equilibrium of the wedge, the sum of the vertical upward forces must be equal to the sum of the vertical downward forces; also the sum of the horizontal forces acting towards the left must be equal to the sum of those acting towards the right. The algebraic expressions for these conditions are:

$$r . AC . \cos\theta = p_1 . AB + p_t . BC. \quad \ldots\ldots \quad (1)$$

$$r . AC . \sin\theta = p_2 . BC + p_t . AB. \quad \ldots\ldots \quad (2)$$

To simplify (1), divide by AC, giving

$$r . \cos\theta = p_1 . \frac{AB}{AC} + p_t . \frac{BC}{AC}$$

$$= p_1 . \cos\theta + p_t . \sin\theta.$$

Divide this by $\cos\theta$, giving

$$r = p_1 + p_t . \tan\theta. \quad \ldots\ldots \quad (3)$$

Equation (2) may be simplified in a similar manner by dividing first by AC and then by $\sin\theta$, giving

$$r = p_2 + p_t . \cot\theta. \quad \ldots\ldots \quad (4)$$

Equations (3) and (4) are simultaneous equations, from which the values of r and θ may be obtained by the ordinary rules of algebra; thus, as the right-hand sides are equal, we have

$$p_1 + p_t . \tan\theta = p_2 + p_t . \cot\theta,$$

or

$$p_1 - p_2 = p_t(\cot\theta - \tan\theta)$$

$$= 2p_t . \cot 2\theta,$$

or

$$\cot 2\theta = \frac{p_1 - p_2}{2p_t}. \quad \ldots\ldots \quad (5)$$

Again, writing equations (3) and (4) thus,

$$r - p_1 = p_t \tan\theta, \quad \ldots\ldots \quad (6)$$

$$r - p_2 = p_t \cot\theta, \quad \ldots\ldots \quad (7)$$

and taking products, we have

$$(r - p_1)(r - p_2) = p_t^2. \quad \ldots\ldots \quad (8)$$

The solution of this quadratic equation may be obtained in the usual manner, giving

$$r = \frac{p_1 + p_2 \pm \sqrt{(p_1 - p_2)^2 + 4p_t^2}}{2}. \quad \ldots\ldots \quad (9)$$

The two roots of r in (9) indicate two principal stresses; also equation (5) gives two values of 2θ differing by 180° for which the cotangents are equal, and hence indicates two sections differing by 90°. The determination of which root of r acts on one section or the other may be obtained by inserting one of the calculated values of r in either (6) or (7); the resulting value of $\tan\theta$ or $\cot\theta$ will indicate the particular section on which this value of r acts.

In the above, both p_1 and p_2 have been taken as pulls; if either or both be pushes, the sign of p_1 and p_2 or both should be reversed in (5) and (9). A positive value for r indicates pull and a negative value indicates push. If any of the given stresses p_1, p_2 or p_t be absent in the data, write zero where these missing values occur in the equations found above.

For example, taking a cube having shearing stresses p_t only (p. 127), $p_1 = 0$, $p_2 = 0$.

Equation (5) gives $$\cot 2\theta = \frac{0}{2p_t} = 0;$$

$$\therefore\ 2\theta = 90^\circ \text{ or } 270^\circ,$$

$$\theta = 45^\circ \text{ or } 135^\circ.$$

Equation (9) gives $$r = \pm\frac{\sqrt{4p_t^2}}{2}$$

$$= \pm p_t.$$

The principal axes are therefore the diagonals of the cube, and the principal stresses are a push and a pull each equal to the given shear stress, thus agreeing with the results already obtained in a different manner.

Stress on a section inclined to the principal axes. Having determined the principal axes of stress and the principal stresses, the stresses on other sections may be found by the following construction. Reference is made to Fig. 280.

Let OA and OB be the principal axes of stress (Fig. 280 (a)), and let OA $= p_1$ and OB $= p_2$ be the stresses acting on the sections OB and OA respectively. To find the stress acting on any other section OK, carry out the following construction. With centre O and radii OA and OB describe circles; draw ON perpendicular to OK, cutting these circles in N and E respectively. Draw NC parallel to OB, and also ED parallel to OA and cutting NC in D. Join OD; OD will represent the stress p acting on OK.

To prove this, draw EF parallel to OB, and let the angle KOB, which is equal to the angle NOA, be called θ. Due to p_1 there will be an oblique stress of magnitude $p_1 \cos\theta$ acting on OK (p. 121). Now $\cos\theta$ is given by $\frac{\mathrm{OC}}{\mathrm{ON}}$ in the diagram and ON is equal to p_1 to scale; hence OC represents the oblique stress. Again, due to p_2 there will be an oblique stress of magnitude $p_2 \cos(90^\circ - \theta) = p_2 \sin\theta$ acting on OK. But $\sin\theta$ is given by $\frac{\mathrm{EF}}{\mathrm{OE}}$, and OE is equal to p_2 to scale; hence the latter oblique stress is given by EF, which is equal to CD. The resultant of these oblique stresses, represented by OC and CD respectively, will be OD, which accordingly gives the stress p on OK. The construction employed for finding D is a well known method of finding points on the circumference of an ellipse having OA and OB for its semi axes. The ellipse is shown in Fig. 280 (*a*), and is called the **ellipse of stress.**

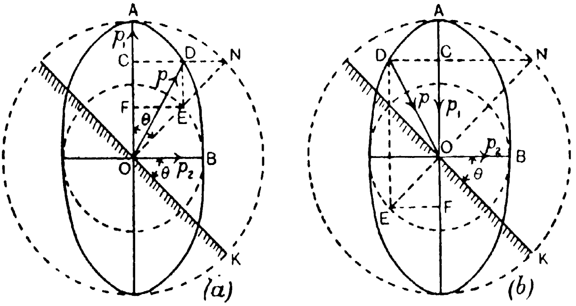


FIG. 280.—Ellipse of stress.

Both principal stresses have been taken as pulls in the above construction. Had one been a push, as p_1 (Fig. 280 (*b*)), and the other a pull, then the construction is modified by producing ND to cut the remote circumference of the ellipse as shown.

Maximum shearing stress. An important fact depends on the noting that the angle EDN (Figs. 280 (*a*) and (*b*)) is 90°, and that therefore D lies always on the circumference of a circle having EN for its diameter. The radius of this circle will be $\frac{1}{2}(\mathrm{ON} - \mathrm{OE}) = \frac{1}{2}(p_1 - p_2)$ for principal stresses of the same kind (Fig. 280 (*a*)), and will be $\frac{1}{2}(\mathrm{ON} + \mathrm{OE}) = \frac{1}{2}(p_1 + p_2)$ for unlike principal stresses (Fig. 280 (*b*)). The stress p may be resolved into normal and shear stresses in each case (Fig. 281 (*a*) and (*b*)), indicated by OG and OH respectively. It both cases is represented by the radius of the circle having EN for diameter; hence, for like stresses,

$$\text{Maximum shear stress} = \tfrac{1}{2}(p_1 - p_2), \quad \ldots\ldots\ \ldots\ldots\ldots (1)$$

and for unlike stresses,

$$\text{Maximum shear stress} = \tfrac{1}{2}(p_1 + p_2). \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

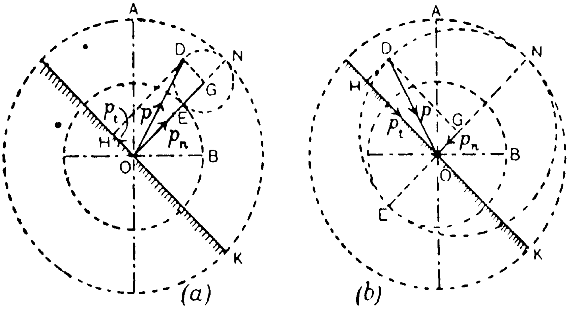


FIG. 281.—Normal and shearing stresses.

These equations require further examination. Both (1) and (2) refer to sections taken perpendicular to the paper, and give the maximum shearing stresses for such sections. Fig. 282 (*a*) shows a bar under axial pull stress p_1 and transverse pull stress p_2, both stresses in (*a*) being in the plane of the paper. The principal axes of stress are OX and OY; the maximum shearing stress for sections perpendicular to the paper in (*a*) will be $\tfrac{1}{2}(p_1 - p_2)$. Examine now Fig. 282 (*b*), showing a side elevation of the bar; p_2 acts perpendicular to the plane of the paper, and it will be evident that the section AB, at 45° to the axis, has a shearing stress of magnitude $\tfrac{1}{2}p_1$ acting on it (p. 124). Hence AB is the section of the bar which carries a shearing stress greater in magnitude than that on any other section.

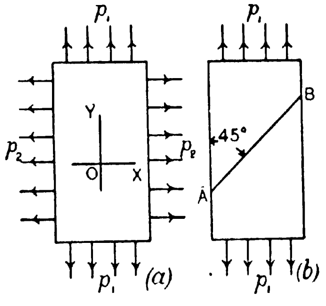


FIG. 282.—Maximum shear stress in a bar under longitudinal and transverse pulls.

It will be noted therefore that with like principal stresses, *e.g.* the longitudinal and circumferential stresses in a boiler shell, the greater principal stress alone determines the value of the maximum shearing stress, and the latter has a value equal to one-half of the greater principal stress. In the case of unlike principal stresses the maximum shear stress must be calculated from $\frac{1}{2}(p_1+p_2)$.

The points above noted are of importance in dealing with crank shafts and other cases where the combinations of loading give rise to unlike principal stresses. The experimental work of Guest and others shows that elastic break-down occurs when the shearing stress attains a certain value in many ductile materials, as has been noted already, and the results above discussed enable us to determine the relation of the maximum shear stress to the loading.

Shaft under combined bending and torsion. An example of this kind of loading will be found in any crank shaft. Considering a solid shaft:

Let M = the maximum bending moment on the shaft, in lb.-inches;
T = the maximum torque, in lb.-inches;
R = the radius of the shaft, in inches.

It is understood that M and T occur both at the same cross section. The stresses due to these may be found from:

$$M=\frac{p_1\pi R^3}{4}, \quad \text{or,} \quad p_1=\frac{4M}{\pi R^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$T=\frac{p_t\pi R^3}{2}, \quad \text{or,} \quad p_t=\frac{2T}{\pi R^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

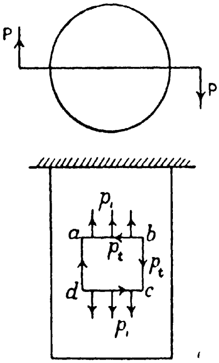


FIG. 283.—Shaft under combined torque and bending.

Reference to Fig. 283, in which a rectangle *abcd* has been sketched on the shaft surface, shows that p_2 is absent. The principal stresses may be calculated from equation (9) (p. 260):

$$r=\frac{p_1+p_2\pm\sqrt{(p_1-p_2)^2+4p_t^2}}{2}$$

$$=\frac{p_1\pm\sqrt{p_1^2+4p_t^2}}{2}. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

This result indicates unlike principal stresses, as the quantity under the square root sign is greater than p_1^2.

In the **Rankine hypothesis, the maximum principal stress is the criterion of break-down,** and this assumption may be applied to brittle materials. The Rankine equation may be obtained as follows. Taking the larger principal stress, viz. $r=\frac{p_1+\sqrt{p_1^2+4p_t^2}}{2}$, and substituting from (1) and (2), we have

$$r=\frac{1}{2}\left\{\frac{4M}{\pi R^3}+\sqrt{\left(\frac{4M}{\pi R^3}\right)^2+4\left(\frac{2T}{\pi R^3}\right)^2}\right\}$$

$$=\frac{1}{2}\left\{\frac{4M}{\pi R^3}+\frac{4}{\pi R^3}\sqrt{M^2+T^2}\right\}$$

$$=\frac{2}{\pi R^3}\{M+\sqrt{M^2+T^2}\},$$

or

$$\frac{r\,.\,\pi R^3}{2}=M+\sqrt{M^2+T^2}. \quad \ldots \quad (4)$$

The left hand side of this result has the same form as the expression for the moment of resistance of a shaft to torsion, the only difference lies in the fact that r is a push or pull stress, whereas, in the torque expression, a shear stress appears. It may be said that if a pure torque T_e were applied to the shaft, of magnitude given by (4), a shear stress would be produced thereby equal in magnitude to the maximum principal stress. Hence,

$$T_e=M+\sqrt{M^2+T^2}. \quad \ldots \quad (5)$$

The result is convenient for practical use, and is usually referred to as **Rankine's formula.**

If the **maximum shear stress** be taken as determining the point of failure, the reduction is as follows:

From equation (3) (p. 264), remembering that one value of r is push and the other pull:

$$r_1=\frac{p_1+\sqrt{p_1^2+4p_t^2}}{2}=\tfrac{1}{2}p_1+\sqrt{\tfrac{1}{4}p_1^2+p_t^2},$$

$$r_2=-\left\{\frac{p_1-\sqrt{p_1^2+4p_t^2}}{2}\right\}=-\tfrac{1}{2}p_1+\sqrt{\tfrac{1}{4}p_1^2+p_t^2}.$$

Also, Maximum shearing stress $=q=\dfrac{r_1+r_2}{2}$

$$=\frac{2\sqrt{\tfrac{1}{4}p_1^2+p_t^2}}{2}$$

$$=\sqrt{\tfrac{1}{4}p_1^2+p_t^2}. \quad \ldots \quad (6)$$

Inserting the values of p_1 and p_t in terms of M and T, we have

$$q=\sqrt{\frac{1}{4}\left(\frac{4M}{\pi R^3}\right)^2+\left(\frac{2T}{\pi R^3}\right)^2}$$

$$=\frac{2}{\pi R^3}\sqrt{M^2+T^2},$$

or

$$\frac{q\pi R^3}{2}=\sqrt{M^2+T^2}.$$

Let T_e be a torque which, if applied alone, would produce a shear stress equal to q. Then

$$T_e = \frac{q\pi R^3}{2} = \sqrt{M^2+T^2}. \quad \ldots\ldots\ldots\ldots\ldots(7)$$

It will be noted that this expression gives an equivalent twisting moment of smaller value than that permitted by the Rankine equation (5).

Springs. Springs are pieces intended to take a large amount of strain, and are used for minimising the effects of shocks, for storing energy, and for measuring forces. The load on any spring is kept well within the elastic limit; hence the change of length, or the distortion, of the spring will be proportional to the load applied. Springs vary in form, depending on the purpose for which they are intended; a few common forms are discussed below.

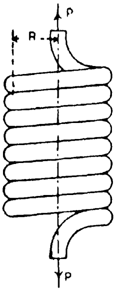


FIG. 284.—Helical spring under pull.

Helical springs. **Helical springs** are made by coiling a rod or wire of the material, generally steel, into a helix. If the spring is to be under pull, the coils of the unstrained spring are made so as to lie close together; open-coiled helical springs are necessary in cases where the load is to be applied as a push, causing the spring to become shorter. Reference is made to Fig. 284, which shows a close-coiled helical spring under pull, and made of material having a round section. It may be assumed that the effect of the load is to put the material of the spring under pure torsion. Bending is also present, and must be taken into account in open-coiled springs, but is small enough to be disregarded in the close-coiled spring under consideration.

Let P = load applied, lb.;

R = the mean radius of the helix, inches;

r = the radius of the section, inches.

Any cross section of the wire will be subjected to a torque given by

$$T = PR \text{ lb.-inches.} \quad \ldots\ldots(1)$$

Consider a short piece of the helix lying between two cross sections AB and CD (Fig. 285), and imagine AB to be fixed rigidly. Let F be the centre of the section CD, and take a horizontal radius which, when

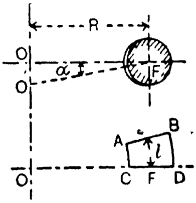


FIG. 285.

produced, cuts the axis of the spring at O. The effect of the torque will be to cause CD to twist through an angle relative to AB, and FO will rotate into the position FO′, the point O undergoing a deflection OO′. Let the mean length of the portion considered be l; then the angle of twist may be written from the equation for that of a shaft (p. 255).

$$a = \frac{2Tl}{C\pi r^4} = \frac{2PRl}{C\pi r^4}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Again,

$$a = \frac{OO'}{R};$$

$$\therefore\ OO' = aR = \frac{2PRl}{C\pi r^4}R$$

$$= \frac{2PR^2l}{C\pi r^4}. \quad \ldots\ldots \quad (2a)$$

Now OO′ gives the extension of the spring along its axis owing to the straining of the small portion considered. The total extension will be the sum of the quantities such as OO′ for the whole length of material in the helix, and can be obtained by writing the total length of wire instead of l in (2a). In the case of a close-coiled spring the total length will be given with sufficient accuracy by multiplying the mean circumference of the helix by the number of complete turns N.

$$\text{Length of wire in helix} = 2\pi RN. \quad \ldots\ldots \quad (3)$$

Hence,

$$\text{Total extension of spring} = \frac{2PR^2}{C\pi r^4} \cdot 2\pi RN$$

$$= \frac{4PR^3N}{Cr^4} \quad \ldots\ldots \quad (4)$$

$$= \frac{8PD^3N}{Cd^4}, \quad \ldots\ldots \quad (5)$$

where
D = mean diameter of helix, inches;
d = diameter of wire, inches;
P = load applied, in lb.;
C = the modulus of rigidity, lb. per square inch;
N = number of complete coils.

The result shows, as had been anticipated, that the extension is proportional to the load applied.

An equation connecting the shearing stress with the extension may be obtained from (4). Thus,

$$\text{Total extension of spring} = \frac{4PR^3N}{Cr^4}$$

$$= \frac{4R^2N}{Cr^4} \cdot PR.$$

Now $$\mathrm{PR}=\mathrm{T}=\frac{p_t\pi r^3}{2}\ \text{(p. 253).}$$

Hence, $$\text{Total extension}=\frac{4\mathrm{R}^2\mathrm{N}}{\mathrm{C}r^4}\cdot\frac{p_t\pi r^3}{2}$$

$$=\frac{2\pi\mathrm{R}^2\mathrm{N}}{\mathrm{C}r}\cdot p_t \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

This result enables the maximum extension to be found for a given spring when a given safe shear stress p_t lb. per square inch must not be exceeded.

Beginning with no load on the spring, the gradual application of a load P lb., producing an extension ϵ inches, will require the performance of a quantity of work given by (see p. 325)

$$\text{Work done}=\text{average force}\times\epsilon$$

$$=\tfrac{1}{2}\mathrm{P}\times\epsilon.$$

Inserting the value of ϵ given in (6), we have

$$\text{Work done}=\mathrm{P}\times\frac{\pi\mathrm{R}^2\mathrm{N}}{\mathrm{C}r}p_t$$

$$=\mathrm{PR}\times\frac{\pi\mathrm{RN}}{\mathrm{C}r}p_t$$

$$=\frac{p_t\pi r^3}{2}\times\frac{\pi\mathrm{RN}}{\mathrm{C}r}p_t$$

$$=\frac{p_t^2}{2}\cdot\frac{\pi^2r^2\mathrm{RN}}{\mathrm{C}}\ \text{inch-lb.}\quad\ldots\ldots\ldots\ldots(7)$$

This work is stored in the extended spring, and represents the energy which can be given out when the spring is recovering its original length, on the assumption of perfect elastic qualities.

The above formulae, being based on those for a shaft of round section, should be used only for helical springs made of round wire. A formula which may be used for the extension of a spring of square section, having sides equal to s inches, is

$$\text{Extension of spring}=\frac{44\mathrm{PR}^3\mathrm{N}}{\mathrm{C}s^4}\quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Helical spring under torsion. Helical springs are loaded occasionally under torsion in the manner indicated in Fig. 286, where a spring AB is subjected to equal opposing couples by means of forces applied to arms attached to the ends of the spring. It is evident that the material of the coil is subjected to bending and that the torque produced by the couples is balanced at any cross section of the wire by the moment of resistance of that section to bending. The neutral axis of any cross section will be parallel to the axis of the helix (Fig. 287). Further, the change of curvature of the helix produced by the application of the torque will follow the same law as that for a beam (p. 166).

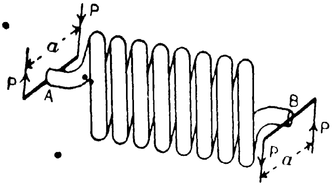


FIG. 286.—Helical spring under torsion.

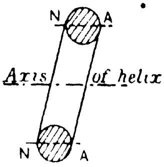


FIG. 287.

Let $T = Pa =$ torque applied, lb.-inches;
$R_1 =$ initial mean radius of helix, inches;
$R_2 =$ final mean radius of helix, inches;
$N_1 =$ initial number of complete coils in helix;
$N_2 =$ final number of complete coils in helix;
$L =$ length of wire in helix, inches;
$I_{NA} =$ moment of inertia of section of wire, inch units.

Then
$$\text{Initial curvature} = \frac{1}{R_1}.$$
$$\text{Final curvature} = \frac{1}{R_2}.$$

Suppose that the tendency is to increase the number of coils, then R_2 will be less than R_1.

$$\text{Change of curvature produced by } T = \frac{1}{R_2} - \frac{1}{R_1}.$$

By use of the equation,
$$\text{Change of curvature} = \frac{M}{EI_{NA}} \text{ (p. 166)},$$

we have
$$\frac{1}{R_2} - \frac{1}{R_1} = \frac{T}{EI_{NA}}. \qquad (1)$$

Again, assuming that the coils lie fairly close together, we have for the length of the helix,
$$L = 2\pi R_1 N_1$$
$$= 2\pi R_2 N_2.$$

Hence,
$$\frac{1}{R_1} = \frac{2\pi N_1}{L},$$
$$\frac{1}{R_2} = \frac{2\pi N_2}{L},$$

Substituting these values in (1) gives

$$\frac{2\pi N_2}{L} - \frac{2\pi N_1}{L} = \frac{T}{EI_{NA}}$$

$$N_2 - N_1 = \frac{TL}{2\pi EI_{NA}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This expression gives the angle as a fraction of a revolution through which B will rotate relative to A when the torque is applied (Fig 286). To obtain the angle of twist in degrees we have

$$\text{Angle of twist} = 360\frac{TL}{2\pi EI_{NA}}$$

$$= 180\frac{TL}{\pi EI_{NA}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

It will be noted from this result that the angle of twist is proportional to the torque, a property which leads to the use of springs of this type in certain cases, for example, the hair spring controlling the escapement of a chronometer. The use of such a spring permits the balance wheel to alter its angle of swing somewhat without altering the time in which it vibrates. The same kind of spring is often used for controlling the movement of the drum in engine indicators, as its property produces a more even stretching of the string driving the drum, and in consequence a less erratic distortion of the diagram drawn on the paper surrounding the drum.

The maximum torque which may be applied without exceeding a stated stress, f, may be found as in a beam (p. 146) from

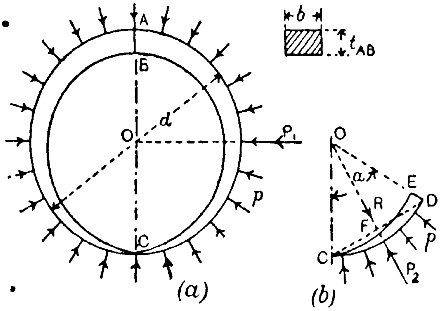
$$T = \frac{f}{m} I_{NA} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

These results may be applied to helical springs under torsion and made of wire having circular, square or rectangular sections.

Piston rings. Spring rings are often used for the purpose of the prevention of leakage past the piston in steam, gas and oil engines. A common way of making spring rings of moderate size is to turn a ring of uniform section, making the diameter somewhat larger than that of the cylinder. A piece is then cut out of the ring sufficient to allow the ring to be sprung into the cylinder, when the ends will come together. Cast iron is often used as the material. In this method of manufacture, the ring does not take a truly circular form when sprung to the diameter of the cylinder, and does not exert a uniform pressure all round the cylinder wall. To secure uniformity in the pressure and a truly circular shape, the thickness of the ring must be varied. The breadth will of course be uniform, as the ring fits accurately a groove turned in the piston.

Fig. 288 (*a*) shows a piston ring of varying thickness; the split is situated at C, and the ring as drawn has been sprung into a cylinder so that the gap at C is closed. The ring is subjected to a uniform radial pressure as shown.

Let d = the diameter of the cylinder, inches;
p = the pressure in lb. per square inch of rubbing surface;
b = the breadth of the ring, inches;
t_{AB} = the thickness of the ring at AB, diametrically opposite C, in inches.



(*a*) (*b*)

FIG. 288.—Piston ring giving uniform bearing pressure.

The half ring on the right-hand side between A and C is under similar conditions of loading to those of a boiler shell (p. 95). Hence, we may write for the resultant force on it:

$$P_1 = pd \times b \text{ lb.}$$

This force will produce a bending moment on AB of amount

$$M_{AB} = P_1 \times \tfrac{1}{2}d$$
$$= \tfrac{1}{2}pd^2b.$$

The relation of the maximum stress f at AB and the thickness of the ring there will be given by

$$M_{AB} = \frac{fbt^2_{AB}}{6} \text{ (see p. 152)},$$

$$\tfrac{1}{2}pd^2b = \frac{fbt^2_{AB}}{6},$$

$$t^2_{AB} = \frac{3pd^2}{f},$$

$$t_{AB} = d\sqrt{3}\sqrt{\frac{p}{f}}. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

The relation between the thickness at any other section and that at AB may be found from the consideration that the ring is to be circular both before and after springing it into the cylinder. Hence the change of curvature all round it will be uniform. Now,

$$\text{Change of curvature} = \frac{M}{EI},$$

and, since E is constant for a given material, it follows that for uniform change in curvature

$$\frac{M}{I} = \text{a constant.} \qquad \ldots\ldots \qquad (2)$$

To obtain the bending moment at any section such as DE, consider the portion of the ring lying between C and DE (Fig. 288 (*b*)). Join CD, and let the angle COD be α. The resultant pressure P_2 acting on the arc CD may be found in the following way. A solid piece of the same breadth of the ring, viz. *b*, bounded by the chord and arc CD will be in equilibrium if subjected to hydrostatic stress *p*. The resultant pressure R on the chord produced by the hydrostatic stress is

$$R = p \times CD \times b,$$

and this must be equal and opposite to P_2. Hence,

$$P_2 = pb \times CD = 2pb \times DF.$$

Again, $$M_{DE} = P_2 \times DF = 2pb \times DF^2.$$

Also, $$DF = DO \sin \tfrac{1}{2}\alpha = \tfrac{1}{2}d \sin \tfrac{1}{2}\alpha;$$

$$\therefore\ M_{DE} = 2pb\frac{d^2}{4}\sin^2\frac{\alpha}{2}$$

$$= \tfrac{1}{2}pbd^2 \sin^2\frac{\alpha}{2}. \qquad \ldots\ldots\ldots\ldots \qquad (3)$$

Also, $$I = \frac{bt^3_{DE}}{12}. \qquad \ldots\ldots\ldots\ldots \qquad (4)$$

Hence, substituting the values of (3) and (4) in (2), we have

$$\frac{\tfrac{1}{2}pbd^2 \sin^2\frac{\alpha}{2}}{\frac{bt^3_{DE}}{12}} = \text{a constant,}$$

or, since *p*, *b* and *d* are constant for a given ring under a given pressure,

$$\frac{\sin^2\frac{\alpha}{2}}{t^3_{DE}} = \text{a constant.} \qquad \ldots\ldots\ldots\ldots \qquad (5)$$

For the section AB, α is 180°, and sin $\frac{1}{2}\alpha$ will be unity. Hence,

$$\frac{\sin^2\frac{\alpha}{2}}{t^3_{DL}} = \frac{\sin^2 90^\circ}{t^3_{AB}} = \frac{1}{t^3_{AB}};$$

$$\therefore\ t^3_{DL} = t^3_{AB}\sin^2\frac{\alpha}{2},$$

$$t_{DL} = t_{AB}\left(\sin\frac{\alpha}{2}\right)^{\frac{2}{3}} \dots\dots\dots\dots\dots\dots \ (6)$$

This result enables the thickness of any section to be calculated after first having determined the thickness at AB.

Carriage spring. Carriage springs are constructed of a number of plates of gradually diminishing length, clamped together at the

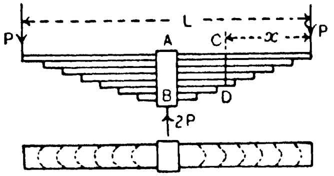


FIG. 289.—Carriage spring.

middle and loaded as shown in Fig. 289. Generally the strips have the same breadth and thickness. The material will be under bending.

Let P = the load in lb., applied at each end ;
L = the distance between the loads, inches ;
N = the number of strips ;
b = the breadth of each strip, inches ;
t = the thickness of each strip, inches.

The maximum bending moment will occur at the middle section AB, and will be given by

$$M_{AB} = \tfrac{1}{2}PL \text{ lb.-inches.}$$

This bending moment will be balanced by the total moment of resistance obtained by adding together the moments of resistance of all the strips. Assuming that each strip touches the strip immediately above it throughout its whole length, both before and after loading, it follows that all the strips will experience equal changes in curvature on the spring being loaded. Considering the curvature at AB, we have

$$\text{Change of curvature of any strip} = \frac{\text{bending moment on strip}}{\text{EI}}$$

$$= \frac{\text{moment of resistance of strip}}{\text{EI}}$$

$$= \text{a constant for all the strips.}$$

Hence, as E and I are both constant, it follows that all the strips have equal moments of resistance.

Let f = maximum stress on any strip at the section AB, lb. per square inch.

Then, Moment of resistance of each strip $= \dfrac{fbt^2}{6}$,

Total moment of resistance at AB $= N\dfrac{fbt^2}{6}$ lb.-inches.

Hence,

$$M_{AB} = N\frac{fbt^2}{6}, \quad \ldots\ldots\ldots\ldots(1)$$

$$\tfrac{1}{2}PL = N\frac{fbt^2}{6},$$

$$P = \tfrac{1}{3}N\frac{fbt^2}{L}, \quad \ldots\ldots\ldots\ldots(2)$$

or

$$f = 3\frac{PL}{Nbt^2}. \quad \ldots\ldots\ldots\ldots(3)$$

The profile of the spring in the elevation may be arranged so as to secure that this value of the maximum stress on any strip shall be constant throughout its length. Considering any section CD, let the number of strips be N_{CD}. Then, from (1),

$$M_{CD} = N_{CD}\frac{fbt^2}{6},$$

or

$$Px = N_{CD}\frac{fbt^2}{6}.$$

If f is constant, the only variables in this expression will be x and N_{CD}. Hence,

$$N_{CD} \propto x, \quad \ldots\ldots\ldots\ldots(4)$$

or, the number of strips and hence the depth of the spring vary as the distance from the end. The profile in the elevation will therefore be triangular (Fig. 290). As this is an awkward shape to

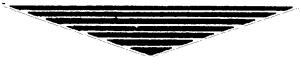


FIG. 290.—Ideal profile of a carriage spring.

produce, the ends of the strips are shaped usually as shown dotted in plan in Fig. 289, which produces practically the same result.

The deflection of the spring may be calculated in the following manner:

$$\text{Change of curvature of any strip} = \frac{\text{its moment of resistance}}{\text{EI}}$$

$$= \frac{\dfrac{f}{\frac{1}{2}t}\text{I}}{\text{EI}}$$

$$= \frac{2f}{\text{E}t}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

As f is constant throughout the length of the strip, the change of curvature throughout will be uniform. Supposing the strips to be straight at first, each strip will bend into the arc of a circle when the spring is loaded. The conditions as regards any one strip might be attained by subjecting that strip separately to a uniform bending moment $\frac{1}{\text{N}} \cdot \frac{1}{2}\text{PL}$. Hence,

$$\frac{1}{\text{R}} = \frac{\frac{1}{\text{N}} \cdot \frac{1}{2}\text{PL}}{\text{EI}}$$

$$= \frac{\text{PL}}{2\text{NE}\dfrac{bt^3}{12}}$$

$$= \frac{6\text{PL}}{\text{NE}bt^3}. \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

Now, for a beam bent into a circular arc, the deflection is given by

$$\Delta = \frac{\text{L}^2}{8\text{R}} \text{ (p. 186).}$$

Hence,

$$\Delta = \frac{\text{L}^2}{8} \cdot \frac{6\text{PL}}{\text{NE}bt^3}$$

$$= \frac{3}{4} \cdot \frac{\text{PL}^3}{\text{NE}bt^3}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

In the above, the frictional resistances of the strips rubbing on each other has been neglected. The effect of this will be to make the spring appear to be stiffer, as evidenced by a deflection smaller than that calculated, when the load is being increased. When the load is being removed, the deflection will be found to be somewhat larger than that calculated. Of course, work will be absorbed by these frictional resistances, with the effect that any vibrations communicated to the spring by impulsive forces, or shock, will die out more rapidly than would be the case with a spring formed out of a single piece of material.

EXERCISES ON CHAPTER XI.

1. A mild-steel shaft is 6 inches diameter. If the safe shear stress allowed is 10,000 lb. per square inch, what torque may be applied?

2. Find the diameter of a solid round shaft of mild steel to transmit a torque of 12,000 lb.-inches with a safe shear stress of 9000 lb. per square inch.

3. A hollow shaft has an outside diameter of 18 inches and an inside diameter of 6 inches. Calculate the torque for a safe shear stress of 4·5 tons per square inch.

4. A solid shaft has the same weight and the same length as the shaft given in Question 3 and is made of similar material. Calculate the safe torque which may be applied. Give the value of the ratio—Torque for the hollow shaft : torque for the solid shaft.

5. What torque may be applied to a tube 3 inches in external diameter, of metal 0·125 inch thick, if the stress is not to exceed 10,000 lb. per square inch?

6. The shaft given in Question 1 is 60 feet in length. What will be the angle of twist when the maximum permissible torque is applied? Take $C = 13{,}000{,}000$ lb. per square inch.

7. Find the angle of twist for the shaft given in Question 3 when the shear stress is 4·5 tons per square inch. The shaft is 100 feet in length. Take $C = 5500$ tons per square inch.

8. What horse-power may be transmitted by a solid shaft 3 inches in diameter at 120 revolutions per minute? The shear stress is 8000 lb. per square inch.

9. What diameter of steel shaft is required in order to transmit 20 horse-power at 250 revolutions per minute?

10. AB and BC are two sections of a body meeting at 90°. Normal pull stresses of 5 and 4 tons per square inch act on AB and BC respectively. Shearing stresses of 3 tons per square inch act from A towards B and from C towards B. Find the principal stresses and the principal axes of stress. Draw a diagram showing the axes and stresses.

11. Answer Question 10 if the normal stress of 5 tons per square inch on AB is a push.

12. A mild-steel shaft 3 inches in diameter has a bending moment of 4000 lb.-inches together with a twisting moment of 6000 lb.-inches. Calculate the following: (*a*) The equivalent torque according to Rankine; (*b*) the equivalent torque on the maximum shear stress hypothesis; (*c*) the maximum and minimum principal stresses; (*d*) the maximum shearing stress.

13. Supposing that a constant bending moment of 4000 lb.-inches be applied to a shaft 3 inches in diameter, what torque may be applied if the maximum shear stress is limited to 10,000 lb. per square inch?

14. A cylindrical boiler is 7 feet in diameter and is made of plates 0·5 inch thick. The steam pressure is 100 lb. per square inch. (*a*) Find the stresses on longitudinal and circumferential sections; also the stresses on sections at 30, 45 and 60 degrees to the axis. (*b*) What is the maximum shear stress on the plate?

15. A helical spring is made of round steel wire 0·25 inch in diameter. The mean radius of the helix is 1·25 inches; number of complete turns 120; the spring is close-coiled. Take C = 12,000,000 lb. per square inch, and find the pull required to extend the spring one inch.

16. A helical spring, material of circular section, has to extend 1 inch with a pull of 50 lb. The mean radius of the helix is 2 inches, and the length of the helical part of the spring is one foot. Assume that the coils are close together, and find the diameter of the wire. C = 12,000,000 lb. per square inch.

17. Suppose that the spring given in Question 15 is put under torsion by couples applied at its ends. Find the torque required to twist the spring through one radian. E = 30,000,000 lb. per square inch.

18. A helical spring is made of steel of square section, 0·3 inch edge, close-coiled. The mean radius of the helix is one inch, and there are 20 complete turns. Take C = 12,000,000 lb. per square inch, and find the pull required to extend the spring one inch.

19. A piston ring for a cylinder 24 inches in diameter has to give a uniform pressure of 2 lb. per square inch of rubbing surface. Find the maximum thickness of the ring if the stress is not to exceed 6000 lb. per square inch. Find also the thickness at a section 90° from the split.

20. A carriage spring of length 30 inches is made of steel plates 2·5 inches wide by 0·25 inch thick. Find the number of plates required to carry a central load of 800 lb. if the maximum stress is limited to 12 tons per square inch. Find the deflection under this load if E = 30,000,000 lb. per square inch.

21. A load is applied to the crank fixed to a wrought-iron shaft 6 inches diameter and 20 feet long, which twists the ends to the extent of 2°; assuming the modulus of transverse elasticity (or coefficient of rigidity) to be 4000 tons per square inch, what is the extreme fibre-stress? (I.C.E.)

22. A closely-coiled spiral spring has 24 coils; the mean diameter of the coil is 4 inches and the diameter of the wire from which the spring is made is 0·5 inch. Determine the axial load which will elongate this spring 6 inches if the modulus of rigidity is 12,000,000 lb. per square inch. (B.E.)

23. A hollow steel shaft is to be used to transmit 1000 H.P. at 90 revolutions per minute; the internal diameter of the shaft is to be $\frac{3}{4}$ of the external diameter. The maximum twisting moment exceeds the mean by 20 per cent. If the maximum intensity of shear stress is not to exceed 4·5 tons per square inch, find the external diameter of the shaft. (L.U.)

24. At a certain point in a loaded body the principal stresses are a tension of 5 tons per square inch and a pressure of 3 tons per square inch, the latter acting in a horizontal direction. Another load is then applied to the body, giving rise to a second stress system, the principal components of which at the same point are a tension of 3 tons per square inch and a pressure of 4 tons per square inch, the latter acting at an angle of 40° to the horizontal. Find the magnitudes and directions of the principal stresses of the resultant stress system. There is no stress at right angles to the plane of the paper. (B.E.)

CHAPTER XII.

EARTH PRESSURE.

Earth pressure. Questions regarding the pressure of earth enter into the design of foundations and of retaining walls for holding back earth. It is not possible to obtain exact solutions owing to the variable properties of the material, and also to the fact that the properties are altered very considerably by the presence or absence of water mixed with the earth.

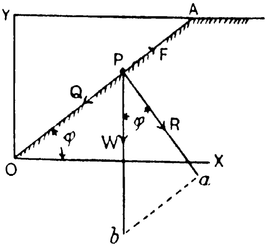


FIG. 291.—Natural slope of earth.

Referring to Fig. 291, if a mass of earth be cut to a vertical face OY, it will weather down by breaking away of the earth until a permanent surface OA is attained ultimately. Let ϕ be the angle which OA makes with the horizontal, and consider a particle of earth resting on the slope at P. Its weight W may be resolved into two forces, one R perpendicular to the slope and another Q acting down the slope. Balance is obtained by the force of friction F acting up the slope, F being equal to Q. Defining the coefficient of friction μ as the ratio of F and R when sliding is just on the point of taking place (p. 353), *i.e.*

$$\mu = \frac{F}{R},$$

the triangle of forces P*ab* gives

$$\frac{Q}{R} = \frac{F}{R} = \frac{ab}{Pa} = \tan \phi.$$

Hence, $$\mu = \tan \phi. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The coefficient of friction may range from 0·25 to 1·0 for earth sliding on earth, ϕ ranging from 14 to 45 degrees.

Rankine's theory of earth pressure. The effect of the weight W resting on the slope OA is to produce a stress on OA having an angle of obliquity equal to ϕ when sliding is just possible. ϕ may be called the **natural angle of repose** of the earth; sliding will not occur if the angle of slope has any value less than ϕ.

In the Rankine theory, it is asumed that **the shearing effects at any section in the earth follow the ordinary frictional laws, and that the obliquity of stress on any section of the earth cannot exceed the natural angle of repose of the earth.**

Referring to Fig. 292, AB is the horizontal earth surface and *abcd* is a small rectangular block of earth having its top and bottom faces horizontal. Let the area of the top face be one square foot, let y be the depth below the surface and let w be the weight of the earth in lb. per cubic foot. The stress p_1 on the top face will be produced by the weight of the superincumbent column of earth, and will be given by $p_1 = wy$ lb. per square foot. ... .(2)

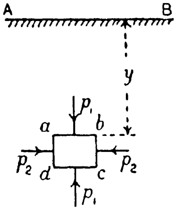


FIG 292.—Conjugate stresses, earth surface level.

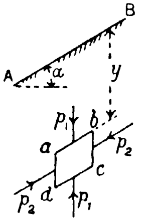


FIG. 293 —Conjugate stresses, earth surface sloping.

The stress p_2 acting on the vertical faces must be determined from the relation mentioned above, viz. ϕ must not be exceeded on any section of the block.

In Fig. 293 the earth surface is sloping at an angle α to the horizontal, and *ab* and *cd* are at the same slope, *bc* and *ad* being vertical. The stress p_1 will be given by

$$p_1 = \frac{wy}{\text{area of top face } ab} = wy \cos \alpha \text{ lb. per square foot. ......(3)}$$

It is evident that p_1, p_1 acting on *ab* and *cd* respectively balance each other, neglecting the weight of the block; hence p_2 and p_2 must balance independently, and must therefore act in the same straight line. It therefore follows that p_2 must be parallel to *ab*. p_1 parallel to *bc* and p_2 parallel to *ab* are called **conjugate stresses.** p_2 is determined by the same consideration as before, viz. ϕ must not be exceeded.

In Fig. 294, OA and OB represent principal stresses p_1 and p_2 respectively, and the construction is shown for obtaining the stress p on a section OK (p. 261). ON is perpendicular to OK, NP and MP are parallel respectively to the principal axes of stress OB and OA, and PO is the stress on OK. P lies always on the circumference of the circle described on MN as diameter. The angle of obliquity of p as shown is PON; the maximum angle of obliquity will occur when OP is tangential to the circle NPM as shown by OT. The angle COT will correspond with the value of ϕ in earthwork problems. From Fig. 294, we have

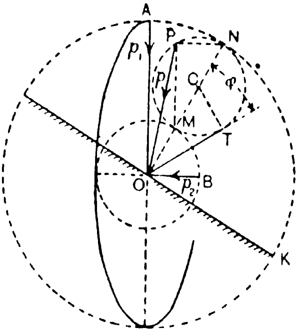


FIG. 294.—Maximum angle of obliquity of stress.

$$\sin\phi = \frac{CT}{OC}$$

$$= \frac{\frac{1}{2}(p_1 - p_2)}{\frac{1}{2}(p_1 + p_2)}$$

$$= \frac{p_1 - p_2}{p_1 + p_2} \quad \ldots \ldots \quad (4)$$

Pressure on retaining walls by Rankine's theory. The foregoing principles may be applied to give a simple graphical solution for the earth pressure on retaining walls. In Fig. 295 XY is the vertical earth face of a retaining wall, the earth surface being horizontal and level with the top of the wall. Produce the horizontal base of the wall and select a point O on it. Draw OA vertically and make it equal to $p_1 = wH$ lb. per square foot. Draw OT, making the angle ϕ with OA. Find, by trial, a circle having its centre C in OA, to pass through A and to touch OT. This circle will cut OA in B, and will correspond to the circle NPM in Fig. 294.

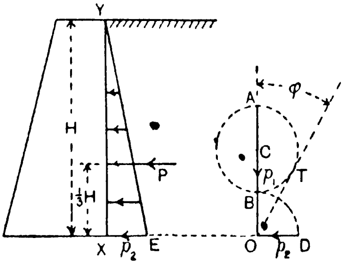


FIG. 295.—Earth pressure on a wall, earth surface level, by Rankine's theory.

Make OD equal to OB, and DO will represent the other principal stress p_2. The stress p_2 will be transmitted horizontally through the earth along OX, and an equal stress p_2 will be produced on the wall at X. Make XE equal to p_2, and join YE. The stress diagram for the face of the wall will be YXE. The average stress will be $\frac{1}{2}p_2$, and if one foot length of wall be taken, the total pressure P will be

$$P = \tfrac{1}{2}p_2 H \text{ lb.}$$

P will act at a point $\frac{1}{3}$H from the foot of the wall.

Fig. 296 illustrates the procedure if the earth surface is surcharged, or inclined to the horizontal, at an angle α. Draw XO parallel to the earth surface. Draw OA vertically, and make OA equal to $p_1 = wH \cos \alpha$ (p. 280). Draw OM perpendicular to XO, and draw also OT, making the angle ϕ with OM. Find, by trial, a circle having its centre C in OM, to pass through A and to touch OT. This circle cuts OM in M and B, and will correspond to the circle NPM in Fig. 294. Join MA and BA: these will correspond with NP and PM in Fig. 294; hence the principal axes of stress will be parallel to MA and BA respectively, and the principal stresses will be represented by OM and OB respectively. Draw OD and OE parallel respectively to BA and AM; make OD equal to OM and OE equal to OB. The ellipse of stress passes through D and E, and cuts XO produced in F. Hence FO is the value of p_2. The quarter DFE alone of the ellipse need be drawn.

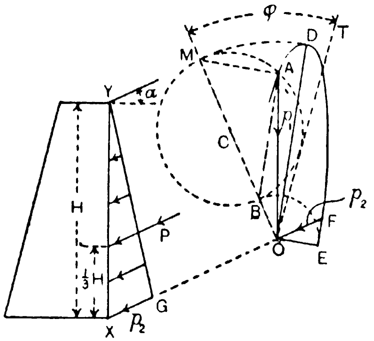


FIG. 296.—Earth pressure on a wall, earth surface surcharged, by Rankine's theory.

Draw the stress diagram for the wall by making XG equal to p_2 and joining GY. The average stress will be $\frac{1}{2}p_2$, and for one foot length of wall we have

$$P = \tfrac{1}{2}p_2 H.$$

P acts parallel to the earth surface, and is at a height $\frac{1}{3}$H above the foot of the wall.

If the earth surface be not surcharged, a simple formula may be obtained for the stress on the wall at any depth:

Let $p_1 = wh$ = the earth pressure on a horizontal foot at a depth h feet.

p_2 = the pressure on the wall at the same depth.

Then, from equation (4), p. 281,

$$\sin\phi = \frac{p_1 - p_2}{p_1 + p_2}.$$

And

$$\frac{1+\sin\phi}{1-\sin\phi} = \frac{2p_1}{2p_2} = \frac{p_1}{p_2},$$

$$\therefore p_2 = \left(\frac{1-\sin\phi}{1+\sin\phi}\right)p_1$$

$$= \frac{1-\sin\phi}{1+\sin\phi}wh.$$

If the earth surface is surcharged at an angle to the horizontal equal to ϕ, then $p_1 = wh\cos\phi$, and it may be shown that the other conjugate stress, p_2, is equal to p_1 and acts on the wall at an angle ϕ to the horizontal.

If the angle of surcharge is α, the following equation may be used in order to find the value of p_2:

$$p_2 = wh\cos\alpha\left\{\frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}\right\}.$$

Wedge theory of earth pressure. Let AB (Fig. 297) be the vertical face of a retaining wall, and let AC be the surface of the earth; also let BC be a plane making the angle ϕ with the horizontal. Considering the wedge of earth BAC, imagine that its particles are cemented together so as to form a solid body. Under this condition, the wedge would just rest without slipping on the inclined plane BC if the wall were removed; in other words, so far as the wedge BAC is concerned, there is no pressure on the wall. Again, considering an indefinitely thin wedge ABA′, at rest between the plane BA′ and the wall, as its weight is negligible, there will be no pressure on the wall. Hence the pressure on the wall, being zero for the inclined planes BC

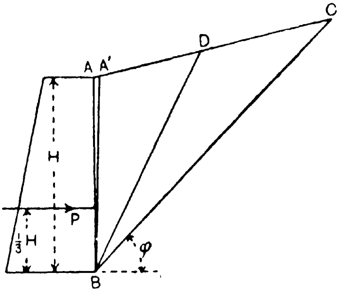


FIG. 297.—Wedge theory of earth pressure on a wall.

and BA′, will attain a maximum value for some plane such as BD lying between BC and BA′. If the wall were removed, the earth would break away at once along the section BD and the wedge ABD would fall, subsequently weathering would remove the wedge DBC. BD is called the **plane of rupture**; the force acting on the wall may be obtained by considering the weight of ABD and the reaction of the earth lying under the section BD.

The force P which the earth communicates to the wall may be assumed to be horizontal, thus ignoring any friction between the vertical face of the wall and the earth; also P may be assumed to act at $\frac{1}{3}$H from the base of the wall (Fig. 298). W is the weight of the wedge ABD, and is calculated by taking account of one foot length of the wall. The reaction Q of the earth underneath BD acts at the angle ϕ to the normal OE to the section BD. These three forces meet at O and are in equilibrium. If θ is the angle DBC, the angle between the lines of W produced and Q will be equal to θ. *abc* is the triangle of forces for W, P, and Q, from which we have

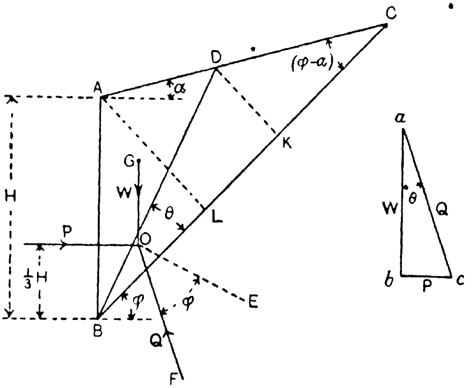


FIG. 298.—Equilibrium of the wedge ABD.

$$\frac{P}{W} = \tan\theta,$$

or $$P = W\tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Draw DK and AL, each perpendicular to BC. Then, if w is the weight of the earth in lb. per cubic foot,

$$\begin{aligned} W &= \text{area ABD} \times w \\ &= (\text{area BAC} - \text{area BDC})w \\ &= (\tfrac{1}{2}\text{BC}.\text{AL} - \tfrac{1}{2}\text{BC}.\text{DK})w \\ &= \tfrac{1}{2}w.\text{BC}(\text{AL} - \text{DK}). \end{aligned}$$

Let DK be called x. Then

$$W = \tfrac{1}{2}wBC(AL - x). \quad \text{.......................(2)}$$

Also, $$\tan\theta = \frac{DK}{BK} = \frac{x}{BC - KC};$$

and $$KC = DK\cot(\phi - \alpha)$$
$$= x\cot(\phi - \alpha).$$

Hence, $$\tan\theta = \frac{x}{BC - x\cot(\phi - \alpha)}. \quad \text{...............(3)}$$

Substitute the values of (2) and (3) in (1), giving

$$P = \tfrac{1}{2}w.BC.\frac{(AL - x)x}{BC - x\cot(\phi - \alpha)}. \quad \text{.........(4)}$$

The whole of the quantities involved in this expression, with the exception of x, are constant for a given wall, the earth having a known value for ϕ and a given slope at the surface. The maximum value of P may be found by differentiating the right-hand side and equating the result to zero. Thus,

$$\frac{d}{dx}\left\{\frac{AL.x - x^2}{BC - x\cot(\phi - \alpha)}\right\}$$
$$= \frac{(AL - 2x)\{BC - x\cot(\phi - \alpha)\} + (AL.x - x^2)\cot(\phi - \alpha)}{\{BC - x\cot(\phi - \alpha)\}^2}.$$

This will be zero when the numerator is zero. Hence,

$$AL.BC - AL.x\cot(\phi - \alpha) - 2x.BC + 2x^2\cot(\phi - \alpha)$$
$$- AL.x\cot(\phi - \alpha) + x^2\cot(\phi - \alpha),$$
$$AL.BC - 2x.BC = -x^2\cot(\phi - \alpha),$$
$$AL.BC - xBC = x.BC - x^2\cot(\phi - \alpha)$$
$$= x\{BC - x\cot(\phi - \alpha)\}.$$

By reference to Fig. 298, it will be noticed that this may be written

$$AL.BC - x.BC = x.BK,$$

or $$2\triangle ABC - 2\triangle BDC = 2\triangle DKB,$$

or $$\triangle BAD = \triangle DKB. \quad \text{..................(5)}$$

The condition for the maximum value of P is therefore that the area of the triangle BAD should be equal to the area of the triangle DKB.

From (1), $$P = W\tan\theta$$
$$= w.\triangle BAD.\tan\theta$$
$$= w.\triangle BKD.\tan\theta$$
$$= w.\tfrac{1}{2}BK.x.\frac{x}{BK}$$
$$= \tfrac{1}{2}wx^2. \quad \text{.................................(6)}$$

Graphical solutions by the wedge theory. The following geometrical constructions may be used for the determination of x:

CASE 1.—**Earth surface level with the top of the wall.** Reference is made to Fig. 299. Draw BC making the angle ϕ with the horizontal. Draw BE perpendicular to BC and cutting the earth surface produced in E. Make EF equal to EA. Then BF is equal to x. Draw FD parallel to BC and join BD; BD will be the plane of rupture. P will be found by measuring BF $= x$ to the same scale as that used in drawing the wall and inserting the value in (6). Apply P horizontally at $\frac{1}{3}$H from the base.

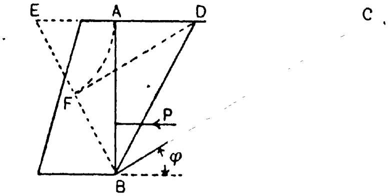


FIG. 299.—Graphical solution, wedge theory, earth surface level.

CASE 2.—**Earth surface surcharged at an angle α.** Draw BC (Fig. 300) making the angle ϕ with the horizontal. Draw BE perpendicular to BC and cutting the earth surface produced in E. On BE describe a semicircle, and draw AF perpendicular to BE. Make EG equal to EF; then BG is equal to x. Draw GD parallel to BC and join DB; DB will be the plane of rupture. Calculate the value of P and apply it as in Case 1.

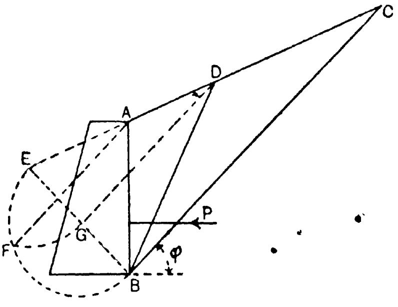


FIG. 300.—Graphical solution, wedge theory, earth surcharged.

CASE 3.—**Earth surface surcharged at the angle φ.** In Fig. 301, draw BE perpendicular to the earth surface and cutting it produced in E. Then BE is equal to *x*. P is calculated and applied as before.

CASE 4.—**Earth surface surcharged at an angle α and friction between the earth and the wall considered.** Draw BC (Fig. 302) at the angle φ to the horizontal to cut the earth surface in C. On BC as diameter describe a semicircle. Make AD equal to AB, and draw DE perpendicular to BC. Make BF equal to BE, and draw FG parallel to AD. Make FK equal to FG. Join BG. Then the pressure P on one foot length of the wall is equal to the weight of the prism of earth having an area in square feet equal to the area FGK and a length of one foot. P will act at $\frac{1}{3}$H from the base of the wall, and will be inclined at an angle φ to the horizontal. The plane of rupture is BG.

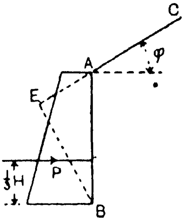


FIG. 301.—Earth surface surcharged at φ, wedge theory.

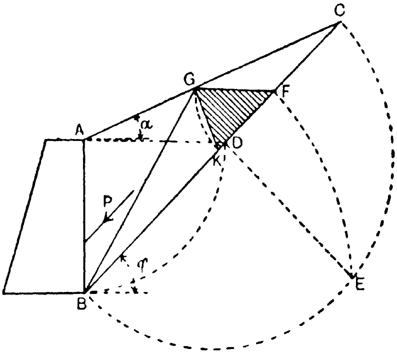


FIG. 302.—Wedge theory; solution when friction of earth on wall is taken account of.

It is assumed in the last case that the value of φ is the same for earth sliding upon earth and for earth sliding upon masonry.

Distribution of normal pressure on the base of the wall. Having found P by application of one of the above methods, the resultant pressure on the base of the wall may be found in the manner shown in Fig. 303. W is the weight of one foot length of the wall, acting vertically through its centre of gravity G. P and W intersect at O, and R is their resultant. For stability, R should pass within the middle third DE of the base of the wall (p. 239).

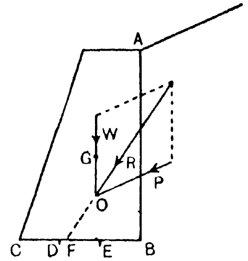


Fig. 303.—Resultant pressure on wall base.

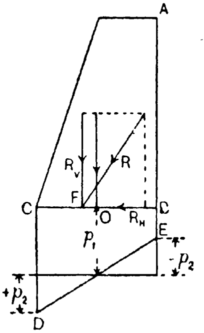


Fig. 304.—Distribution of normal stress on the wall base.

In Fig. 304, F is the point in which R intersects the base of the wall, and O is the middle of the base. R may be resolved into two forces, R_V and R_H; the latter produces shearing stress on the base, having a somewhat indefinite distribution; the former produces normal stress. To determine the latter, shift R_V from F to O, and apply a compensating couple $R_V \times FO = M$. R_V acting at O will produce a uniform normal stress p_1 of value given by

$$p_1 = \frac{R_V}{\text{area of wall base}} = \frac{R_V}{BC} \text{ lb. per square foot.}$$

M will produce a stress which will vary from a push p_2 at C to an equal pull p_2 at B. These may be found from

$$M = \frac{p_2}{m} I,$$

where m is $\frac{1}{2}BC$ and I is the moment of inertia of 1 foot length of the wall base taken with reference to the axis passing through O and perpendicular to the plane of the paper.

$$I = \frac{1 \times BC^3}{12}.$$

Hence,
$$R_V \times FO = \frac{p_2}{\frac{1}{2}BC} \cdot \frac{BC^3}{12} = \frac{p_2 BC^2}{6},$$

$$p_2 = \frac{6R_V . FO}{BC^2}.$$

A stress diagram CBED is drawn in Fig. 304, in which

$$CD = p_1 + p_2 \quad \text{and} \quad BE = p_1 - p_2.$$

Rankine's theory applied to foundations. In Fig. 305 is shown a wall the weight of which is supported by a vertical reaction coming from the earth on which it rests. Consider one foot length of the wall, and find its weight W lb. The vertical stress p_1 on the earth will be

$$p_1 = \frac{W}{\text{area of base AB}}$$

$$= \frac{W}{AB} \text{ lb. per square foot.}$$

The horizontal stress p_2 acting on the vertical faces of a small rectangular block of earth immediately under the foot of the wall will be found from the consideration that the angle ϕ must not be exceeded by the obliquity of the stress. Make OC to represent p_1; draw OT making the angle ϕ with OC, find by trial a circle CTF having its centre E in OC, passing through C and touching OT; make OG equal to OF; then GO is equal to p_2. Part of the ellipse of stress has been drawn, although this is not required in the construction. The stress p_2 is transmitted horizontally through the earth, and will act on the vertical faces of a small rectangular block of earth at K. There will be a stress p_3 acting on the horizontal faces of this block and caused by the weight of the column of earth resting on the top face of the block. p_3 is found by a second application of the same construction. Make KH equal to p_2; draw KL making the angle ϕ with KH; the circle HLM has its centre in KH, passes through H and touches KL. Make KN equal to KM, when NK will be equal to p_3.

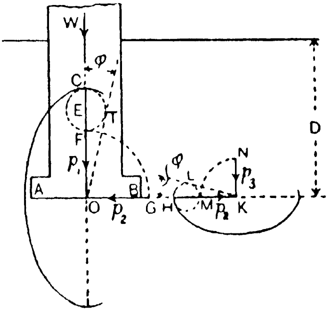


FIG. 305.—Rankine's theory applied to foundations.

Let D be the depth of the foot of the wall below the earth surface, and let w be the weight of the earth in lb. per cubic foot. Then

$$p_3 = wD \text{ lb. per square foot;}$$

$$\therefore \ D = \frac{p_3}{w} \text{ feet.}$$

This result gives the minimum depth of the foundation, and represents the case of the earth surrounding the wall being just on the point of heaving up. The actual depth of the foundation may be obtained by application of a factor of safety.

D may be found by calculation from equation (4), p. 281. Thus,

$$\sin\phi=\frac{p_1-p_2}{p_1+p_2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \frac{1+\sin\phi}{1-\sin\phi}=\frac{2p_1}{2p_2}=\frac{p_1}{p_2},$$

$$p_2=p_1\left(\frac{1-\sin\phi}{1+\sin\phi}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also,

$$\sin\phi=\frac{p_2-p_3}{p_2+p_3};$$

$$\therefore p_3=p_2\left(\frac{1-\sin\phi}{1+\sin\phi}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$=p_1\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2, \text{ from } (2);$$

Again,

$$D=\frac{p_3}{w}=\frac{p_1}{w}\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2$$

$$=\frac{W}{w.AB}\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

EXAMPLE. A wall carries a weight of 800 tons. The area of the foot of the wall is 200 square feet. Find the minimum depth of foundation if the weight of the earth is 120 lb. per cubic foot and if ϕ is 30°.

$$p_1=\frac{800}{200}=4 \text{ tons per square foot.}$$

$$D=\frac{p_1}{w}\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2$$

$$=\frac{4\times 2240}{120}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)^2$$

$$=8{\cdot}3 \text{ feet.}$$

EXERCISES ON CHAPTER XII.

1. Given principal stresses of 6 tons and 3 tons per square foot, both pushes, find the angle of greatest obliquity of stress.

2. A retaining wal. for earth, 12 feet high, has its earth face vertical. The surface of the earth is horizontal and is level with the top of the wall. Find the total force per foot length on the wall by Rankine's theory, taking the weight of the earth as 110 lb. per cubic foot and ϕ as 40°.

3. Answer Question 2 if the earth surface is surcharged at 20° to the horizontal.

4. Answer Question 2 by application of the wedge theory.

5. Answer Question 3 by application of the wedge theory.

6. Answer Question 3 by the wedge theory, taking account of the friction between the earth and the wall. It may be assumed that ϕ has the same value for earth sliding on earth and for earth sliding on masonry.

7. A masonry retaining wall for earth has its earth face vertical, and the earth is surcharged at an angle of 30° to the horizontal. The wall is 9 feet high, 2 feet broad at the top, and 5 feet broad at the base. The earth weighs 110 lb per cubic foot and ϕ is 30°. Find the total earth pressure on the wall by the wedge theory.

8. In Question 7, the masonry weighs 120 lb. per cubic foot. Find the resultant pressure on the horizontal base of the wall. Does it pass within the middle third of the base? Find the maximum and minimum normal stresses on the base, and draw a diagram showing the distribution of normal stress.

9. A wall and the load which it carries produce a stress of 3 tons per square foot on the earth underneath the wall. If the weight of earth is 110 lb per cubic foot and if ϕ is 35°, find the minimum depth of the foundation below the surface of the earth.

10. A brick wall 25 feet high, of uniform thickness and weighing 120 lb. per cubic foot, has to withstand a wind pressure of 56 lb. per square foot. What must be the thickness of the wall in order to satisfy the condition that there shall be no tension in any joint of the brickwork? (I.C.E.)

11. Concrete exerts on earth at the bottom of a trench a downward pressure of 2 tons per square foot; the earth weighs 130 lb. per cubic foot and its angle of repose (in Rankine's theory) is 30°; what is the least safe depth below the earth's natural surface of the bottom of the concrete? Why are we unable to make much practical use of the theory of earth pressure? (B.E.)

12. A concrete retaining wall is trapezoidal in cross section, 24 feet high; thickness at top, 3 feet; at base, 10 feet; the back face, which is subjected to earth pressure, being vertical. The wall is not surcharged. If the concrete weighs 140 lb. per cubic foot, the earth-filling behind the wall 125 lb. per cubic foot, and if the angle of repose of the earth is 22 degrees, investigate the stability of the wall. (B.E.)

13. Give the assumptions upon which Rankine's theory of earth pressure is based. Show that the intensity of horizontal pressure on a retaining wall at a depth d feet below the horizontal earth surface is

$$\frac{1-\sin\phi}{1+\sin\phi}wd,$$

where w is the weight of 1 cubic foot of earth and ϕ is the angle of repose of the earth. A practical rule takes the pressure as equivalent to that given by a fluid weighing 20 lb. per cubic foot. Find the angle of repose corresponding to this, assuming w equals 100 lb. per cubic foot. (L.U.)

CHAPTER XIII.

TESTING OF MATERIALS.

Wires under pull. A simple apparatus is illustrated in Fig. 306 and will enable the elastic properties of wires under pull to be studied. Two wires, A and B, are hung from the same support, which should be fixed to the wall as high as possible in order that long wires may be used. One wire, B, is permanent and carries a fixed load W_1, in order to keep it taut. The other wire, A, is that under test, and may be changed readily for another of different material. The test wire may be loaded with gradually increasing weights W. The extension is measured by means of a vernier D, clamped to the test wire and moving over a scale E, which is clamped to the permanent wire. The arrangement of two wires prevents any drooping of the support being measured as an extension of the wire.

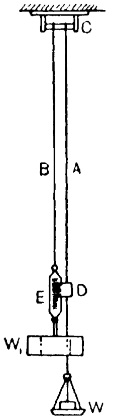


FIG. 306.—Apparatus for tensile tests on wires.

EXPT. 15.—**Elastic stretching of wires.** See that the wires are free from kinks. Measure the length L in inches from C to the vernier. Measure the diameter of the wire. State the material of the wire and also whatever is known of its treatment before it came into your hands. Apply gradually increasing loads to the wire A, and read the vernier after the application of each load. Stop the test when it becomes evident that the extensions are increasing more rapidly than the loads. Tabulate the readings thus:

TENSION TEST ON A WIRE.

Load, lb.	Vernier reading	Extension, inches.

Plot the loads in column 1 as ordinates and the corresponding extensions in column 3 as abscissae (Fig. 307). It will be found that a straight line will pass through most of the points between O and a point A, after which the line turns towards the right. The point A indicates the break-down of Hooke's law.

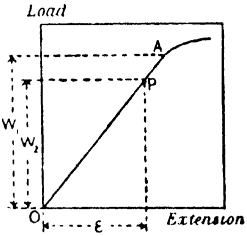


FIG. 307.—Graph of a tensile test on a wire.

Let W_1 = load in lb. at A in Fig. 307.

d = the diameter of the wire in inches.

Then,

Stress at elastic break-down

$$= \frac{W_1}{\frac{1}{4}\pi d^2} \text{ lb. per square inch.}$$

Select a point P on the straight line OA (Fig. 307), and measure W_2 and ϵ from the diagram.

Let W_2 = load in lb. at P,

ϵ = extension in inches at P,

L = length of test wire in inches.

Then, Young's modulus $= E = \frac{\text{stress}}{\text{strain}} = \frac{W_2}{\frac{1}{4}\pi d^2} \cdot \frac{L}{\epsilon}.$

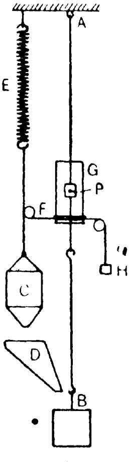


FIG. 308.—Apparatus for testing wires to rupture.

Several wires of different material should be tested in a similar manner.

In Fig. 308 is shown in outline a simple form of machine for testing wires to breaking; the machine is fitted with an arrangement whereby an autographic diagram is produced, *i.e.* a diagram is drawn by the apparatus showing the loads and corresponding extensions.

AB is the test wire, fixed at A and carrying a receptacle B at its lower end. The load is applied by means of lead shot, stored in another receptacle C, which is fitted with an orifice and a control shutter at its lower end; D is a shoot for guiding the shot into B. C is hung from a helical spring E, which is extended when C is full and shortens uniformly as the weight is removed by the shot running out of C. A cord F is attached to E, passes round a guide pulley and also two or three times round a drum G, and has a small weight H attached in order to keep it tight. A piece of paper is wrapped round G, and circumferential movements of this paper will be proportional to the load removed from C and applied to the test wire. A small guided frame carrying a pencil is attached to the test wire at P; vertical movements of the pencil will indicate the extensions of the portion of test wire between A and P. In action, a curve is drawn on the paper which shows loads horizontally and extensions vertically.

EXPT. 16.—**Tensile test to rupture.** Arrange the apparatus and fit the test wire; see that all the arrangements are working properly. Draw the lines of zero extension and zero load by rotating the drum for the first and by moving the pencil frame vertically for the second. Measure the diameter of the test wire and the length from A to P. Allow the shot to run into B until the test wire breaks. To obtain the breaking load, weigh the receptacle B together with its contents.

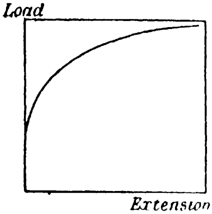


FIG. 309.—Autographic record of a test on copper wire.

Let W = breaking load in lb.,

d = diameter of the wire in inches.

Then, Breaking stress $= \frac{W}{\frac{1}{4}\pi d^2}$ lb. per square inch of original cross-sectional area.

In Fig. 309 is given a reproduction of a diagram after removal from a machine of this kind. The scale of loads may be found by placing different weights in C and observing the resulting movements of the paper on the drum. The diagram shown is for copper wire, and the point of elastic break-down may be stated roughly from it.

Experiments should be made on several wires of different materials, such as copper, brass and iron.

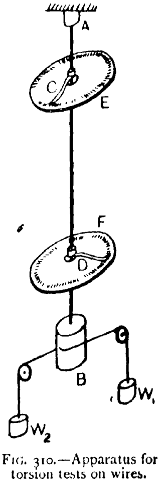


FIG. 310.—Apparatus for torsion tests on wires.

Wires under torsion. Apparatus by means of which may be measured the angle of twist produced in a wire by a given torque is illustrated in Fig. 310. AB is a test wire, firmly fixed at A to a rigid clamp and carrying a heavy cylinder at B. The cylinder serves to keep the wire tight, and also provides means of applying the torque. The torque must be applied as a couple in order to avoid bending, and is produced by means of cords wound round B; these cords pass over guide pulleys, and carry equal weights W_1 and W_2 at the ends. Pointers C and D are clamped to the wire, and move as the wire twists over fixed graduated scales E and F. The angle of twist produced in the portion CD of the wire is thus indicated.

EXPT. 17.—**Torsion test on wires.** Arrange the apparatus as shown. State the material of the wire; measure its diameter d and the length L between the pointers C and D, both in inches. Measure also the diameter D of the cylinder B, in inches. Apply gradually increasing loads, and read the scales E and F after each load is applied. Tabulate the readings.

EXPERIMENT ON TWISTING.

Load, $W_1 = W_2$, lb	Torque, W_1D, lb.-inches	Angle of twist, degrees.

Plot the torques in column 2 as ordinates and the corresponding angles of twist as abscissae. A typical diagram is given in Fig. 311, from which it will be observed that the graph is practically a straight line, indicating that the angle of twist is proportional to the torque. Select a point P on the straight line, and measure the torque T lb.-inches and the angle α from the diagram. If the diagram is plotted in degrees, convert α to radians. The value of the modulus of rigidity of the material may be calculated.

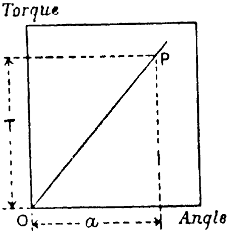


FIG. 311.—Graph of a torsion test on a wire.

Let

T = the torque, in lb.-inches;
L = the length of the wire, in inches;
R = the radius of the wire, in inches;
α = angle of twist, in radians;
C = the modulus of rigidity, in lb. per square inch.

Then, from equation (4), p. 255, we have

$$\alpha = \frac{2TL}{\pi R^4 C},$$

or

$$C = \frac{2TL}{\alpha \pi R^4}.$$

Several wires of brass, copper and steel should be tested. In each case, any information regarding the previous history of the wire should be noted.

Helical springs under pull. The extensions of a helical spring under pull may be investigated by use of the apparatus illustrated in Fig. 312. A is the spring under test; it is hung from a hook at the top of a stand. A graduated scale B is hung from the spring, and carries a hook on which loads W may be placed. The vertical movements of the scale indicate the extensions of the spring, and are read by means of a telescope at C.

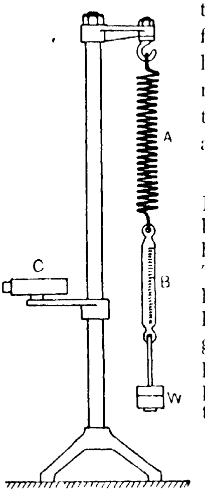


FIG. 312.—Apparatus for testing helical springs.

EXPT. 18.—**Extensions of helical springs.** Make a helical spring by coiling round a round bar, or mandril, some wire for which you have found C previously, as directed on p. 295. Test this spring under gradually increasing loads, noting the extension produced by each load. Plot loads and extensions; these should give a straight line if the extensions are proportional to the loads. Select a point on the plotted line, and read the load W lb. and the corresponding extension ϵ inches.

Let

D = the mean diameter of the helix, inches;
d = the diameter of the wire, inches;
N = the number of complete turns in the helix.

Then, from equation (5), p. 267,

$$\epsilon = \frac{8WD^3N}{Cd^4},$$

or

$$C = \frac{8WD^3N}{\epsilon d^4} \text{ lb. per square inch.}$$

Calculate the value of C from this equation, and compare it with the value of C found by the direct method of applying torque.

Other springs made of wire of circular section are supplied. Make similar experiments, and find the value of C for each spring.

Springs of material having a square section are also supplied. If the side of the square is s in inches, find the numerical values of the coefficient c for each spring by inserting experimental values in the following equation:

$$\epsilon = c\frac{WD^3N}{s^4}.$$

Maxwell's needle. A useful piece of apparatus for making vibrational experiments on wires is the Maxwell's needle shown in Fig. 313 (a). The wire AB is fixed firmly at A and is clamped at B to a brass tube C. Four inner tubes D, E, F and G of equal lengths can be pushed into C; the total length of the four tubes is equal to the length of C. Two of the short tubes are empty, and the other two are closed at the ends and are loaded with lead shot. Experiments are made by first having the loaded tubes at D and G and the empty ones at E and F. A few degrees of twist are given to the wire, and the needle is then allowed to oscillate horizontally. The time taken to execute, say 100 vibrations, is observed, and hence the time of one vibration is obtained. The tubes are then exchanged by placing the loaded pair at E and F and the empty pair at D and G, and the experiment repeated in order to find the time of one vibration. The distribution of mass in the system has been altered without altering the actual quantity of matter, and the second time will be found to be shorter than the first.

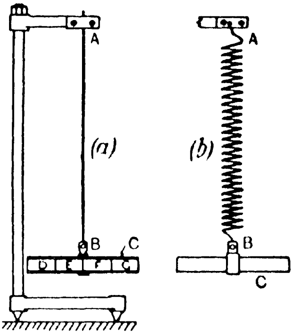


FIG. 313.—Maxwell's needle.

Let t_1 = the time in seconds to execute a vibration, the needle starting from the end of a swing and coming back again to the same position, loaded tubes at D and G.

t_2 = the corresponding time in seconds when the loaded tubes are at E and F.

m_1 = the mass in pounds of one loaded tube.

m_2 = the mass in pounds of one empty tube.

a = the half length of C in feet.

L = the length of the test wire, in inches.

d = the diameter of the test wire, in inches.

g = the acceleration due to gravity = 32·2 feet per second per second.

C = modulus of rigidity of material of wire, lb. per sq. inch.

Then

$$C = \frac{1536\pi La^2}{gd^4}\left(\frac{m_1 - m_2}{t_1^2 - t_2^2}\right)$$

EXPT. 19.—**Determination of C by Maxwell's needle.** Test several wires of different materials by this method, and calculate C for each. If wires of the same material have been tested for the values of C by other methods, compare the results.

Torsional oscillations of a helical spring. Maxwell's needle may be used for determining the value of Young's modulus for a wire of given material. The wire is first wound into a helical spring and arranged as shown in Fig. 313 (*b*), where C is the Maxwell's needle. Take the same symbols as before, with the addition of the following:

R = the mean radius of the helix, in inches.
N = the number of complete turns in the helix.
E = Young's modulus, in lb. per square inch.

Then
$$E = \frac{6144\pi^2 RNa^2}{gd^4}\left(\frac{m_1 - m_2}{t_1^2 - t_2^2}\right).$$

EXPT. 20.—**Determination of E by torsional oscillations of a spring.** Twist the needle through a small horizontal angle, taking care not to raise or lower it while doing so. On being released, it will execute torsional oscillations. Ascertain the times as before for the loaded tubes in the outer position and also in the inner position. Measure the dimensions required, and calculate E from the above equation. No correction is required for the mass of the spring in this experiment.

Longitudinal vibrations of a helical spring. Using the same apparatus, illustrated in Fig. 313 (*b*), the value of the modulus of rigidity may be found for the material of the spring. The spring, loaded with the needle, is pulled downwards a little and released; it will then execute vibrations vertically.

Let t = time in seconds to execute one vibration from the lowest position and back to the starting-point.
M = the mass of the needle, or other load, hung on + one-third the mass of the spring, in pounds.
N = the number of complete turns in the helix.
R = the mean radius of the helix, in inches.
d = the diameter of the wire, in inches.
C = the modulus of rigidity, lb. per square inch.

Then
$$C = \frac{64}{3}\,\frac{\pi^2 MNR^3}{gt^2d^4}.$$

EXPT. 21.—**Determination of C by longitudinal vibrations of a spring.** Use a spring made of wire which has been tested already for the value of C by the direct method of torque (p. 295), and also by the method of torsional oscillations (p. 297). Find C for the material by application of the method described above, and compare the results by the three methods.

The direct determination of Poisson's ratio, $\frac{1}{m}$, and also of the bulk modulus K for a material presents considerable difficulty. These may be calculated easily from the known experimental values of E and C by use of the following relations:

$$\text{Poisson's ratio} = \frac{1}{m} = \frac{E-2C}{2C}.$$

$$\text{Bulk modulus} = K = \frac{EC}{3(3C-E)}.$$

Take the results for E and C which you have obtained for wires of the same material, and calculate $\frac{1}{m}$ and K for each material.

EXAMPLE. A series of tests on steel wires gave average values as follows: E = 13,500 and C = 5500 tons per square inch. Find the values of Poisson's ratio and of the bulk modulus.

$$\frac{1}{m} = \frac{E-2C}{2C}$$

$$= \frac{13{,}500 - 11{,}000}{11{,}000} = \underline{\frac{1}{4 \cdot 4}}$$

$$K = \frac{EC}{3(3C-E)}$$

$$= \frac{13{,}500 \times 5500}{3(3 \times 5500 - 13{,}500)} = \underline{8250} \text{ tons per sq. inch.}$$

Elastic bending of beams. The apparatus shown in Fig. 314 is capable of giving very accurate experimental results on the elastic

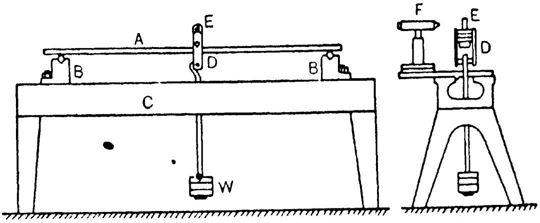


FIG. 314 —Apparatus for elastic bending of beams.

bending of beams. The test beam A rests on steel knife-edges supported by blocks B, B. The blocks may be bolted at any distance apart on a lathe bed C. The load W is applied by means of a shackle D having a steel knife-edge which rests on the beam. The piece E, carried by the shackle, is pierced by a hole which is covered by a piece of transparent celluloid having a fine line ruled on it. This line is observed through a micrometer microscope F, and will travel over the eyepiece scale as the beam deflects. The value of a scale division of the eyepiece scale may be ascertained by use of a scale engraved on the vertical pillar of the microscope; a rack and pinion movement permits of vertical movement of the microscope up or down the pillar.

For testing beams fixed at the ends, the knife-edges at B, B are removed; these are merely dropped into V grooves on the top of the blocks. The test beam now rests on the top of the blocks (Fig. 315), and is held down firmly at each end by a strong cast-iron cap and four studs.

The angle of slope at any position of the beam may be measured by means of the arrangement shown in Fig. 316. A is a small three-legged stool carrying a mirror and rests on the test beam. B is a reading telescope having a hair line in the eyepiece, and is used to observe the reading of a scale C reflected to the telescope by the aid of the mirror at A.

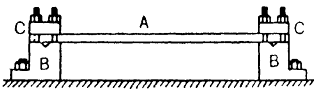


FIG. 315.—Test beam fixed at ends.

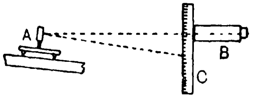


FIG 316.—Apparatus for measuring the slope of a beam.

The apparatus may be used for a large number of experiments; the following indicates some of the more simple.

EXPT. 22. Take a bar of mild steel of rectangular section about 2 inches × 1 inch and about 3·5 feet in length. Arrange it as a beam simply supported on a span of 3 feet and loaded at the centre of the span. Apply gradually increasing loads, and measure the deflection at the centre of the span after the application of each load. Verify these readings by removing the loads, one at a time, and observing the deflections after the removal of each load. Tabulate these readings, and plot loads and deflections. If the resulting diagram is a straight line, then the deflections of the beam are proportional to the load. Select a point on the plotted line, and note the load W lb. and corresponding deflection Δ inches; also let L be the span in inches Calculate the value of Young's modulus for the material, using the equation given on p. 169, viz.:

$$\Delta=\frac{WL^3}{48EI}.$$

For the given section, breadth B and depth D, both in inches,

$$I = \frac{BD^3}{12};$$

$$\therefore \Delta = \frac{WL^3}{4EBD^3},$$

or $$E = \frac{WL^3}{4\Delta BD^3} \text{ lb. per square inch.}$$

Expt. 23. Use the same piece of material (*a*) as a cantilever, (*b*) as a beam fixed at both ends. In each case measure the deflections for loads gradually increased and gradually diminished. Plot the results and determine Young's modulus, making use of the following equation for case (*a*):

$$\Delta = \frac{WL^3}{3EI} \quad \text{(p. 168).}$$

For case (*b*) use $$\Delta = \frac{WL^3}{192EI} \quad \text{(p. 178).}$$

Compare the values of E obtained by the three methods employed.

Expt. 24. Arrange a test bar as a cantilever (Fig. 317). Let the load be applied at B, and arrange the three-legged mirror stool at C,

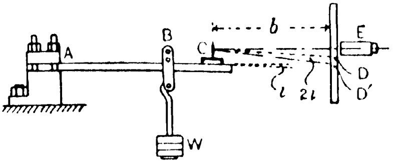


Fig. 317.—Slope of a cantilever.

a scale divided decimally in inches at D, and a reading telescope at E. On loading the cantilever a certain angle of slope will occur at B; as there is no load, and consequently no bending moment between B and C, whatever slope exists at B will occur uniformly between B and C. Hence the slope measured at C will be the slope at the point of application of B. The slope at B may be calculated from

$$i_B = \frac{WL^2}{2EI} \text{ radians} \quad \text{(p. 168).}$$

If the piece of material used in the previous bending experiments is employed in this experiment, E is known, and hence i_B may be calculated for any given load. To verify the calculation, observe the scale readings for gradually increasing and gradually diminishing loads; plot the results, and select from the diagram the value of i_B corresponding to the value of W used in the calculation.

In reducing the scale readings to radians, it must be noted that if the mirror at C tilts through an angle i, the ray of light CD will travel through an angle of magnitude $2i$. Let a be the change of scale reading due to an increment of load, and let b be the distance from the mirror to the scale, both in inches. The angle turned through by the ray CD will be

$$DCD' = \frac{a}{b} \text{ radian};$$

$$\therefore \ 2i = \frac{a}{b},$$

$$i = \frac{a}{2b} \text{ radian}.$$

Ten-ton testing machine. In Fig. 318 is shown in outline the principal parts of a testing machine constructed by Messrs. Joshua

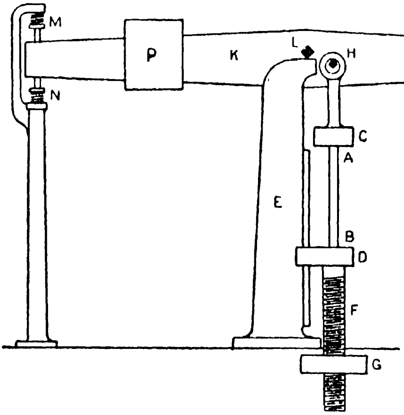


FIG. 318.—Ten-ton Buckton testing machine.

Buckton to the design of Mr. J. H. Wicksteed. As illustrated, the machine is arranged for applying **pull.** The test piece, AB, is held by grips in two crossheads C and D; D is guided by the main column E of the machine, and may be drawn downwards by means of a screw F and a wheel G; the latter serves as a nut for F, and is prevented from moving vertically. The rotation of G is effected by gearing and belt drive from some source of power; open and crossed belts permit of either direction of rotation being given to G. The belts are under the control of the operator by means of striking gear. The upper crosshead C is hung from a knife-edge H fixed in the beam K. The beam is supported by a knife-edge L resting on the top of the column E. Its movement in a vertical plane is limited by spring-buffer stops M and N. A counter-poise P can be moved along the beam by means of a screw and hand-wheel under the control of the operator until the pull transmitted through the test piece to the beam is equilibrated. The magnitude of the pull is shown by the position of the counterpoise in relation to a scale of pounds which is attached to the beam.

For applying **push** to the test piece, the machine is modified as shown in Fig. 319. The specimen AB is placed between crossheads Q and R, the former being connected to the screw F and the latter being hung from the beam.

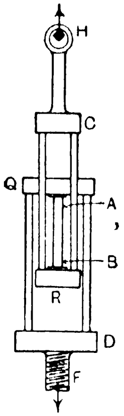


FIG. 319.—Arrangement for applying push.

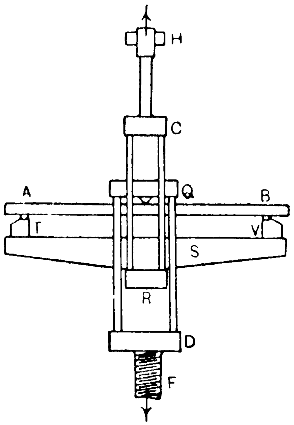


FIG. 320.—Arrangement for bending tests.

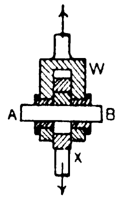


FIG. 321.—Shearing device.

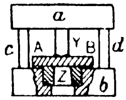


FIG. 322.—Punching device.

In carrying out **bending tests**, arrangements are made as shown in Fig. 320. The test beam AB rests on supports T, V, which in turn are carried by a beam S secured to the crosshead R. The weighing beam on the machine thus supports the beam under test. A central load is applied by means of a ram attached to the crosshead Q and drawn downwards by means of the screw F.

Simple **shearing tests** are carried out by means of the appliance illustrated in Fig. 321. The piece X may slide inside W; the test piece AB is pushed into cylindrical steel dies carried by W; X has another steel die which bears on the central portion of the test piece.

The machine is arranged for pull as in Fig. 318; W is attached to C and X to D. On operating the machine, the test piece is put under double shear under much the same conditions as a rivet in a double-strapped butt-joint (p. 102).

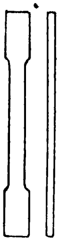


Fig. 323.—Flat test piece

Fig. 322 shows an appliance which may be used for **punching tests.** The upper block *a* can move vertically relative to the lower block *b*, and is guided by pins *c* and *d*. *a* carries a punch Y and *b* has a die Z. AB is the test piece. The machine is arranged for compression as shown in Fig. 319, and the punching appliance is placed between the crossheads Q and R.

The same machine may be used for **torsion tests,** but it will be found more convenient to have a separate torsion machine. One such is described on p. 316.

A flat bar tension test piece is shown in Fig. 323. The enlarged ends ensure that fracture shall not take place in the grips. Fig. 324 shows the pair of steel wedge grips used for holding each end of this test piece. The grips have serrated faces for gripping securely the specimen. Round test pieces may be gripped in a similar manner, but a better plan is to have each end of the specimen A screwed into a holder B (Fig. 325); the holder has a nut C resting in a spherical seat formed in D, and permits of better alignment of the specimen in the machine than is possible with wedge grips. Both patterns of grip are used for ductile materials.

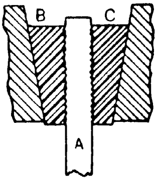


Fig. 324.—Wedge grips.

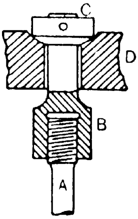


Fig. 325.—Spherical seated screwed grip

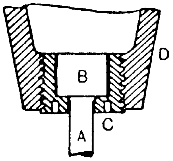


Fig. 326.—Grip for brittle materials.

For holding hard non-ductile materials like cast iron, the holder shown in Fig. 326 is employed. The specimen A is round, and has each end enlarged as shown at B. A split nut C screwed into the holder D supports A.

The arrangement shown in Fig. 327 will be found to give satisactory working in compression tests. The ends of the specimen AB re screwed into holders C and D. Hard teel balls are placed at E and F in conical depressions, and enable the load to be applied ery nearly axially.

Columns made of cycle tubes provide a arge range of useful tests. The arrangement when both ends are rounded is shown in Fig. 328. Conical hard steel plugs C and D are inserted in the ends of the tube AB and bear on hard steel seats E and F. It will be found useful to carry out a series of ests on specimens having a range of ratios of L to k. The breaking loads for these hould be plotted in the manner described on p. 235. Great care should be taken in order to secure initial straightness, and the oad should be applied as smoothly as possible n order to avoid shocks which would precipitate rupture.

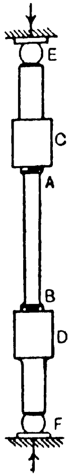


FIG. 327.—Test piece arranged for compression.

Autographic recorder. The autographic ecorder fitted to the machine in the laboratory at West Ham was designed by Professor Barr of Glasgow University, and is shown in outline in Fig. 329. AB is the test piece under pull, and has two clamps D and E attached to it at a measured distance apart. A cord F is attached to D, passes over a pulley at E and thence to a drum C. Any extension of the test piece between D and E will be shown by rotation of the drum. The drum C has a paper wrapped round it on which the diagram of loads and extensions is drawn by a pencil G. Horizontal distances on this paper will represent extensions of the portion DE of the test piece.

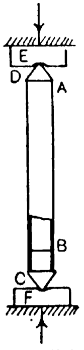


FIG. 328.—Tubular test column.

The pencil G is given vertical movements proportional to the load on the specimen by means of the following mechanism. The counterpoise of the machine is driven along the beam by means of the operating wheel H and gearing connected to the spindle K. The same spindle is connected also by gear wheels L to a screwed spindle M, on which is threaded a guided frame N carrying the pencil G. Vertical movements of the pencil will therefore be reduced copies of horizontal movements of the counterpoise, and thus will represent to scale the load on the specimen.

In testing ductile materials there are generally two points where the piece stretches so rapidly that the beam of the machine is certain to drop on to the lower buffer-stop; these are the yield point and the part of the test where local contraction is occurring preparatory to fracture. Should the beam drop on the buffer-stop, a portion of the

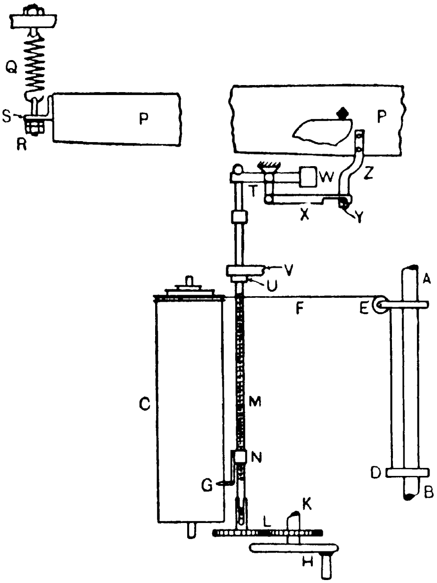


FIG. 329 —Autographic recorder.

diagram will be lost, as the load on the specimen is no longer represented by the position of the counterpoise on the beam. To obtain the complete diagram, a special spring Q is suspended from the end column of the machine (Fig. 329). Adjustable lock nuts are provided at R, and a bracket S is fixed to the end of the machine beam P.

As the beam descends, S will come into contact with R and the spring Q will extend, thus removing some of the load from the test piece. The movement of the beam while extending Q is utilised for lowering the pencil G by an amount proportional to the load removed from the test specimen. The screwed spindle M is capable of vertical movement, and is held up in normal circumstances by means of a lever T and balance weight W, the collar U thus being pressed against the fixed bracket V. A rod X is connected to the lever T, and has its end hooked to engage a pin Y fixed to a lever Z which is secured to the machine beam P. As the beam descends, S comes into contact with R and Y arrives at the hooked end of X simultaneously. Further movement of the beam will extend Q and lower M, and so will lower the pencil by an amount proportional to the load taken off the specimen by the spring.

It will be evident that the apparatus can be used for the production of an autographic record of any of the tests made in the machine; the cord which rotates the drum is connected in each case to the part the movements of which are to be recorded as horizontal distances on the paper.

Extensometers. In tension tests which do not exceed the elastic limit, it is necessary to attach some form of extensometer to the specimen for the purposes of detecting and measuring the very small extensions which occur. The instrument devised by Sir J. A. Ewing is probably the most useful in general practice, and is shown in outline in Fig. 330. The test piece AB has clamped to it two blocks or levers C and D, by means of pairs of pointed pinching screws at E and F. C and D are connected by a bar G which is pivoted to D at H and is pulled against C at its upper end by means of a spring M; the end of G has a ball K formed on it which beds in a conical recess in the end of the micrometer screw L. At the other end of C is suspended a rod N having a ball at its upper end; this ball is pulled upwards into a conical recess by means of a light spring O. The lower end of N is guided by pins on D and carries a fine hair line at P. This hair line is observed through a micrometer microscope Q.

Suppose that the test piece extends under pull and that the rod G remains unaltered in length. The hair line P will be displaced upwards relative to the microscope, and so will appear to travel over the eyepiece scale. Each scale division represents approximately one five-thousandth of an inch, and it is easy with a little experience to subdivide each division into ten parts, thus enabling readings to be taken to the nearest fifty-thousandth of an inch.

The precise value of a scale division of the microscope is ascertained as follows: After focussing the instrument and reading the scale, the micrometer L is given one complete turn. As its pitch is 0·02 inch, the effect is to change the length of G by this amount. The arms of the lever C on either side of the specimen are equal; hence P will be moved relative to the microscope by 0·02 inch. The microscope scale is read again, and the difference between this and the original reading corresponds to a movement of P of 0·02 inch. Now in use, G remains unaltered in length, and the movement of P is produced by the extension of the specimen; the effect of the levers is to produce a movement at P equal to double the extension of the specimen. Accordingly 0·01 inch extension of the specimen will produce a movement of 0·02 inch at P. Hence the scale divisions movement in the microscope found as directed above correspond to 0·01 inch extension. Once focussed and calibrated, the instrument requires no further adjustment during the test unless the extension is sufficiently large to run the risk of moving the hair line beyond the limit of the microscope scale. In this event, it may be brought to a working position again by giving the micrometer L one turn, when the test will proceed as before. The loads are applied best in equal increments, and the reading of the microscope taken after application of each increment. The following record of a tensile test may assist in indicating the methods of noting the observations and of reducing the results:

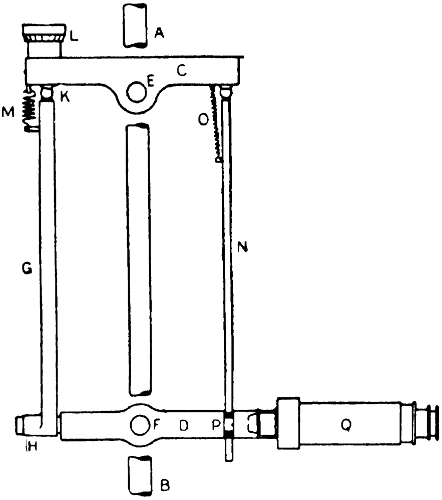


FIG. 330.—Ewing's extensometer.

TENSILE TEST ON A MILD-STEEL SPECIMEN.

Laboratory No. A, M.S., 14.10.10.

Form of test piece; round, with swelled ends; ends rough turned to 0·75 inch diameter; body turned and polished. A length of 10 inches of the body was marked off at 1 inch intervals by light centre punch dots.

Diameter of specimen, 0·445 inch.
Area of cross section, 0·1556 inch.

Elastic test with Ewing's extensometer.

CALIBRATION OF INSTRUMENT.

Micrometer screw	Microscope scale.		
	Original	Final	Difference
1 turn	30	77·2	47·2
1 turn	50	97·2	47·2
1 turn	40	87·2	47·2

47·2 microscope scale divisions are equivalent to an extension of 0·01 inch.

∴ 1 microscope scale division $= \frac{1}{4720} =$ 0·0002118 inch.

The load was applied in increments of 100 lb. A number of these readings are omitted in the following table in order to economise space. None of the omitted readings depart from the plotted curve (Fig. 331).

LOG OF TEST.

Load, lb.	Microscope scale	Load, lb.	Microscope scale.
100	30·0	4700	65·3
500	33·0	4800	66·1
1100	37·9	4900	67·1
1600	41·9	5000	68·0
2000	44·8	5100	68·8
2500	48·3	5200	69·3
3000	52·1	5300	70·2
3500	56·0	5400	71·1
4000	59·8	5500	72·1
4500	63·8	5600	73·0
4600	64·6	5700	74·5 creeping to 83·5

The loads and scale readings are shown plotted in Fig. 331. It will be observed that the line ceases to be straight at A, which point accordingly indicates the break-down of Hooke's law.

Load at elastic break-down, 4700 lb.

Stress at elastic break-down $= \dfrac{4700}{0{\cdot}1556 \times 2240}$

$= 13{\cdot}48$ tons per sq. inch.

Load at which creeping started $= 5700$ lb.

Stress " " $= \dfrac{5700}{0{\cdot}1556 \times 2240}$

$= 16{\cdot}35$ tons per sq. inch.

The creeping of the hair line in the instrument marks the commencement of a stage in which the beam of the testing machine would exhibit a tendency to descend slowly towards the lower buffer-stop while a constant load is maintained on the specimen. When this stage is developed fully, the material is said to be in a plastic state, and the point is called the yield point. The autographic record (Fig. 332) shows the yield point clearly.

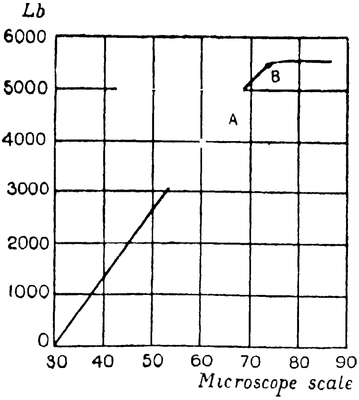


FIG. 331.—Elastic limit tensile test; mild steel.

To determine Young's modulus:

From the diagram (Fig. 331), a load of 4000 lb. produced an extension corresponding to 29 microscope scale divisions.

Extension of specimen $= 29 \times 0{\cdot}0002118$

$= 0{\cdot}00614$ inch.

Gauge points of extensometer are 8 inches apart.

$$\text{Strain} = \frac{0{\cdot}00614}{8} = 0{\cdot}000768.$$

$$\text{Stress} = \frac{4000}{0{\cdot}1556} = 25{,}700 \text{ lb. per sq. inch.}$$

$$\text{Young's modulus} = E = \frac{25700}{{\cdot}000768} = 33{,}500{,}000 \text{ lb. per sq. inch.}$$
$$= 14{,}900 \text{ tons per sq. inch.}$$

Test to maximum load. The extensometer being removed, the autographic recorder was connected to the specimen at 10 inch gauge points, and the load was increased from zero until the test piece began to form a neck preparatory to breaking. The resulting diagram is shown in Fig. 332, and gives the yield load as 6600 lb. From this we calculate

$$\text{Yield stress} = \frac{6600}{0{\cdot}1556 \times 2240} = 18{\cdot}93 \text{ tons per sq. inch.}$$

The maximum load which the specimen could carry was 9600 lb. Hence,

$$\text{Breaking stress} = \frac{9600}{0{\cdot}1556 \times 2240} = 27{\cdot}5 \text{ tons per sq. inch.}$$

The autographic record (Fig. 332) shows an interesting point regarding the effects of overstrain (*i.e.* straining beyond the yield point) on the elastic properties of the material. After the specimen had been stretched 1·2 inches on a length of 10 inches, the load was removed. Reapplication of the load caused the diagram to rise from zero along a practically straight line until the former curve was reached again at a load of about 9500 lb. Yielding along the curve then continued as before. The overstraining had hardened the material and raised the yield load from 6600 lb. to about 9500 lb., *i.e.* only slightly below the ultimate load.

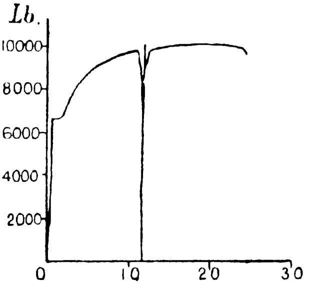


FIG. 332 —Autographic record; mild steel under tension.

The test piece was removed from the machine and the lengths of the intervals between the centre punch dots were measured; also the diameters at each dot. From these the curves in Fig. 333 were plotted. It will be noted that both extensions and diameters vary considerably, and illustrate the necessity for stating the distance between the gauge points as well as the percentage extension of a test piece. A good method is to measure the total extension on a length of 10 inches, also the extension on the 2 inches interval which includes the fracture; the difference between these will be the general extension on the remaining 8 inches of the specimen. These extensions, expressed as percentages, give useful information regarding the ductility of the material. Another measure of the ductility may be obtained by measuring the cross-sectional area of the fracture. The loss in area may be found from this measurement, and may be expressed as a percentage of the original sectional area.

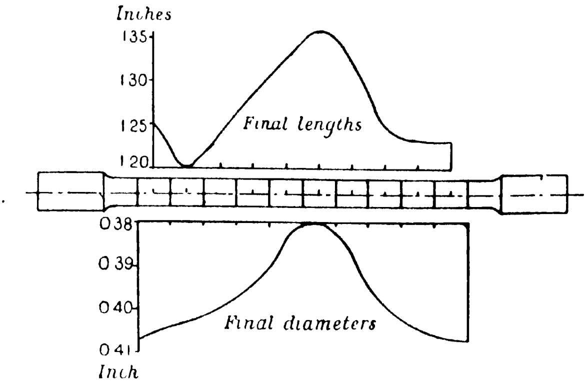


FIG. 333.—Dimensions of a mild steel specimen after a tension test

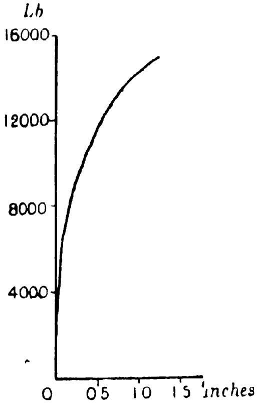


FIG. 334.—Autographic record; Delta metal under tension.

Fig. 334, copied from the autographic record of a specimen of Delta metal under pull, is given as illustrating totally different characteristics from the mild-steel diagram shown in Fig. 332. In particular, the absence of any yield point will be noticed.

Bending tests. The records given in Figs. 335 and 336 illustrate an instructive test made on a mild-steel bar having a span of 36 inches, breadth 2·01 inches and depth 2·015 inches. The bar was arranged as shown in Fig. 320 and bent by application of a central load until the deflection was 2·6 inches. The record (Fig. 335) shows that yielding was reached at about 7200 lb. The test was arrested at about 2·1 inch deflection, and the load brought to zero and then reapplied; the diagram shows that the new yield load is about 9000 lb.

Then the load was removed entirely and the bar turned over; central loading was applied again so as to straighten the bar. The diagram given in Fig. 336 shows the result. There is practically no part of this test where the elastic law is followed, a fact which will be understood readily when it is realised that the bar came out of the former test badly overstrained both on its compression side and on its tensile side, and, in the effort to recover some of the deflection imposed on it, the material became self-stressed throughout. The second test began therefore with the material in a complicated state of stress. This test was also arrested at about 2·2 inch deflection. On reapplication of the load, a yield load of about 10,000 lb. will be observed in the diagram. Had the bar been annealed after straightening, it is probable that a diagram somewhat resembling Fig. 335 for the first test would be obtained. The tendency of the annealing is to remove self-stressing from the material.

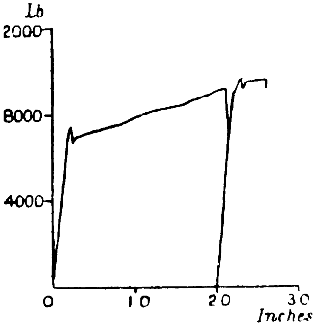


FIG. 335.—Mild steel bar under bending; first test.

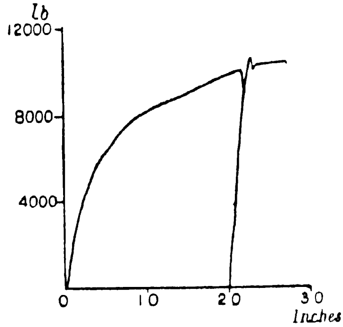


FIG. 336.—Mild steel bar under bending; second test.

Fig. 337 has been copied from the autographic record obtained in testing a cast-iron bar under bending. The bar was rectangular in section, 2 inches wide and $1\frac{1}{32}$ inches deep, span 20 inches. Rupture occurred with a central load of 3200 lb., the maximum deflection recorded being 0·2 inch. It will be noted that the load and deflection remain approximately proportional up to fracture. The contrast of the ductile and brittle materials is rendered clear by inspection of Figs. 335 and 337. The mild-steel bar could not be broken by bending; the cast-iron bar could take a very small deflection only.

The usual test for timber is by bending under similar conditions to those noted above. The specimens should be of as large size as is possible, then the effect of any local flaws such as shakes and knots will not be emphasised, as would be the case with a smaller specimen containing the same flaws. In Fig. 338 are given copies of records of bending tests on yellow deal, teak and oak. The yellow deal specimen was arranged with the annual rings nearly horizontal, and failed by shearing horizontally along the fibres and round the annual rings. The dimensions and results are given in the following table:

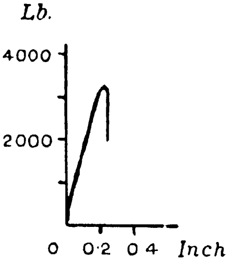


FIG. 337.—Cast-iron test bar under bending

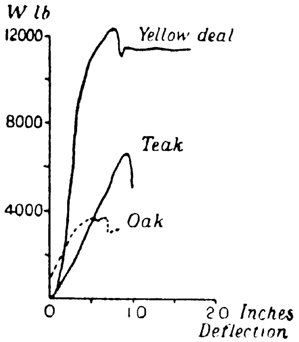


FIG. 338.—Bending tests on timber.

BENDING TESTS ON THREE TIMBER SPECIMENS.

Material.	Span × breadth × depth, inch units.	Central breaking-load, lb.	Deflection at centre, inch.
Yellow deal	24 × 2·9 × 3·4	12,600	0·8
Teak	60 × 3·14 × 4·26	6,720	0·92
Oak	24 × 1·35 × 2·25	3,500	—

In reducing the results of tests on cast-iron and timber specimens, it is usual to state the value of the **coefficient of rupture.** This coefficient represents the value which the maximum stress at rupture due to bending would have if Hooke's elastic law were followed throughout. For a beam of rectangular section supported at the ends and having the load applied at the middle of the span, the calculation will be as follows:

Let W = maximum load, in lbs.
L = the span, in inches.
b = the breadth, in inches.
d = the depth, in inches.

Then
$$M = \frac{f}{m} \cdot I,$$
$$\frac{WL}{4} = \frac{f}{\frac{1}{2}d} \cdot \frac{bd^3}{12} = \frac{fbd^2}{6},$$
and
$$\text{Coefficient of rupture} = f = \frac{3}{2}\frac{WL}{bd^2}.$$

Shearing tests. Autographic records of two shearing tests carried out in the apparatus described on p. 303 are given in Figs. 339 and 340. The former is for a mild-steel specimen and the latter is for a specimen of gun-metal. It should be noted that pure shear is not obtained with this apparatus, the specimen being under bending as well as shearing. In order to minimise the bending effect, the specimens should be turned to fit the bored holes in the dies. The results of the tests are given below.

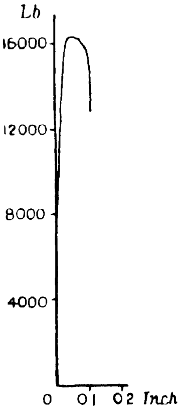


FIG. 339.—Mild steel under shearing.

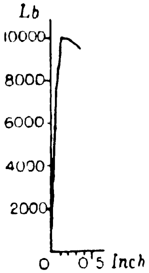


FIG. 340.—Gun-metal under shearing.

SHEARING TESTS.

Material.	Diameter, inch.	Cross-sectional area, square inch.		Shearing load, lb.	Shearing strength, tons per sq. inch.
		Actual.	Under shear		
Mild steel	0·496	0·194	0·388	16,400	18·9
Gun-metal	0·500	0·196	0·392	10,190	11·6

Punching tests. In punching a hole in a piece of material, the action of the punch is first to increase the pressure on the material until the plastic stage is reached; in this stage, some of the metal flows from under the punch into the surrounding plate, the plate immediately under the punch becoming thinner. This effect continues until, partly by the increasing force on the punch and partly by the diminishing thickness of the plate, the rupturing shear stress on the material is attained and a wad is pushed out. The following results of a punching test may be of interest; the autographic record is given in Fig. 341.

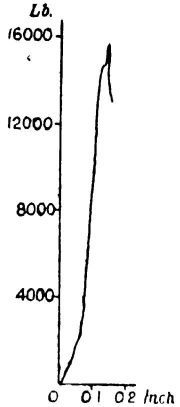


FIG. 341—Punching test on wrought iron

PUNCHING TEST ON A WROUGHT-IRON PLATE.

Thickness of the plate = 0·265 inch.

Diameter of the wad punched out = 0·38 inch.

Area under shear stress $= \pi dt = \pi \times 0{\cdot}38 \times 0{\cdot}265$
$= 0{\cdot}317$ square inch.

Maximum load on the punch = 15,750 lb.

Maximum shearing stress $= \dfrac{15{,}750}{0{\cdot}317 \times 2240} = 22{\cdot}2$ tons per sq. inch.

Thickness of the plate round the hole after punching = 0·268 inch.
Thickness of the wad = 0·257 inch.
Loss of thickness of material in the wad = 0·008 inch.
Gain of thickness of material round the hole = 0·003 inch.

Total work done in punching the hole, represented by the area of the autographic record, is about 810 inch lb.

Avery torsion machine. An outline diagram of this machine is given in Fig. 342, where AB is the test piece. The end B is connected to a worm wheel C, which may be rotated by means of a worm D and hand wheel E. The wheel C has 90 teeth; hence each quarter turn of the hand wheel twists the specimen through one degree. The torque is measured by means of a system of levers FG, MK and NP, connected to the end A of the specimen. NP carries a counterpoise Q, which may be run along the lever by means of a screw and hand wheel, and shows the torque by its position relative to a scale attached to the lever. The scale reads from zero to 1000 lb.-inches; to obtain higher torque the counterpoise is run back to zero, and a load R is suspended from the end of the lever and is sufficient to give a torque of 1000 lb.-inches. The test then proceeds to 2000 lb.-inches by traversing the counterpoise. This process repeated enables 5000 lb.-inches torque to be obtained, the lever MK resting on the knife-edge M during this stage. To increase the torque further, the knife-edge M is lowered and L is raised simultaneously by means of a lever; the effect of this is to double the value of the scale divisions on the lever NP. The effects of the loads at R also are doubled. To reset the torque at 5000 lb.-inches, hang two weights at R (equivalent now to 4000 lb.-inches) and set the counterpoise at 500 (equivalent to 1000 lb.-inches). The test then proceeds as before, the capacity of the machine being now 10,000 lb.-inches.

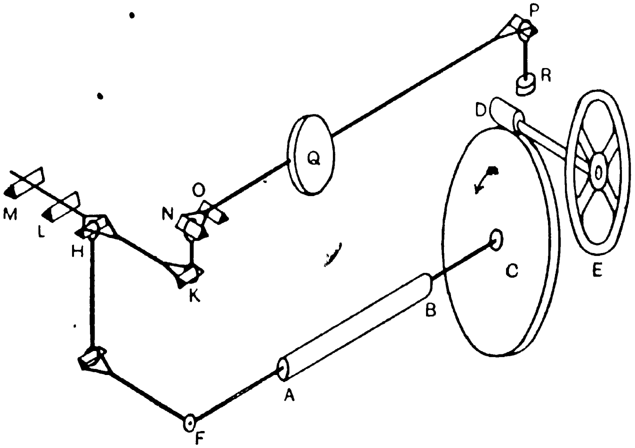


FIG. 342.—Arrangement of Avery torsion machine.

In testing a piece to destruction, readings are taken of the torques and of the corresponding angles of twist by counting the number of teeth passed by the worm-wheel; each tooth represents 4 degrees of twist. Plotting these readings will give a curve such as is illustrated in Fig. 343. The principal results of this test are given below.

TORSION TEST ON A MILD-STEEL SPECIMEN.

Laboratory No. 6, M.S., 22.3.10.
Original diameter, 0·756 inch.
Original length, 5·625 inches.
Diameter after fracture, 0·754 inch.

Length after fracture, 5·750 inches.
Yield torque, estimated by the dropping of the beam, 2000 lb.-inches.
Breaking torque, 6400 lb.-inches.
Angle of twist at yield, 3·5 degrees on 5·75 inches length.
Angle of twist at breaking, 1896 degrees on 5·75 inches length.
Mean torque, from diagram, 5650 lb.-inches.
Total work done in fracturing specimen, 187,000 inch-lb.
Work done per cubic inch of material, 74,200 inch-lb.

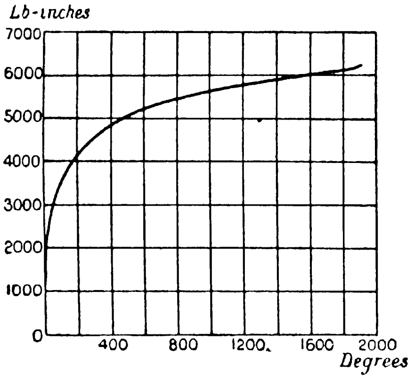


Fig. 343.—Graph of a torsion test on mild steel.

The form of the test specimen is indicated at AB in Fig. 344; the same diagram also illustrates an appliance whereby angles of twist

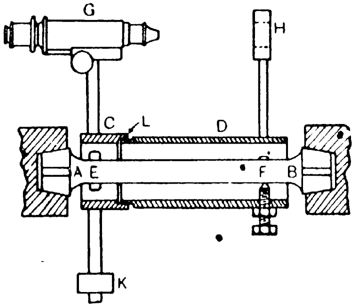


Fig. 344.—Apparatus for measuring angles of twist within the elastic limit.

within the elastic limit may be measured. C and D are two pieces of wrought-iron steam tube turned and bored at L to an easy fit.

Three steel pinching screws nip the specimen at E and other three screws engage it at F. The angle of twist is measured on the length of the specimen between E and F. C carries a micrometer microscope G, balanced by means of a weight K, and D carries a rod having a small piece of transparent celluloid at H. A radial line is scratched on the celluloid, and is sighted through the microscope.

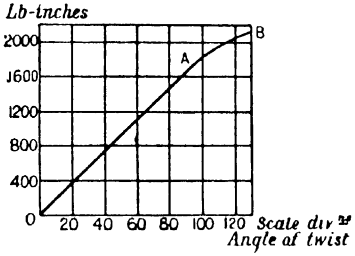


FIG. 345.—Graph of an elastic torsion test, mild steel.

The circumferential movement of the line is given by the scale readings of the microscope; these reduced to inches, and divided by the radius of the mark sighted on the scratched line, will give angles of twist in radians. These numbers may be plotted as shown in Fig. 345, which illustrates the results obtained in testing the following specimen.

ELASTIC TORSION TEST ON A MILD-STEEL SPECIMEN.

Laboratory No. 2, M.S., 17.2.10.
Diameter, 0·753 inch.
Gauge points, 7·5 inches.
Value of one scale division of the microscope, 0·000377 radian.
Hooke's law broke down at 1750 lb.-inches of torque (point A in Fig. 345).
Angle of twist at break-down of Hooke's law, 0·0358 radian.
Maximum stress at break-down of Hooke's law, 9·3 tons per square inch.
From the diagram, torque T = 1650 lb.-inches, when the angle of twist α is 0·0336 radian.
Hence,

$$\text{Modulus of rigidity} = C = \frac{2TL}{\pi R^4 \alpha} = 5200 \text{ tons per square inch.}$$

"Creeping" of the specimen was noticed first distinctly when the torque was 2000 lb.-inches, *i.e.* at this load the machine beam would begin to show an inclination gradually to droop under a steady load. The point is marked B in Fig. 345, and, as will be seen, occurs a considerable interval beyond the point A of elastic break-down.

The total work done up to the elastic limit will be found by taking the product of the mean torque and the angle of twist, and is 30·3 inch-lb.

To obtain the resilience, *i.e.* the work done per cubic inch of material, divide the total work by the volume of the specimen between the gauge points. The result is 9·3 inch-lb.

Cement testing. Portland cement is made by mixing chalk and fine clay in certain proportions, burning the mixture at a clinkering temperature, and finely grinding the resulting product. Cement of this kind is much used for making concrete for constructional purposes. Concrete consists of an aggregate of clean broken stones, etc., to which sufficient clean sand is added to fill completely the voids between the stones. A quantity of Portland cement is intimately mixed with these, sufficient to coat the surface of every stone and every particle of sand with cement. Water is added, and the whole is mixed thoroughly in order to produce a plastic mass, which is rammed into moulds prepared to give the required structural shape.

The qualities which Portland cement should possess have been laid down by the Engineering Standards Committee, and the tests should be carried out in accordance with the terms of their specification,[1] a copy of which should be in the hands of the experimenter.

The fineness to which the cement has been ground is of great importance, and is tested by means of sieves, one having 5776 holes per square inch and another having 32,400 holes per square inch. These sieves are made in a special way of wire having standard diameters in terms of the specification. The residue left on the first sieve should not exceed 3 per cent., and on the latter 18 per cent.

The specific gravity of the cement is taken now in place of weighing the cement in bulk. This may be ascertained by use of a specific gravity bottle having a graduated stem and containing a measured quantity of turpentine (cement will not set in turpentine). A measured weight W of cement is introduced into the bottle, and its volume V may be observed from the rise in level of the turpentine in the stem of the bottle. Then

$$W = Vw\rho,$$

where w is the weight of a cubic unit of water and ρ is the specific gravity of the cement. The specific gravity should be not less than 3·15 for cement freshly burned and ground. 3·10 is permitted at a period not less than four weeks after grinding.

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[1] *British Standard Specification for Portland Cement*, Crosby, Lockwood & Son. Revised 1910.

The strength of cement is determined usually by means of tensile tests, although cement in practice is generally under compression. Tensile tests may be carried out in a comparatively small machine, while compression tests require a machine capable of exerting great pressure. When compression tests are made, the test briquettes are generally cubical; briquettes for tensile tests are prepared in moulds having a shape in accordance with that laid down in the standard specification. Considerable experience is required in order satisfactorily to gauge or mix the cement intended for test briquettes. The quantity of water to be used depends on the kind of cement, and greatly influences the strength of the cement. The student can test this easily by preparing several briquettes having water percentages of from 18 to 25, and testing these for tensile strength.

The moulds should rest on an iron plate while being filled; no severe mechanical ramming should be necessary if the correct percentage of water has been used. During the first 24 hours after filling, a damp cloth should be placed over the moulds. The briquettes are removed from the moulds then and placed in clean water until the strength test is carried out. The temperature throughout should be near 60° Fah. The tensile strength of neat cement briquettes (*i.e.* briquettes made of cement alone, without sand or other material) at 7 days from gauging should not be less than 400 lb. per square inch. Briquettes consisting of one part by weight of cement to three parts by weight of Leighton Buzzard sand, prepared in accordance with the terms of the standard specification, should have a tensile strength of 150 lb. per square inch at 7 days after gauging.

The setting time is tested by means of a standard needle having a flat point one millimetre square and having a total weight of 300 grams. The cement is taken as set when the application of the needle fails to make an impression.

The soundness of the cement is tested by the Le Chatelier method. A cylindrical mould having an axial split and furnished with two long pointers is filled with cement, as directed in the standard specification. This is kept in water for 24 hours, and then the distance between the ends of the pointers is measured. The mould and cement are then boiled for 6 hours and allowed to cool. The distance is measured again, and the increase should not exceed a stated amount.

Cubical cement and concrete blocks, bricks and stones are tested under compression. It is best to prepare the top and bottom surfaces by smoothly coating them with plaster of Paris in order to give level parallel surfaces for the testing machine plates to bear upon. Generally, the fracture is by shearing on planes roughly at 45° to the horizontal. Broken cement compression briquettes generally resemble two square based pyramids standing apex on apex.

EXERCISES ON CHAPTER XIII.

1. The following is the experimental record of a test on a specimen of cast iron. The object of the experiment was to determine the compression value of E for the material: Ewing's extensometer was employed. The specimen was turned and polished.

Diameter of specimen, 0·474 inch; gauge points, 8 inches; calibration of extensometer, 1 scale division $= \frac{1}{4750}$ inch.

Load, lb.	200	300	400	500	600	700	800	900	1000	1100
Scale readings, load increasing	50·0	48·2	46·7	45·0	43·5	41·9	40·1	38·6	37·2	36·1
Scale readings, load decreasing	50·0	48·2	47·0	45·6	44·0	42·0	40·1	38·5	37·2	36·1

Find the value of E.

2. Tensile tests were carried out on a turned and polished specimen of gun-metal. The following observations were made: Diameter of specimen, 0·534 inch; gauge points, 8 inches; calibration of Ewing's extensometer, 1 scale division $= \frac{1}{4800}$ inch. The extensometer readings are given below:

Load, lb.	0	100	200	300	400	500	600
Scale readings	40·0	42·0	43·4	44·8	46·2	48·0	49·1
Load, lb.	700	800	900	1000	1100	1200	1300
Scale readings	50·8	52·0	53·5	55·0	56·3	58·1	60·1
Load, lb.	1400	1500	1600	1700	1750	1800	0
Scale readings	61·8	63·3	66·0	68·3	69·3	71·0	44·5

Creeping was first observed at 1600 lb. load.

Find the value of E. What is the stress when Hooke's law breaks down for this material? How much permanent set was given?

3. The gun-metal specimen given in Question 2 was tested to breaking under tension after five weeks rest. The following observations were made:

Breaking load, 5480 lb.; load at which the beam of the machine dropped, 3600 lb.; stretch on a length of 8 inches, 0·85 inch; stretch on a length of 2 inches, including the fracture, 0·3 inch ; diameter at fracture, 0·479 inch. Reduce these observations, following the procedure indicated on p. 311.

4. A mild-steel bar of square section 2 inches × 2 inches was arranged as a beam of 60 inches span, simply supported ; the load was applied at the middle of the span, and the deflections at the load were measured by means of a micrometer microscope, the calibration of which gave one eyepiece-scale division = 0·065 mm. The following observations were taken :

Load, lb.	500	1000	1500	2000	2500	3000	3500
Eyepiece scale divisions	0	29	51	72·8	95	117·5	139·7

Load, lb.	3700	3800	3900	4000	4100	4200
Eyepiece-scale divisions	149	153·5	158·5	163·5	169·2	175·5

Find the value of E for the material, also the maximum stress in the bar when Hooke's law broke down.

5. A mild-steel bar 1·478 inches broad × 0·091 inch deep was arranged as a cantilever, the load being 16·3 inches from the support. Deflection and slope at the load were measured by means of the apparatus illustrated in Figs. 314 and 317. The calibration of the micrometer microscope used for observing the deflections gave one eyepiece-scale division = 0·6 mm. In the slope observations a scale of millimetres was used ; distance from the mirror to the scale = 566 mm. The following observations were taken

Load, lb.	0	2	4	6	8	10	12	14
Deflection scale	7·93	7·63	7·3	7·0	6·7	6·4	6·1	5·8
Slope scale	103·2	102·5	101·9	101·1	100·4	99·8	99·0	98·2

The beam theory gives for the ratio of deflection to slope of a cantilever carrying a load at its free end :

$$\frac{\Delta}{i}=\frac{WL^3}{3EI}\times\frac{2EI}{WL^2}=\frac{2}{3}L.$$

Compare the experimental ratio of $\Delta : i$ with that calculated.

6. A cast-iron test bar was tested under bending ; span 36 inches, simply supported ; breadth 1·02 inches ; depth 2·0[illegible] inches. The observations gave the central breaking load = 4120 lb. and the maximum central deflection = 0·5 inch. Find the coefficient of rupture.

7. The following particulars relate to tests on a model reinforced concrete column. Height of column 24 inches ; section square, of 3 inches edge ; main reinforcement, four mild-steel bars, each 0·31 inch diameter, arranged at the corners of a square of $1\frac{11}{16}$ inches edge ; secondary reinforcement, thirteen horizontal lacings of iron wire, 0·067 inch diameter, about 2 inches pitch. Concrete mixture, cement 7 lb., granite chips 14 lb., water 3 lb. The column was made in a wooden mould, removed five days after making and tested fourteen days after making; it was kept damp throughout this time.

Observations taken: Hooke's law broke down sensibly at 9000 lb. load; 8000 lb. load shortened the column to the extent of 0·0154 inch; the column ruptured when the load reached 20,670 lb.

Taking $m=15$, find the stress in the steel and in the concrete when Hooke's law broke down. Assuming the elastic laws to hold up to rupture, find these stresses at rupture. What is the value of E for the complete column?

8. The following observations were made during a torsion test on a mild-steel specimen: Diameter of specimen, 0·714 inch; gauge points of strain indicator, 7·81 inches; calibration of indicator, one scale division =0·04 degree twist.

Torque, lb.-inches	0	200	400	600	800	1000
Scale divisions	576	569	562	554	546	538
Torque, lb.-inches	1100	1200	1300	1400	1450	1500
Scale divisions	534	530	526	520	517	514

Find the value of C; also the stress at break-down of Hooke's law and the resilience in inch-lb. per cubic inch of material.

9. A test was made in order to determine C for a copper wire by the torsional oscillation method, using Maxwell's needle. Employing the symbols explained on p. 297, the following observations were taken

m_1, pounds	m_2, pounds	a, feet.	d, inch	L, inches	t_1, sec	t_2, sec
0·457	0·053	0·5	0·048	25	4·3	2·73

Find the value of C for this material.

10. Using the same Maxwell's needle, particulars of which are given in Question 9, the following observations were made during a test for the determination of E for steel wire by the torsional oscillations of a helical spring. Diameter of wire, 0·081 inch; mean radius of helix, 0·4945 inch; number of complete turns in helix, 133; $t_1=4·21$ seconds; $t_2=2·666$ seconds. Find the value of E.

11. The spring given in Question 10 was tested by the longitudinal vibration method in order to determine C. Mass of load hung from spring, 1·575 pounds; mass of spring, 0·6048 pound; time of one complete vibration, 0·631 second. Find the value of C.

12. Use the results obtained in Questions 10 and 11, and calculate the value of the bulk modulus K for the material of the spring; find also the value of Poisson's ratio.

PART II.

MACHINES AND HYDRAULICS.

CHAPTER XIV.

WORK, ENERGY, POWER, SIMPLE MACHINES

Work. **Work** is said to be done by a force when it acts through a distance. If a body A (Fig. 346) is at rest under the action of two equal forces P and R, no work is being done by either force; if the body is moving with constant speed towards the right, work is being done by P against the resistance R. Work is measured by the product of the magnitude of the force and the distance through which it acts, the latter being measured along, or parallel to, the line of the force. In the case of a car travelling along a level road (Fig. 347) no work is done by the weight W, nor by the reactions of the ground, as none of these forces advance through any distance in the directions of their lines of action. Work is done by the weight of the car in descending an incline (Fig. 348). If the total height of descent is H, then the work done by W will be WH. Or, the solution may be obtained by resolving W into components P and Q respectively, parallel and at right angles to the incline. Q does no work while the car is descending; P does work to the amount P × AB.

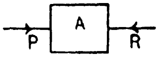

FIG. 346.

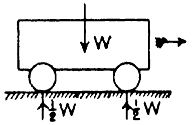

FIG. 347.

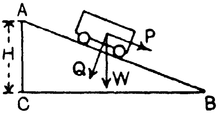

FIG. 348.

The unit of work in general use in this country is the **foot-pound,** and is performed when a force of one pound weight acts through a distance of one foot. The foot-ton, inch-pound, and inch-ton are used occasionally. Metric units of work are the gram-centimetre, the kilogram-centimetre and the kilogram-metre.

Energy. Energy means **capability of doing work**, and is measured by stating the units of work capable of being performed. There are many different forms of energy, such as **potential energy**, said to be possessed by a raised body in virtue of the fact that its weight may perform work while the body is descending; **kinetic energy**, which a body possesses when in motion and gives up while coming to rest; **elastic energy**, possessed by a body under strain and given out while coming back to its original form or dimensions; **heat energy**, which a body may give up in cooling to a lower temperature, **chemical energy**, which may be present in a substance owing to its constituents being capable of combining in such a way as to liberate energy in the form of heat; **electrical energy**, possessed by a body by virtue of its electric potential being higher than that of surrounding bodies.

Conservation of energy. The experience of all observers shows that the following general law is true: **Energy cannot be created nor destroyed, but can be converted from one form into another form.** This law is known as the **conservation of energy.** If no waste of energy were to occur during the conversion, a given quantity of energy in one form could be converted into an equal quantity in a different form. Exact equality never is obtained in practice; there is always waste, sometimes to a very large extent. For example, in converting the energy available in coal into mechanical work by means of a steam boiler and engine, it is common to find wasted 90 per cent. of the energy available, only 10 per cent. appearing in the desired form.

In measuring heat energy, the British thermal unit may be used, one such unit being the quantity of heat required to raise the temperature of one pound of water through one degree Fahrenheit. The pound-calorie unit is likely to be used more extensively in future, and is the quantity of heat required to raise the temperature of one pound of water through one degree Centigrade. The experiments of Dr. Joule and others show that an expenditure of 778 foot-pounds of energy will produce one British thermal unit; 1400 foot-pounds is the energy equivalent to a pound-calorie unit. Mechanical energy may be converted into heat without very large waste occurring (for example, in mechanically stirring water), but the reverse operation is always accompanied with great waste.

Power. Power means **rate of doing** work. The British unit of power is the **horse-power**, and is developed when work is being done at the rate of 33,000 foot-pounds per minute. The horse power in any given case may be calculated by dividing by 33,000 the work done per minute in foot-pounds.

The electrical power unit is the **watt**, and is the rate of working when an electric current of one ampere flows from one point of a conductor to another, the potential difference between the points being one volt. The product of amperes and volts gives watts. 746 watts are equivalent to the mechanical horse-power. When the amperes and volts are stated, we have

$$\text{Mechanical horse-power} = \frac{\text{amperes} \times \text{volts}}{746}.$$

The Board of Trade unit of electrical energy is one kilowatt maintained for one hour. One horse-power maintained for one hour would produce 33,000 × 60 or 1,980,000 foot-pounds. The kilowatt-hour would therefore produce energy given by

$$\text{Energy} = 1{,}980{,}000 \times \tfrac{1000}{746}$$
$$= \underline{2{,}654{,}000} \text{ foot-pounds.}$$

Machines. A machine is an arrangement designed for the purpose of taking in energy in some definite form, modifying it, and delivering it in a form more suitable for the purpose in view. Machines for raising weights are arranged conveniently in most mechanical laboratories, and experiments on such are very instructive. Fig. 349 shows, in outline, a small crab which may be taken as a type of such machines. A load W lb. is suspended from a cord wrapped round a drum, and is raised by the action of another load P lb. attached to a cord coiled round an operating wheel. The wheel and drum are connected by means of toothed wheels, so that P descends as W ascends.

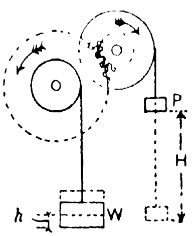

FIG. 349 Outline diagram of an experimental crab.

The **velocity ratio** of such a machine is defined as the ratio of the distance moved by P while W ascends a measured distance. Let H and *h* be these distances respectively in inches (Fig. 349); they may be measured direct in the machine. Then

$$\text{Velocity ratio} = V = \frac{H}{h} \quad . \quad . \qquad (1)$$

Let P be so adjusted that it will descend with steady speed on being started by hand, thus raising a load W. The **mechanical advantage** of the machine is defined by

$$\text{Mechanical advantage} = \frac{W}{P} \quad . \quad . \qquad (2)$$

By the principle of the conservation of energy, if no waste of energy occurs in the machine, the work done by P would be equal to the work done on the load. Suppose, in these circumstances, that the same working force P is employed, a larger load W_1 lb. could be raised than would be the case in the actual machine. W_1 may be calculated as follows:

$$\text{Work done by P} = \text{work done on } W_1,$$
$$PH = W_1 h,$$
$$W_1 = P\frac{H}{h} = PV$$
$$= P \times \text{the velocity ratio.} \quad \ldots\ldots \quad \ldots \quad (3)$$

The effect of frictional and other sources of waste in the actual machine has been to diminish the load from W_1 to W. Hence,

$$\textbf{Effect of friction} = F = W_1 - W$$
$$= PV - W. \quad \ldots \quad \ldots\ldots \quad \ldots\ldots \quad (4)$$

Efficiency of machines. The energy supplied to the machine is PH inch-lb. (Fig. 349), the energy actually given out by the machine is Wh inch-lb. The **efficiency** of the machine is defined by

$$\text{Efficiency} = \frac{\text{energy given out}}{\text{energy supplied}}$$
$$= \frac{Wh}{PH} = \frac{W}{P} \times \frac{1}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$
$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

The efficiency thus stated will be always less than unity. Efficiency is often given as a percentage, obtained by multiplying the result given in (5) by 100. 100 per cent. efficiency could be obtained only under the condition of no energy being wasted in the machine, a condition impossible to attain in practice.

From equation (3), we have

$$W_1 = P\frac{H}{h},$$
$$\frac{W_1}{P} = \frac{H}{h} = V. \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (6)$$

A result which shows that **the mechanical advantage of an ideal machine having no waste of energy is equal to the velocity ratio.**

For machines of the type described above, the following equation may be stated:

Energy supplied = energy given out + energy wasted in the machine.

Occasionally machines have to be considered in which there are internal springs or other devices for storing energy. In such cases the equation becomes:

Energy supplied = energy given out + energy stored in the machine + energy wasted in the machine.

A machine is said to be running light when no energy is being given out. If no energy is being stored in a machine running light, then the energy supplied must be sufficient to make good the energy wasted in overcoming the resistances in the machine.

Reversal of machines. A machine in which the frictional resistances are small may reverse if P is removed. To investigate this point, consider the machine when W is being raised (Fig. 349):

$$\text{Energy supplied} = PH,$$

$$\text{Energy given out} = Wh,$$

$$\text{Energy wasted} = PH - Wh. \quad \ldots\ldots \quad (7)$$

Now let P be removed and let the conditions be such that W is just able to reverse the machine. Let W descend through a height h. Then

$$\text{Energy supplied and wasted in the machine} = Wh. \quad \ldots\ldots\ldots (8)$$

Assuming that this waste has the same value as when W is being raised, we have, from (7) and (8),

$$PH - Wh = Wh,$$

$$PH = 2Wh.$$

Or,

$$\text{Efficiency} = \frac{Wh}{PH} = \frac{1}{2}.$$

Hence, when W is being raised, the efficiency will be 50 per cent. for reversal to be possible if P is removed. Any value of the efficiency exceeding 50 per cent. would be accompanied by the same effect.

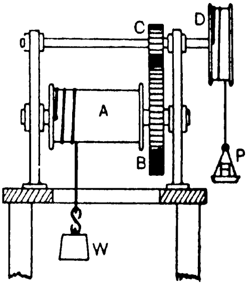

FIG. 350.—A small lifting crab.

The following record of tests on a lifting crab will serve as a model for carrying out experiments on laboratory machines.

EXPERIMENT ON A SMALL LIFTING CRAB.

Date of test, 10th February, 1911.

The machine used was constructed by students in the workshops of the West Ham Technical Institute. Its general arrangement in "single gear" is shown in Fig. 350. A weight W is suspended from a cord wrapped round a drum A. Motion is communicated to A by means of toothed wheels B and C; these are of gun-metal with machine-cut teeth. Energy is supplied by a descending weight P, which is attached to a cord wrapped round a wheel D.

The object of the experiments was the determination of the mechanical advantage and efficiency for various loads.

By direct measurement of the distances moved by P and W, the velocity ratio was found to be $V = 8 \cdot 78$.

The weight of the hook from which W was suspended is 1·75 lb. The weight of the scale pan in which were placed the weights making up P is 0·665 lb.

The machine having been first oiled, the weights W and P were adjusted so as to secure descent of P with steady speed. The results obtained are given below.

RECORD OF EXPERIMENTS AND RESULTS.

(1) W lb., including weight of hook	(2) P lb., including weight of scale pan.	(3) Load W_1 if no frictional resistances, $W_1 = PV$ lb.	(4) Effect of friction, $F = (W_1 - W)$ lb	(5) Mechanical advantage, $\frac{W}{P}$.	(6) Efficiency, per cent, $\frac{(5)}{V} \times 100$.
8·75	1·785	15·7	6·95	4·9	55·8
15·75	2·665	23·4	7·65	5·9	67·2
22·75	3·565	31·3	8·55	6·38	72·6
29·75	4·405	38·7	8·95	6·74	76·6
36·75	5·335	46·8	10·05	6·89	78·5
43·75	6·215	54·6	10·85	7·04	80·0
50·75	7·115	62·5	11·75	7·14	81·2
57·75	8·065	70·8	13·05	7·16	81·6
64·75	8·915	78·4	13·65	7·26	82·7
71·75	9·815	86·2	14·45	7·30	83·2
78·75	10·705	94·1	15·35	7·36	83·7
85·75	11·59	101·8	16·05	7·40	84·3
92·75	12·515	110	17·25	7·41	84·4
99·75	13·405	118	18·25	7·43	84·6
106·75	14·285	125·4	18·65	7·47	85·0
113·75	15·205	133·8	20·05	7·48	85·2
120·75	16·065	141	20·25	7·51	85·5
127·75	16·965	149	21·25	7·53	85·7

Curves are plotted in Fig. 351 showing the relation of P and W and also that of F and W. It will be noted that these give straight lines. Curves of mechanical advantage and of efficiency in relation to W are shown in Fig. 352. It will be noted that both increase rapidly when the values of W are small and tend to become constant when the value of W is about 120 lb. The efficiency tends to attain a constant value of 86 per cent.

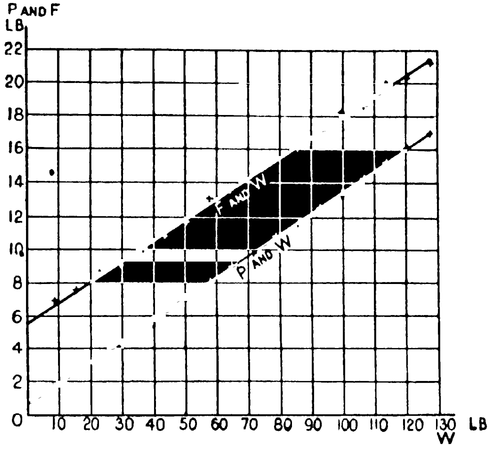

FIG. 351.—Graphs of F and W, and P and W, for a small crab.

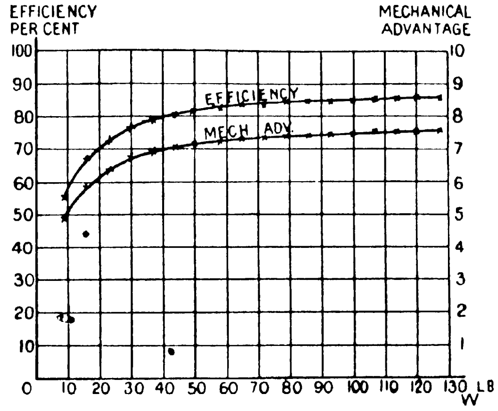

FIG. 352.—Graphs of efficiency and mechanical advantage for a small crab.

As both of the curves showing the relation of P and of F with W are straight lines, it follows that the following equations will represent these relations :

$$P = aW + b, \quad \ldots \quad (1)$$

$$F = cW + d, \quad \ldots \quad (2)$$

where a, b, c and d are constants to be determined.

Select two points on the PW graph, and read the corresponding values of P and W.

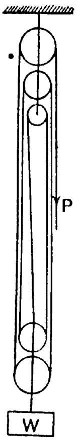

Fig. 353.—Pulley blocks.

$$P = 3{\cdot}5 \text{ lb. when } W = 22{\cdot}7 \text{ lb.}$$
$$P = 16{\cdot}0 \text{ lb. when } W = 120{\cdot}0 \text{ lb.}$$

Hence, from (1), $3{\cdot}5 = 22{\cdot}7a + b$,

$$16 = 120a + b.$$

Solving these simultaneous equations, we obtain

$$a = 0{\cdot}128,$$
$$b = 0{\cdot}64;$$
$$\therefore\ P = 0{\cdot}128W + 0{\cdot}64 \ldots\ldots\ldots\ldots (3)$$

Similarly,

When F = 8 lb., W = 20 lb.
When F = 18 lb., W = 100 lb.

Hence, from (2), $8 = 20c + d$,

$$18 = 100c + d.$$

The solution of these gives

$$c = 0{\cdot}125,$$
$$d = 5{\cdot}5.$$

Hence, $$F = 0{\cdot}125W + 5{\cdot}5. \ldots\ldots\ldots (4)$$

Suppose the machine to be running light, then W = 0, and the corresponding values of P and F obtained from (3) and (4) are

$$P = 0{\cdot}64 \text{ lb.},$$
$$F = 5{\cdot}5 \text{ lb.}$$

The interpretation is that a force of 0·64 lb. is required to work the machine when running light, and that, if there were no frictional waste, a load of 5·5 lb. could be raised by this force.

Hoisting tackle. The fact that the mechanical advantage of a machine, neglecting friction, is equal to the velocity ratio, enables the latter to be calculated easily in cases of hoisting tackle. A few such appliances, which may be found in most laboratories, are here given.

In the **pulley-block** arrangement shown in Fig. 353, let n be the number of ropes leading from the lower to the upper block. Neglecting friction, each of these ropes will support $\frac{1}{n}$th of W, and this will also be the value of P. Hence,

$$V = \frac{W}{P} = \frac{Wn}{W} = n.$$

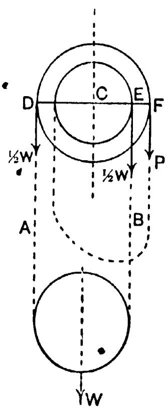

Fig. 354.—Weston's differential blocks.

A set of **Weston's differential blocks** is shown in outline in Fig. 354; the upper block has two pulleys of different diameters, and a chain, shown dotted, is used. The links of the chain passing round these pulleys engage with recesses which prevent slipping. Neglecting friction, each of the chains A and B will support $\frac{1}{2}$W. Taking moments about the centre C of the upper pulleys, and calling the radii R and r respectively, we have

$$\tfrac{1}{2}W \times CD = (P \times CF) + (\tfrac{1}{2}W \times CE),$$
$$\tfrac{1}{2}W(R - r) = PR;$$
$$\therefore\ V = \frac{W}{P} = \frac{2R}{R - r}.$$

Instead of R and r, the number of links which can be fitted round the circumferences of the pulleys may be used: evidently these will be numbers proportional to R and r.

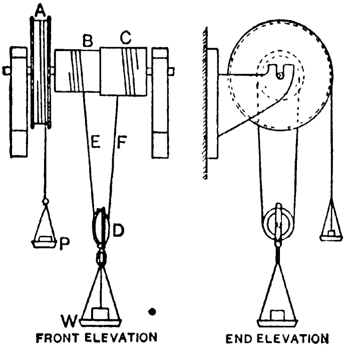

FIG. 355.—Wheel and differential axle

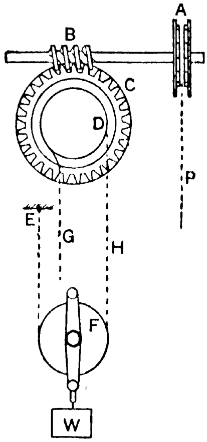

FIG. 356.—Helical blocks.

The **wheel and differential axle** (Fig. 355) is a similar contrivance, but has a separate pulley A for receiving the hoisting rope. Taking moments as before, we have

$$PR_A + \tfrac{1}{2}WR_B = \tfrac{1}{2}WR_C,$$
$$PR_A = \tfrac{1}{2}W(R_C - R_B);$$
$$\therefore\ V = \frac{W}{P} = \frac{2R_A}{R_C - R_B}.$$

A set of **helical** blocks is shown in outline in Fig. 356. A is operated by hand by means of a hanging endless chain and rotates a worm B, which in turn advances the worm wheel C one tooth for each revolution of A. If there be n_C teeth on C, then A will rotate n_C times for one revolution of C, and P will advance a distance $n_C L_A$, which is equal to n_C times the length of the number of links of the hanging chain which will pass once round A. The chain sustaining the load W is fixed at E to the upper block, passes round F, and then is led round D, which has recesses fitting the links in order to prevent slipping. Let L_D be the length of the number of links which will pass once round D. Then in one revolution of D, W will be raised through a height equal to $\frac{1}{2}L_D$. Hence,

$$V = \frac{n_C L_A}{\frac{1}{2} L_D} = \frac{2 n_C L_A}{L_D}.$$

EXPTS. 33 to 37. Experiments on the hoisting appliances described above should be carried out and the results reduced by methods similar to those explained for a small crab on p. 329.

Diagram of work. Since work is measured by the product of force and distance, it follows that the area of a diagram in which ordinates represent force and abscissae represent distances will

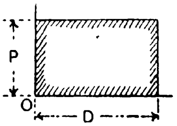

FIG. 357.—Diagram of work done by a uniform force.

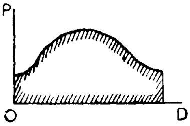

FIG. 358.—Diagram of work done by a varying force.

represent the work done. A uniform force P pounds acting through a distance D feet does work which may be represented by the area of a rectangle (Fig. 357). To obtain the scale of the diagram:

Let 1 inch height represent p lb.;
1 inch length represent d feet.

Then one square inch of area will represent pd foot-pounds of work. If the area of the rectangle is A square inches, then

$$\text{Work done} = pd\text{A foot-lb.}$$

In the case of a varying force, the work diagram is drawn by setting off ordinates to represent the magnitude of the force at different intervals of the distance acted through (Fig. 358). A fair curve drawn through the tops of the ordinates will enable the force to be measured at any stage of the distance. The work done is the product of the average force and the distance, and as the average force is given, to scale, by the average height of the diagram, and the distance, to scale, by the length of the diagram, we have, as before, the work done represented by the area of the diagram. Using the same symbols as before, one square inch of area represents *pd* foot-pounds of work, and the total work done will be given by

$$\text{Work done} = pdA \text{ foot-lb.}$$

The area A may be measured by means of a planimeter, or by use of any convenient mensuration rule (p. 6).

The case of hoisting at steady speed a load from a deep pit is of interest (Fig. 359). Let W_1 lb. be the weight of the cage and load, and let W_2 lb. be the total weight of the vertical rope when the cage is at the bottom, a depth of H feet. At first the pull P lb. required at the top of the rope will be $(W_1 + W_2)$ lb. P will diminish gradually as the cage ascends, and will become equal to W_1 when the cage is at the top. The work diagram for hoisting the cage and load alone is a rectangle ABCD, BC and AB representing W_1 and H respectively; the diagram for hoisting the rope alone is DCE, in which W_2 is represented by CE. From the diagrams, we have

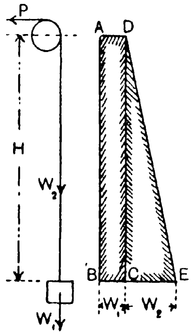

FIG. 359.—Diagram of work done in hoisting a load.

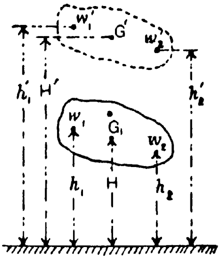

FIG. 360.—Work done in raising a body.

$$\text{Total work done} = (W_1 + \tfrac{1}{2}W_2)H \text{ foot-lb.}$$

Work done in elevating a body. It will be shown now that the work done in raising vertically a given body may be calculated by concentrating the total weight at the centre of gravity. Referring to Fig. 360, let w_1, w_2, etc., be the weights of the small particles of which the body is composed, and let h_1, h_2, etc., be their initial heights above ground level, and let h_1', h_2', etc., be their final heights. Then

$$\text{Work done on } w_1 = w_1(h_1' - h_1)$$

and
$$\text{Work done on } w_2 = w_2(h_2' - h_2), \text{ etc.}$$

Hence,

$$\text{Total work done} = (w_1h_1' + w_2h_2' + \text{etc.}) - (w_1h_1 + w_2h_2 + \text{etc.})$$
$$= \Sigma wh' - \Sigma wh.$$

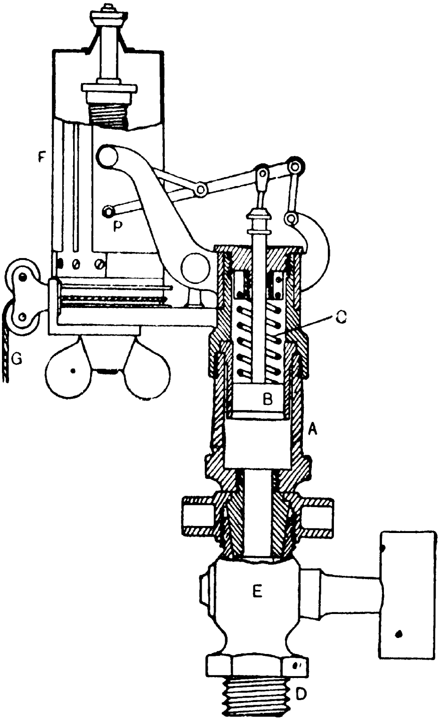

FIG. 361.—Thomson indicator.

Let G and G′ be the initial and final positions of the centre of gravity of the body, situated respectively at heights H and H′, and let W be the total weight of the body. Then

$$WH' = \Sigma wh' \text{ (p. 49)}$$

and
$$WH = \Sigma wh.$$

Hence, $\quad$ Total work done $= WH' - WH$

$$= W(H' - H).$$

Therefore the total work done in raising a body may be calculated by taking the product of the weight of the body and the vertical height through which the centre of gravity has been raised. This method is equivalent to concentrating the total weight at the centre of gravity.

Indicated work and horse-power. An **indicator** is an instrument used in obtaining a diagram of work done in the cylinder of an engine. The essential parts of an indicator are shown in Fig. 361. A small cylinder A is fitted with a piston B, which is controlled by a helical spring C. Connection is made at D to the engine cylinder; E is a stop cock. The piston B is connected by means of a piston rod to a lever system having a pencil fixed at P; the function of the lever system is to guide P in a straight vertical line, and to give it an enlarged copy of the motion of the piston B. As the spring follows Hooke's law, it follows that the movement of P will represent a definite pressure in pounds per square inch for each inch of vertical travel. The pencil moves over a piece of paper wrapped round a drum F. The drum is rotated in one direction by means of a cord G, and is brought back again by means of an internal spring. The cord G is actuated by some reciprocating part of the engine which gives it a reduced copy of the motion of the engine piston. Hence a diagram will be drawn on the paper showing pressures in the engine cylinder by its ordinates, and distances travelled by the engine piston by its abscissae (Fig. 362). The curve *abc* is for the forward travel of the piston, actuated by the steam or other pressure, and the curve *cde* is for the backward travel, and shows the exhaust.

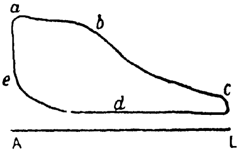

FIG. 362.—Work done during the forward stroke.

The work done during the stroke may be found by first obtaining the average height of the diagram inclosed by the curves in inches and multiplying this by the scale of pressure; the result gives the average pressure on the piston in pounds per square inch.

Let $\quad A =$ the effective area of the piston, in square inches.

$L =$ the length of the stroke, in feet.

$p_m =$ the average pressure, in lb. per square inch.

Then $\quad$ Work done per stroke $= p_m AL$ foot-lb.

In the case of a double-acting steam engine, the diagram of work for the other side of the piston will resemble Fig. 363. The effective area of one side of the piston (Fig. 364) will be $\frac{\pi D^2}{4}$, and of the other side $\frac{\pi}{4}(D^2-d^2)$. Let p'_m be the average pressure for the latter side of the piston. Then

$$\text{Work done per revolution} = \left\{p_m\frac{\pi D^2}{4}+p'_m\frac{\pi}{4}(D^2-d^2)\right\}L \text{ ft.-lb.}$$

The work done per minute will be obtained by multiplying by N, the revolutions per minute, and the **indicated horse-power**, written I.H.P., by dividing the result by 33,000.

Rough calculations are made often by neglecting the piston rod; thus A will be assumed as $\frac{\pi D^2}{4}$ for each side of the piston. A mean pressure p is taken as $\frac{1}{2}(p_m+p'_m)$ and used for both sides of the piston. The calculation for indicated horse-power will be given approximately by

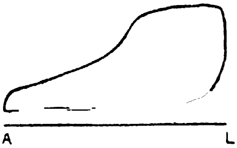

FIG. 363 – Work done during the return stroke

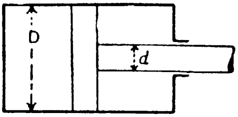

FIG. 364 – Double-acting steam engine cylinder.

$$\text{I.H.P.} = \frac{2pALN}{33000},$$

where N, as before, is the revolutions per minute.

In the case of a gas or oil engine, in which one side only of the piston is used, the other side being open to the atmosphere, the indicator diagram (Fig. 365) is used in the same manner to obtain the mean pressure. The work done during the cycle will be given by

$$\text{Work done} = pAL.$$

Let N_E = number of explosions per minute.

Then
$$\text{I.H.P.} = \frac{pALN_E}{33000}.$$

The indicated horse-power may be taken as a measure of the energy given to the engine piston during a stated time. A fraction only of this can be given out by the engine, the difference representing horse-power expended in driving the engine itself and overcoming the frictional resistances of its mechanism.

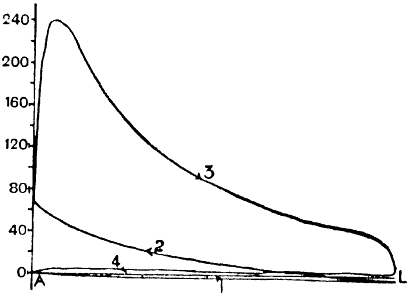

FIG. 365.—Indicator diagram from a gas engine cylinder

Brake horse-power. Provided the engine is not too large, the horse-power which the engine is capable of giving out in doing useful work may be measured by means of a **brake**. The result is called the **brake horse-power,** written B.H.P. It is evident that the efficiency of the engine mechanism will be given by

$$\text{Mechanical efficiency} = \frac{\text{power given out}}{\text{power supplied}}$$

$$= \frac{\text{B.H.P.}}{\text{I.H.P.}}.$$

This may be expressed as a percentage by multiplying by 100.

The horse power expended in overcoming the frictional resistances of the mechanism will be

$$\text{H.P. wasted in the engine} = \text{I.H.P.} - \text{B.H.P.}$$

Work done by a couple. Let equal forces P_1 and P_2 lb., forming a couple (Fig. 366), act on a body free to rotate about an axis at O and let the body make one revolution. As P_1 does not advance through any distance, it does no work. P_2 advances through a distance $2\pi d$ feet, where d is the arm of the couple in feet. Hence,

Work done by the couple per revolution

$$= P_2 \times 2\pi d$$

$$= P_2 d \times 2\pi$$

$$= \text{moment of couple} \times \text{angle of rotation in radians.}$$

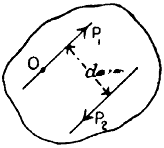

FIG. 366.—Work done by a couple.

The units of this result will be foot-lb. if the moment of the couple is stated in lb.-feet units. It is evident that any axis of rotation perpendicular to the plane of the couple may be chosen without altering the numerical result, because a couple has the same moment about any point in its plane (p. 59). Since the work done will be proportional to the angle of rotation, we have

Work done = (moment of couple in lb.-feet × α) foot-lb.,

where α is the total angle turned through in radians.

Let N = revolutions per minute,

T = moment of couple, in lb.-feet.

Then Angle of rotation = 2πN radians per minute;

Work done per minute = T × 2πN foot.-lb.

Advantage is taken of this result in estimating the brake horse-power of an engine.

Brakes. In the more usual form of brakes, frictional resistance is applied to the flywheel of the engine by means of a band. Rotation of the band is prevented by means of pulls applied by dead weights, or by spring balances. From the observed values of the pulls, the moment of the applied couple may be calculated. This, together with the revolutions per minute, enables the work done per minute and the horse-power to be calculated.

As the work done against the frictional resistance of the band is transformed into heat, and thus will cause the temperature of the wheel to rise, it is often necessary to adopt some means of cooling the wheel.

Rope brakes. A simple form of brake is shown in Fig. 367, and consists of two ropes passed round the wheel and prevented from slipping off sideways by means of wooden brake blocks, four of which are shown. A dead weight W is applied to one end of the ropes and a spring balance applies a force P to the other end. The net resistance to rotation will be (W − P), and this constitutes one force of the couple. The other equal force is Q, and arises from a pressure applied to the wheel shaft by its bearings. The forces W and P are applied at a radius R, measured to the centre of the rope. Hence, the moment of the couple applied is (W − P)R.

Let W = dead load, in lb.

P = spring balance pull, in lb.

R = radius to the rope centre in feet.

N = revolutions per min.

Then Work done per revolution $= (W - P) R \cdot 2\pi$ foot-lb.

" " min. $= (W - P) 2\pi RN$ foot-lb.

$$\text{Brake horse-power} = \frac{(W - P) 2\pi RN}{33000}.$$

In using a brake of this pattern, it is advisable to have W attached by a loose rope to an eyebolt fixed to the floor. This device will prevent any accident should the brake jam or seize.

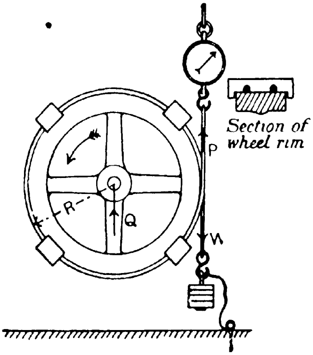

FIG 367 Simple rope brake.

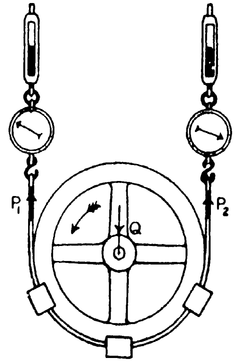

FIG. 368.—Rope brake for small powers.

In cases when the power is small, it may be better to pass the ropes round half the circumference of the wheel only (Fig. 368), using a spring balance at each end. The brake horse-power may be calculated from

$$\text{B.H.P.} = \frac{(P_1 - P_2) 2\pi RN}{33000}.$$

This plan has an advantage in the fact that both spring balances are assisting to sustain the weight of the wheel, and thus partially relieve the shaft bearings of pressure. Hence there will be lower frictional resistances in the engine and a slightly improved mechanical efficiency. A leather strap may be substituted for ropes in this kind of brake.

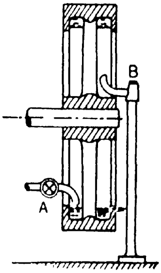

FIG. 369.—Arrangement for cooling the brake wheel.

Cooling of the wheel may be effected by having its rim of channel section (Fig 369) and running cold water in through a pipe A having a regulating valve. Centrifugal action maintains the water in the rim recess, provided the speed of rotation be sufficient. The heated water is removed gradually by means of a scoop pipe B having a sharpened edge, and thus a continuous water circulation is maintained.

Band brake. An excellent form of brake has been designed by Professor Mellanby of the Royal Technical College, Glasgow. An application of this brake to the flywheel of a steam engine of about 15 horse-power in the author's laboratory is shown in Fig. 370. A number of wooden blocks A are arranged round the circumference of the wheel,

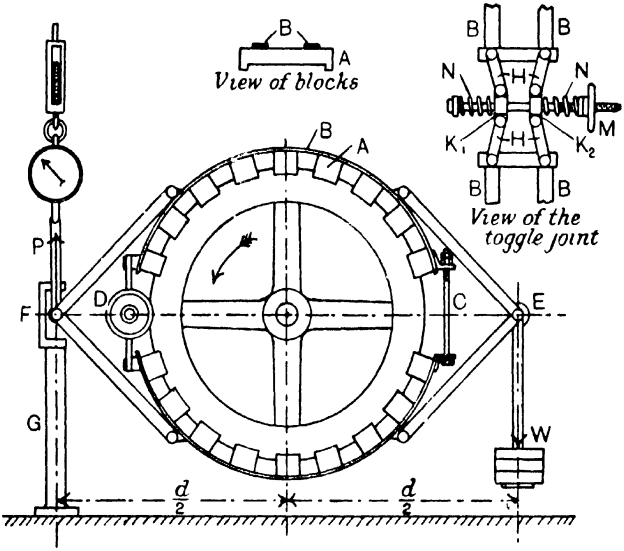

FIG. 370.—Mellanby type of band brake

and are held in position by hoop iron bands B, B. The brake bands are in halves, connected at C by means of long adjusting bolts fitted with lock nuts, and at D by means of toggle joints, by use of which the tension of the bands may be adjusted. A dead load W is hung from a pin E, which is attached to the brake hoops by four rods. A spring balance applies a pull P through a similar arrangement on the other side of the brake. There is a short column G fixed to the floor and slotted at its top end so as to restrict the movements of the pin F. Details of the toggle joint are shown separately. Two blocks K$_1$ and K$_2$ are connected by four links H and pins to the brake bands B. The blocks K_1 and K_2 may be made to approach one another and thus shorten the brake bands by means of the long bolt and the hand wheel M; a feather key in K_1 prevents rotation of the bolt. Helical springs N, N assist the adjustment.

In use, P and W are adjusted very easily so as to be equal. Hence a pure couple is applied to the wheel, and the shaft bearings are relieved of carrying any of the dead load W. The toggle joint adjustment is very good, and enables the frictional resistance of this particular brake to be adjusted within very fine limits. In the original large form, a dash-pot is introduced at G to subdue oscillations of the brake. This has not been found necessary in the smaller brake used by the author.

It will be noted that both P and W offer resistance to rotation.

Let d = horizontal distance between P and W in feet.

Then
$$\text{B.H.P} = \frac{Wd\,.\,2\pi N}{33000} = \frac{Pd\,.\,2\pi N}{33000},$$

provided P and W are adjusted so as to be equal. If they are not exactly equal, then their mean, $\frac{1}{2}(W + P)$, should be taken, giving

$$\text{B.H.P.} = \frac{\frac{1}{2}(W + P)\,2\pi dN}{33000}$$
$$= \frac{(W + P)\pi dN}{33000}.$$

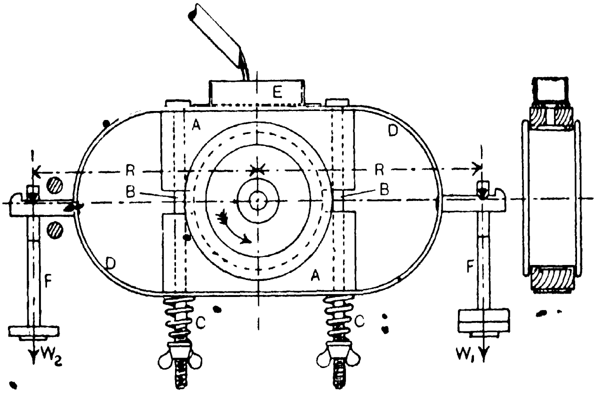

FIG. 371.—Brake for high speeds of rotation.

High-speed brake. In Fig. 371 is shown a brake for high speeds of rotation. The brake wheel consists of a flanged wheel mounted on the second motion shaft of a De Laval steam turbine, and runs at 3750 revolutions per minute. The brake blocks A, A are made of wood, and are pressed to the wheel by means of two bolts B, B fitted with wing nuts and helical springs C, C, the latter rendering it easy to adjust and maintain the desired pressure. A steel band D is fixed to the brake blocks, and serves to keep the parts together when the brake is removed and also for the application of the loads W_1 and W_2. The whole contrivance is balanced when W_1 and W_2 are removed (leaving the suspending rods F, F in position); hence the effective force is $(W_1 - W_2)$ lb. at a radius R feet.

$$\text{B.H.P.} = \frac{(W_1 - W_2)\,2\pi RN}{33000}.$$

Other methods of estimating effective horse-power. Hydraulic brakes have been used for fairly high powers. The principle of such brakes is to fit a badly designed centrifugal pump to the engine shaft. The pump wheel and casing are so constructed as to set up violent eddies in the water, with the result that there is considerable resistance opposed to rotation of the wheel. The pump casing is capable of rotating with the wheel, but is prevented from so doing by an attached lever and dead weight. The moment of this weight gives the couple required for the estimation of the energy absorbed. Brakes of this type were introduced by Professor Osborne Reynolds, and in his hands served not only for determining the horse-power of the engine, but also for the determination of the mechanical equivalent of heat. The latter experiment was carried out by observing the quantity and rise of temperature of the water passed through the brake in a given time.

The brake or effective horse-power of very large engines cannot be determined experimentally by use of a brake. If electrical generators are being driven, a close estimation may be made from the electrical energy delivered from the generator, making allowance for electrical and mechanical waste in the machine.

In electrical installations driven by steam turbines, the electrical horse-power alone can be measured, as no indicator diagrams can be obtained from turbines.

Shaft horse-power. Where turbines are adopted on board ships for driving the propellers, the **shaft horse-power** is measured, and corresponds to the brake horse power. The method consists in measuring the angle of twist in a test length of the propeller shafting by means of a **torsion-meter.** The test length is calibrated carefully before being placed on board, and should be re-calibrated at intervals, so that a curve is available showing the moment of the couple required to produce a given angle of twist. The turning moment on the shaft is obtained from the angle of twist indicated by the torsion-meter.

Let T = turning moment, in lb.-feet.
N = revolutions per min.

Then
$$\text{Shaft horse-power} = \frac{T \times 2\pi N}{33000}.$$

Shaft calibration In Fig. 372 is shown the method employed at the Thames Iron Works engine department for calibrating the test length of shaft; the view is a plan. The shaft AB has flanged ends solid with the shaft, and is bolted at A to a very rigid bracket C; a bearing at D supports the other end. A beam EF is bolted to the end B of the shaft, and couples may be applied by means of the upward pull of a 5-ton Denison weigher at E, and the equal downward force applied by placing weights in a skip hung from F. GH, KL and MN are balanced arms fixed to the shaft, and have verniers and scales at the ends H, L and N which serve to measure the angle of twist independently of the torsion-meter. The arm GH is bolted to the flange at A, and indicates any yielding of the bracket C or of the fixing of the shaft to the bracket. The difference of the readings at L

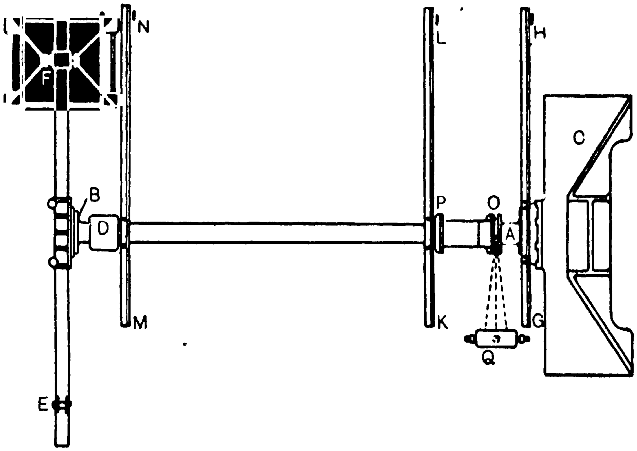

FIG. 372.—Arrangement for calibrating a shaft for shaft horse-power.

and N will give the angle of twist of the shaft between the arms KL and MN, and will not be affected by any yielding of the bracket or other fixings.

The torsion-meter is fixed to the shaft at OP, and is of the Hopkinson-Thring type; the lamp and scale are situated at Q. In Fig. 373 is shown the arrangement of the torsion-meter. C is a sleeve made in halves and clamped to the shaft AB, which it grips at its left-hand end. D is a collar, also made in halves, and clamped to the shaft. The angle through which C twists relative to D is measured by means of a small mirror at E. The mirror may rotate slightly about a radial axis on the collar D, and is controlled by a short rod attached to the sleeve C at F. A ray of light from the lamp H is reflected and changed in direction horizontally by the mirror. Two mirrors are used at E, placed back to back, and the ray is reflected to the scale when E arrives at the top and also when it is at the bottom; when at the top, the ray is reflected to the left part of the scale, and is reflected to the right part when E arrives at the bottom. Owing to the rapid rotation of the shaft, these intermittent rays produce practically a continuous light on the scale. A separate fixed mirror (not shown in the illustration) is attached to the sleeve and serves as a zero pointer on the scale. The scale and lamp are carried on trunnions to facilitate the preliminary adjustment required in order to secure that both zero mirror and movable mirror give the same scale reading when there is no torque on the shaft.

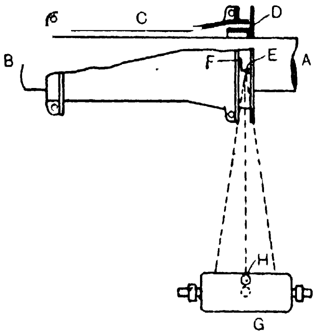

FIG. 373.—Hopkinson Thring torsion-meter

The following records were obtained by Mr. C. H. Cheltnam during a calibration test with the apparatus described above:

Calibration of a Propeller Shaft.

External diameters of the shaft between the vernier arms:

12·25 inches for a length of 6 inches;
11·375 inches for a length of 24·75 inches;
11·25 inches for a length of 134 inches.

Diameter of the hole in the shaft, 6·75 inches.

Distance between the clamping planes of the vernier arms, 164·75 inches.

Radius of the vernier arms, 114·6 inches.

(Since one radian = 57·3 degrees, a movement of 2 inches at the vernier represents one degree twist on a length of shaft of 164·75 inches.)

External diameter of the shaft at the torsion meter, 11·25 inches.

Diameter of the shaft hole at the torsion meter, 6·75 inches.

Distance between the clamping planes of the meter, 33·625 inches

One division on the torsion-meter scale corresponds to an angle of twist of $\frac{1}{638}$ degrees.

LOG OF TEST

No	Torque, lb feet	Vernier readings, inches		Angle of twist, degrees, by verniers	Torsion-meter readings
		No 1	No 2		
1	16,800	0 075	0 300	0 1125	15 2
2	33,600	0·150	0 610	0 2300	30 2
3	50,400	0·225	0 915	0 3450	45 5
4	67,200	0·300	1·225	0 4625	61 0
5	84,000	0 380	1·535	0·5775	77 0
6	100,800	0·460	1·850	0 6950	92 5
7	117,600	0 545	2·160	0 8075	108·0
8	134,400	0 620	2·480	0 9300	124·0

The torques and angles of twist obtained from the vernier readings are plotted in Fig. 374; in Fig. 375 the torsion-meter readings and torques have been plotted; both give straight lines.

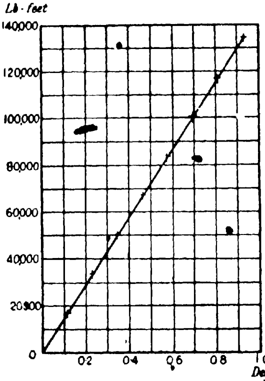

FIG. 374.—Graph of the vernier readings.

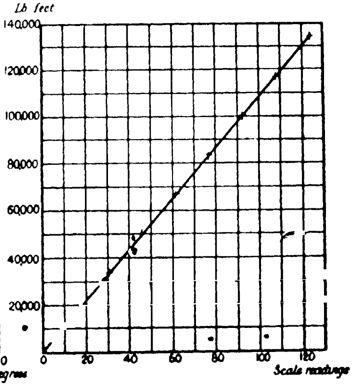

FIG. 375.—Graph of the torsion-meter readings.

To check the meter readings, the modulus of rigidity is calculated (*a*) from the vernier readings, (*b*) from the meter readings.

(*a*) The torque at No. 6 is 100800 × 12 = 1,209,600 lb.-inches, and gives an angle of twist of 0·6950 degree = 0·01213 radian. Using equation (5) (p. 255) for the angle of twist of a hollow shaft, viz.

$$a = \frac{2TL}{\pi(R_1^4 - R_2^4)C} \text{ radians, } \ldots\ldots\ldots\ldots\ldots(1)$$

and modifying it to suit the case of a shaft of three different external radii R_a, R_b, R_c, and corresponding lengths L_a, L_b, L_c, the same torque being applied throughout, we have

$$a = \frac{2TL_a}{\pi(R_a^4 - R_2^4)C} + \frac{2TL_b}{\pi(R_b^4 - R_2^4)C} + \frac{2TL_c}{\pi(R_c^4 - R_2^4)C},$$

or

$$C = \frac{2T}{\pi a}\left[\frac{L_a}{R_a^4 - R_2^4} + \frac{L_b}{R_b^4 - R_2^4} + \frac{L_c}{R_c^4 - R_2^4}\right]$$

$$= \frac{2 \times 1209600}{\pi \times 0{\cdot}01213}\left[\frac{6}{6{\cdot}125^4 - 3{\cdot}375^4} + \frac{24{\cdot}75}{5{\cdot}687^4 - 3{\cdot}375^4} + \frac{134}{5{\cdot}625^4 - 3{\cdot}375^4}\right]$$

$$= \underline{11{,}774{,}000} \text{ lb. per square inch.}$$

(*b*) The torque at No. 6 is 1,209,600 lb.-inches, and gives a scale reading on the meter of 92·5. Hence,

$$a = \frac{1}{638} \times 92{\cdot}5 \times \frac{1}{57{\cdot}3} = 0{\cdot}00253 \text{ radian.}$$

From (1),

$$C = \frac{2TL}{\pi(R_1^4 - R_2^4)a}$$

$$= \frac{2 \times 1209600 \times 33{\cdot}625}{\pi(5{\cdot}625^4 - 3{\cdot}375^4) \times 0{\cdot}00253}$$

$$= \underline{11{,}742{,}000} \text{ lb per square inch.}$$

The agreement of these values of C is close enough testimony to the accuracy of the meter. To obtain the shaft horse-power constant, we have

$$\text{Shaft horse power} = \frac{T \times 2\pi N}{33000},$$

where T is the torque in lb.-feet and N is the revolutions per minute.

At No. 6, the torque is 100,800 lb.-feet and the meter reading is 92·5 scale divisions. Hence,

$$\text{Torque for one scale division} = \frac{100800}{92{\cdot}5}$$

$$= 1090 \text{ lb.-feet.}$$

Let the meter reading be n scale divisions. Then

$$T = 1090n \text{ lb.-feet.}$$

Hence, $$\text{Shaft horse-power} = \frac{1090n \times 2\pi N}{33000}$$

$$= \frac{1}{4.82} nN.$$

EXAMPLE. At the steam trial of the vessel to which this shaft was fitted, the mean meter reading was 98 scale divisions at 300 revolutions per minute. Find the shaft horse-power.

$$\text{Shaft horse power} = \frac{1}{4.82} nN$$

$$= \frac{98 \times 300}{4.82}$$

$$= \underline{\underline{6100.}}$$

Transmission dynamometers are sometimes used for estimating the horse-power required to drive a given machine. The principle of the Froude or Thorneycroft dynamometer is shown in Fig 376. A is a pulley on the line shaft; B is a pulley on a shaft connected to the machine to be driven. A drives B by means of a belt passing round pulleys C and D which are mounted on a frame pivoted at F. When power is being transmitted, the pulls T_1, T_1 of the belt are greater than T_2, T_2; hence a force P applied to the frame at G is necessary in order to preserve equilibrium. Taking moments about F, we have

$$P \times GF = (2T_1 \times FC) - (2T_2 \times FD).$$

The arms FC and FD are usually equal. Hence,

$$P \times GF = 2FC(T_1 - T_2),$$

or $$T_1 - T_2 = \tfrac{1}{2}P \times \frac{GF}{FC}.$$

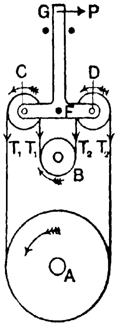

FIG. 376.—Froude or Thorneycroft dynamometer

Let R = the radius of pulley B in feet.
N = revs. per min. of pulley B.

Then $$\text{H.P.} = \frac{(T_1 - T_2)2\pi RN}{33000}.$$

The more usual method now adopted is to drive the machine direct by means of an electro-motor and measure the electrical horse-power consumed.

EXERCISES ON CHAPTER XIV.

1. Calculate what useful work is done in pumping 1000 gallons of water to a height of 60 feet. If this work is done in 25 minutes, what horse-power is being developed? Suppose that the efficiency of the pumping arrangements is 55 per cent., and find what horse-power must be supplied.

2. A load of 4000 lb. is raised at steady speed from the bottom of a shaft 360 feet deep by means of a rope weighing 10 lb. per yard. Calculate the total work done, and draw a diagram of work.

3. A loaded truck has a total weight of 15 tons. The frictional resistances amount to 12 lb. per ton. Calculate the work done in hauling it a distance of half a mile (*a*) on a level track, (*b*) up an incline of 1 in 80.

4. Find the price in pence per 1000 foot-lb. of energy purchased in the following cases:

(*a*) Coal, of heating value 15,000 British thermal units per pound, at 16 shillings per ton.

(*b*) Petroleum, of heating value 19,500 British thermal units per pound, at 10*d.* per gallon weighing 8·2 lb

(*c*) Gas, of heating value 520 British thermal units per cubic foot, at 2·25 shillings per 1000 cubic feet.

(*d*) Electricity, at 1·5*d.* per Board of Trade unit.

5. In a machine used for hoisting a load the velocity ratio is 45, and it is found that a load of 180 lb. can be raised steadily by application of a force of 12 lb. Find the mechanical advantage, effect of friction and the efficiency. Would there be any danger of reversal if the force of 12 lb. were removed?

6. A load of 1200 lb. is raised by means of a rope provided with an arrangement for indicating the pull at any instant. The following observations were made:

Height above ground, feet	0	10	20	35	50	65	80
Pull in rope, lb.	2000	1950	1880	1800	1750	1650	1500

Find approximately the work done on the load.

7. The cylinder of a steam engine is 30 inches in diameter, and the stroke of the piston is 4 feet. The piston rod is 5 inches in diameter. Suppose the mean pressure for both sides of the piston to be 65 lb. per square inch, what will be the horse-power at 75 revolutions per minute?

8. A rope brake is fitted to a flywheel 3 feet in diameter to the rope centre and running at 220 revolutions per minute. It is desired to absorb 7 brake horse-power. What should be the difference in the pulls at the two ends of the rope?

9. A shaft 6 inches in diameter runs at 180 revolutions per minute and transmits 900 horse-power. Assume the torque to be uniform, and calculate its value.

10. In Question 9 a torsion-meter is fitted to a shaft at points 6 feet apart. Taking C to be 5500 tons per square inch, what angle of twist, in degrees, will be indicated by the instrument?

11. In calibrating a propeller shaft by use of the apparatus illustrated in Fig. 372, the following observations were made. External and internal diameters of the hollow shaft, 7 inches and 4·017 inches respectively; distance between the clamping planes of the vernier arms, 51 inches; distance between the clamping planes of the torsion-meter, 25·5 inches; radius of the vernier arms, 10·5 inches. 24,000 divisions on the meter scale correspond to an angle of twist of 1 radian.

Test No.	Torque, lb.-inches	Difference in vernier readings, inches	Angle of twist on a length of 51 inches, by verniers, radian	Torsion meter readings	Angle of twist on a length of 25·5 inches, by meter, radian.
1	0	0·0000		0	
2	80,640	0·1725		19·85	
3	108,864	0·2300		26·75	
4	181,440	0·3875		44·60	
5	254,016	0·5425		62·65	
6	338,688	0·7200		82·75	
7	431,424	0·9250		105.50	

Fill in the blank columns. Plot (*a*) torque and angle of twist by verniers, (*b*) torque and angle of twist by meter. Find and compare the torques required to produce 0·001 radian twist on a length of 51 inches, (*c*) by verniers, (*d*) by meter, (*e*) find the modulus of rigidity from the vernier readings; (*f*) find the shaft horse-power constant from the meter readings. On steam trials the mean torsion-meter reading was 98·5 and the revolutions per minute 665; (*g*) find the shaft horse-power.

12. Estimate in ton-inches the maximum torsion of a shaft driven by an engine of 500 I.H.P. at a speed of 200 revolutions per minute, allowing an efficiency of 85 per cent. and a ratio of maximum to mean turning effort of 1·25. (I.C.E.)

13. A destroyer has a solid circular propeller shaft, 9½ inches in diameter, which makes 400 revolutions per minute. A torsion-meter, fixed to the shaft, shows that the angle of twist over a length of 20 inches is 0·15°. If the modulus of rigidity is 5000 tons per square inch, find the horse-power transmitted through this shaft. (B.E.)

14. Explain how the work done by a varying force can be measured by means of an indicator diagram. The pressure on a piston P working in a cylinder AB of length 3 feet is proportional to its distance from A. If the pressure on the piston at B is 150 lb. weight, draw a diagram showing the pressure in any position, and find the work done as the piston moves from B to A. (L.U.)

15. Describe a differential pulley block. The diameters of the two grooves are 12 and 11·5 inches, what is the velocity ratio? Experiments are made on this pulley block when a load W is lifted by an effort E. When W was 600 lb. E was 26 lb., and when W was 300 lb. E was 18 lb.: what is E probably when W is 800 lb.? What is the efficiency when W is 800 lb.? (B.E.)

16. An electrometer lifts 80 tons of grain 100 feet high; the electric energy costs 40 pence at the rate of 2 pence per unit. How much electric energy is used? What is the efficiency of the lifting arrangements? (B.E.)

CHAPTER XV.

FRICTION.

Definitions. When two bodies are pressed together it will be found that there is a resistance offered to the sliding of one upon the other. This resistance is called the **force of friction**. The force which friction offers always acts contrary to the direction of motion of the body, or, if the body is at rest, the force tends to prevent motion.

Let two bodies A and B (Fig. 377 (*a*)) be pressed together so that the mutual pressure perpendicular to the surfaces in contact is R. Let B be fixed, and let a force P, parallel to the surfaces in contact, be applied. If P is not large enough to produce sliding, or, if sliding with steady speed takes place, B will apply to A a frictional force F equal and opposite to P (Fig. 377 (*b*)). The force F may have any value lower than a certain maximum, which depends on the magnitude of R and on the nature and condition of the surfaces in contact. If P is less than the maximum value of F, sliding will not occur; sliding will be on the point of occurring when P is equal to the maximum possible value of F. It is found that the frictional resistance offered after steady sliding conditions have been attained is less than that offered when the body is on the point of sliding.

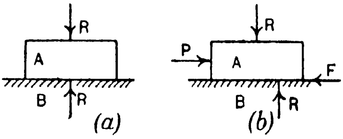

FIG. 377.—Force of friction.

Let F_s = frictional resistance in lb. when the body is on the point of sliding.

F_k = frictional resistance in lb. when steady sliding has been attained,

R = perpendicular pressure in lb. between the surfaces in contact.

Then
$$\mu_s = \frac{F_s}{R},$$
$$\mu_k = \frac{F_k}{R},$$

are called respectively the **static** and **kinetic coefficients of friction.**

Friction of dry surfaces. For dry clean surfaces, experiments show that the following laws are complied with approximately:

The force of friction is practically proportional to the perpendicular pressure between the surfaces in contact, and is independent of the extent of such surfaces and of the speed of rubbing, if moderate. Another way of expressing the same laws is to say that for two given bodies, **the kinetic coefficient of friction is practically constant for moderate pressures and speeds.** It is very difficult to secure any consistent experimental results on the static coefficient of friction; it is roughly constant for two given bodies.

The value of the coefficient of friction in any given case depends on the nature of the materials, especially on the hardness and ability to take on a smooth regular surface, and on the state of the rubbing surfaces as regards cleanliness. Rubbing surfaces are made usually of fair shape and are well fitted to one another. If clean and dry, a film of air may be present between the surfaces and prevent actual contact. Pressure and working may squeeze this film out, and the bodies will then adhere strongly together, or **seize.** Seizing takes place more rapidly with bodies of the same than with those of different materials.

Considerable increase in the speed of rubbing and also heating of the bodies tend to lower the value of the coefficient of friction. For this reason, the frictional force produced by the application of the brakes to the wheels of a locomotive running at high speed is higher during the first few seconds than is ultimately the case after the temperature has risen owing to the conversion of mechanical work into heat. The coefficient rises again when the speed becomes very slow, and may become sufficiently high to cause the wheels to skid just before stopping. The coefficient of friction for light pressures on large areas is a little greater than for heavy pressures on small areas.

The value of the coefficient of friction to be expected in any given case cannot be predicted with accuracy on account of the erratic nature of the conditions. The following table gives average values only; experimental results will often show considerable variance with the tabular values.

Coefficients of Friction.

AVERAGE VALUES.

Metal on metal, dry,	0·2 ;	oiled continuously,	0·05.
Metal on wood, dry,	0·6 ;	greasy,	0·2.
Wood on wood, dry,	0·2 to 0·5 ;	greasy,	0·1.
Hemp ropes on metal, dry,	0·25 ;	greasy,	0·15.
Leather belts on iron pulleys,	0·3 to 0·5.		
Leather on wood,	0·3 to 0·5.		
Stone on stone,	0·7.		
Wood on stone,	0·6.		
Metal on stone,	0·5.		

Fluid friction. For liquids such as water and oils flowing in a pipe, the following laws are followed approximately:

The frictional resistance is independent of the pressure to which the liquid is subjected, and is proportional to the extent of the surface wetted by the liquid.

The resistance is very small at slow speeds; below a certain critical speed the motion of the liquid is steady and the resistance is proportional to the speed; at speeds above this, the liquid breaks up into eddies, and the resistance is proportional approximately to the square of the speed.

The critical speed depends on the nature of the liquid and on its temperature. Rise of temperature of the liquid diminishes the resistance. The resistance is independent of the material of which the pipe or channel is made, but the wetted surface should be smooth; rough surfaces increase the resistance.

Friction in machine bearings. The frictional laws for lubricated machine bearings are intermediate between those for liquids and for dry surfaces. The ideal bearing would have a film of oil of uniform thickness, and would run at constant temperature. There would be no metallic contact anywhere, and the resistance would be that of metal rubbing on oil. In such a bearing, the laws of liquid friction would be followed, and the resistance would be independent of the load and proportional to the speed of rubbing. In ordinary bearings the resistances experienced depend on the success which is achieved in getting the oil into the bearing and in preserving the oil film; the working load is kept sufficiently low to avoid the danger of the film being squeezed out and seizing occurring.

Friction of journals. The value of the coefficient of friction to be expected in any given case depends largely on the method of lubrication. In Beauchamp Tower's* experiments, one method of lubrication adopted was to have an oil bath under the journal

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* *Proc. Inst. Mechanical Engineers*, 1883 *and* 1884.

(Fig. 378). Remarkably steady conditions were obtained, and it was ound that the coefficient of friction could be expressed by

$$\mu = \frac{c\sqrt{v}}{p}, \quad \text{.............................(1)}$$

where c is a coefficient the value of which depends on the kind of ubricant used, v is the speed of rubbing in feet per second, p is the pressure per square inch of projected area of the journal.

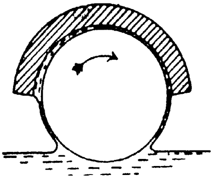

FIG. 378.—Oil bath lubrication.

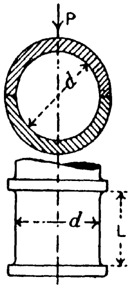

FIG. 379.—Projected area of a journal.

Let P = the total load on the bearing, in lb.
d = diameter of bearing, in inches.
L = length of bearing, in inches (Fig. 379).

Then Projected area of bearing $= dL$,

$$p = \frac{P}{dL} \text{ lb. per sq. inch. .....(2)}$$

The following table gives some of Tower's results:

JOURNAL FRICTION, OIL-BATH LUBRICATION.

Lubricant.	p, lb. per sq. inch.	v, feet per sec	μ	c, mean value for range of loads and speeds given.
Olive oil	520 100	2·61 7·85	0·0008 0·0089	0·29
Lard oil	520 100	2·61 7·85	0·0009 0·009	0·285
Mineral grease	625 153	2·61 7·85	0·001 0·0083	0·425
Sperm oil	310 100	2·61 7·85	0·0011 0·0064	0·205
Rape oil	415 153	2·61 7·85	0·0009 0·004	0·215
Mineral oil	310 100	2·61 6·99	0·0014 0·0073	0·27

It will be noted in (1) above that the coefficient of friction is inversely proportional to p, and hence is independent of the total pressure P on the bearing with oil-bath lubrication. It also follows that the frictional resistance of the bearing will be constant for all working loads, and will vary as the square root of the speed. Thus, referring to Fig. 380:

Let F = the frictional resistance of the bearing, lb.
P = the load on the bearing, lb.

Then, from (1) and (2) above

$$\mu = \frac{F}{P} = \frac{F}{pdL} = \frac{c\sqrt{v}}{p};$$

$$\therefore \ F = cdL\sqrt{v}; \qquad \ldots\ldots \qquad (3)$$

an expression which is independent of P.

With less perfect systems of lubrication, there is a tendency for the oil film to be broken partially, and higher coefficients of friction are obtained. In some cases the coefficient may reach values from 0·03 to 0·08.

Heating of journals. Work is done against the frictional resistance and is converted into heat. Referring to Fig. 380, in which D is the diameter of the journal in feet, we have

$$F = \mu P \text{ lb.}$$

Work done in one revolution $= \mu P\pi D$ foot-lb.
" per minute $= \mu P\pi DN$ foot-lb.,
where N is the number of revolutions per minute.

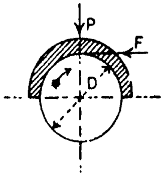

FIG. 380.—Friction of a journal.

Heat produced $= \frac{\mu P\pi DN}{778}$ British thermal units per minute.

This heat is dissipated by conduction and radiation, but the temperature of the bearing will rise during the early period of running. At higher temperatures the oil possesses lower **viscosity**, *i.e.* it flows more easily and offers less resistance to rubbing; hence less work will be done, and consequently less heat will be produced as the temperature rises. The tendency is therefore to attain a steady temperature, in which condition the heat developed will be exactly balanced by the heat carried away by conduction and radiation. It must be noted, however, that the lower viscosity possessed by the oil at the higher temperatures increases its liability to be squeezed out; hence, if steady conditions are to be attained, the load must not be too great and the oil must be of suitable quality. 100° Fahrenheit may be regarded as a safe limit of temperature under full working load. Occasionally the bearings are made hollow, and a water circulation is provided in order to keep the temperature low. Bearing pressures up to 3000 lb. per square inch are used, depending on the nature of the materials, method of lubrication, means of cooling, speed of rubbing, and on the consideration of whether the load always pushes on one side of the bearing or is alternately push and pull. Forced lubrication is often used, the oil being supplied under pressure to the bearings by means of a pump.*

Friction of a flat pivot. The case of a flat pivot or foot-step bearing (Fig. 381) may be worked on the assumption that the coefficient of friction μ is constant for all parts of the rubbing surface; the resultant frictional force F will then be found from

$$F = \mu P \text{ lb.}$$

If it is assumed that the distribution of bearing pressure is uniform, we have

$$\text{Load per unit area} = \frac{P}{\pi R^2}.$$

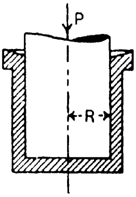

FIG. 381.—Flat pivot bearing.

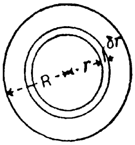

FIG. 382.

Consider a narrow ring (Fig. 382) having a radius r and a breadth δr.

$$\text{Area of the ring} = 2\pi r . \delta r.$$

$$\text{Load on the ring} = \frac{P}{\pi R^2} . 2\pi r . \delta r = \frac{2P}{R^2} . r . \delta r.$$

$$\text{Frictional force on the ring} = \frac{2P\mu}{R^2} . r . \delta r.$$

$$\text{Moment of this force} = \frac{2P\mu}{R^2} . r^2 . \delta r.$$

To obtain the total moment, this expression should be integrated over the whole rubbing surface; thus:

$$\text{Total frictional moment} = \frac{2P\mu}{R^2}\int_0^R r^2 . dr = \frac{2P\mu}{R^2} \cdot \frac{R^3}{3}$$

$$= \tfrac{2}{3}R . \mu P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= F \times \tfrac{2}{3}R. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1')$$

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* An excellent discussion of the theory of lubrication and design of machine bearings will be found in *Machine Design*, Part I., by Prof. W. C. Unwin. Longmans, 1909.

It is seen thus that the resultant frictional resistance F may be taken to act at a radius equal to two-thirds the radius of the bearing.

The frictional moment of a flat pivot may also be solved on the assumption that the wear is uniform and is proportional to the product pv, where p is the bearing pressure in lb. per square inch and v is the speed of rubbing at any given part of the rubbing surfaces. Thus,

$$pv = \text{a constant}.$$

Also the velocity v at any point varies as the radius r. Hence,

$$pr = \text{a constant} = a \text{ say};$$

$$\therefore p = \frac{a}{r}.$$

Considering the narrow ring (Fig. 382):

Load on the ring $= p . 2\pi r . \delta r = 2\pi a . \delta r$.

Friction on the ring $= 2\pi\mu a . \delta r$.

Integrating over the whole rubbing surface, we have

$$\text{Total frictional resistance} = F = 2\pi\mu a . \int_0^R dr$$

$$= 2\pi\mu a R. \qquad \text{..........(2)}$$

Again,

Moment of the friction on the ring $= 2\pi\mu a . r . \delta r$.

The total moment will be obtained by integration of this expression over the whole rubbing surface, thus:

$$\text{Total moment of friction} = 2\pi\mu a . \int_0^R r . dr = 2\pi\mu a . \frac{R^2}{2}$$

$$= \pi\mu a R^2$$

$$= F \times \tfrac{1}{2}R \text{ (from (2))} \qquad \text{......(3)}$$

$$= \tfrac{1}{2}\mu PR. \qquad \text{..................(3')}$$

It is probable that the actual value of the moment of friction will fall between the limits expressed in (1′) and (3).

In the case of a **collar** (Fig. 383), no great error will be made by assuming that F acts at the mean radius $\frac{1}{2}(R_1 + R_2)$. Hence,

$$\text{Moment of } F = \tfrac{1}{2}\mu P(R_1 + R_2) \text{ lb. inches.} \qquad \text{............(4)}$$

Tower's experiments on collar friction show that μ is independent both of speed and pressure unless the pressure is very small. The average value of μ found was 0·036. The bearing pressure should not exceed 50 lb. per square inch.

Tower also experimented with a flat pivot bearing 3 inches in diameter. If Tower's results obtained for the moments of friction be reduced from equation (1), thus:

$$\mu = \text{moment of F} \times \frac{3}{2PR},$$

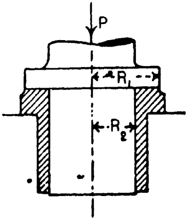


FIG. 383.—Collar bearings.

values of μ are found which vary from about 0·015 at 50 revolutions per minute and 40 lb. per square inch, bearing pressure to about 0·006 at 350 revolutions per minute and 100 lb. per square inch bearing pressure.

Schiele pivot. In the Schiele pivot (Fig. 384), the bearing is curved so as to secure uniform axial wear all over the surface. There is thus less likelihood of the oil film being squeezed out.

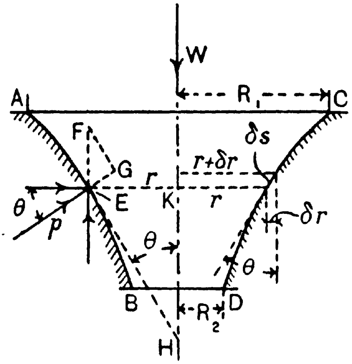


FIG. 384 —Schiele pivot.

Supposing the bearing to wear so that the point F descends ultimately to E, then EF is the axial wear, and is constant for any point on the bearing. EG is normal to the surface at E and FG is parallel to the tangent EH at E. The normal wear is EG, and may be assumed to be proportional to the speed of rubbing. The intensity of normal pressure at E is p; it is assumed that p is constant for all points on the surface. The velocity of rubbing v feet per second at any point evidently will be proportional to the radius r. Hence,

$$\mathrm{EG} \propto v \propto r;$$

$$\therefore \ \mathrm{EG} = kr,$$

where k is a constant. The triangles EFG and HEK are similar. Hence,

$$\frac{\mathrm{EF}}{\mathrm{EG}}=\frac{\mathrm{EH}}{\mathrm{EK}}=\frac{\mathrm{EH}}{r};$$

$$\therefore \ \mathrm{EF}=\frac{\mathrm{EG}\times\mathrm{EH}}{r}=\frac{kr.\mathrm{EH}}{r}$$

$$=k.\mathrm{EH}=\text{a constant.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Hence EH is constant; a curve such as AB having this property is called a **tractrix**.

Considering a narrow ring having a radius r and a horizontal breadth δr (Fig. 384), we have

Horizontal projected area of the ring $= 2\pi r.\delta r$.

Actual area of the ring $= 2\pi r.\delta r.\dfrac{1}{\sin\theta}$.

Also,
$$\frac{1}{\sin\theta}=\frac{\mathrm{EH}}{\mathrm{EK}}=\frac{\mathrm{EH}}{r};$$

$\therefore$ Actual area of the ring $= 2\pi.\delta r.\text{EH}.$
Normal pressure on the ring $= p \times 2\pi.\delta r.\text{EH}.$ .. ..(2)
Friction on the ring $= 2\pi.p\mu.\delta r.\text{EH}.$
Moment of this friction $= 2\pi p\mu.\text{EH}.r.\delta r.$

$$\text{Total moment of friction} = 2\pi p\mu\text{EH}\int_{R_2}^{R_1} r.dr$$
$$= 2\pi p\mu\text{EH}\frac{R_1^2 - R_2^2}{2}$$
$$= \pi p\mu\text{EH}(R_1^2 - R_2^2). \quad ...(3)$$

Again, (2) gives

Normal pressure on the ring $= 2\pi p.\text{EH}.\delta r.$
The vertical component of this $= 2\pi p.\text{EH}.\delta r.\sin\theta$

$$= 2\pi p.\text{EH}.\delta r.\frac{\text{EK}}{\text{EH}}$$
$$= 2\pi p.r.\delta r. \quad ......\ ...(4)$$

The sum of the vertical components for all the rings composing the curved surface of the bearing will be equal to W. Hence,

$$W = 2\pi p\int_{R_2}^{R_1} r.dr$$
$$= 2\pi p\frac{(R_1^2 - R_2^2)}{2}$$
$$= \pi p(R_1^2 - R_2^2). \quad ....(5)$$

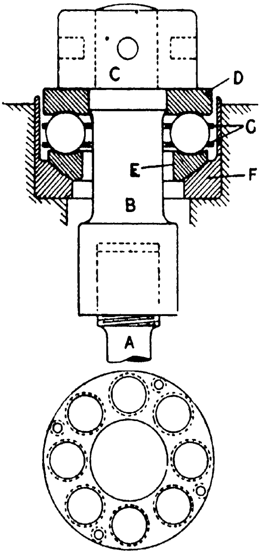
Substitution of this in (3) gives

$$\text{Total moment of friction} = \mu\text{W}.\text{EH}. \quad ......(6)$$

The Schiele pivot is not much used in practice on account of the difficulty of manufacture.

Rolling friction. In rolling friction, such as that of a wheel or roller travelling on a flat surface, the frictional resistances are roughly proportional to the load and inversely proportional to the radius of the wheel or roller. The resistance also depends on the hardness of the materials, and is comparatively small for very hard surfaces. In ball bearings, both balls and ball races are made of hardened steel; the races are best made concave, to a radius about 0·66 the diameter of the balls. This plan both reduces the resistance and enables a heavier load to be carried. In such bearings, the value of μ is practically constant through wide ranges of speeds and loads; 0·0015 is an average value.

Fig. 385 shows a heavy pattern of ball bearing made by The Hoffmann Co. and applied to a shackle B for holding one end of a test piece A undergoing both tension and torsion. The test piece is screwed to the shackle, the end of which is furnished with a nut C, which rests on the top ball race D. The bottom ball race E has its lower face made spherical to fit the corresponding spherical bottom of the cup F. This arrangement permits the test piece to accommodate itself to any want of alignment. A cage G made of two thin plates, secured together by means of four distance pieces, holds the balls in position and prevents them from coming into contact with one another; the cage also prevents any of the balls being lost when the bearing is taken to pieces. A similar bearing is applied to the other end of the test piece. The moment of friction is very small; with a tensile load of four tons and a test piece 1 inch in diameter, it is possible to rotate the whole by simply gripping the test piece with one hand.



Plan of Cage

FIG. 385.—Hoffmann thrust ball bearing

Resultant reaction between two bodies. In Fig. 386 (*a*) is shown a block A resting on a horizontal table BC. The weight W of the block will be constant, and will act in a line perpendicular to BC.

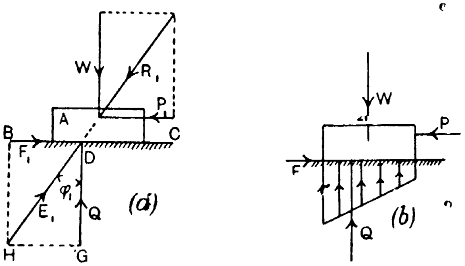


FIG 386.—Reactions at the surfaces in contact.

Let a horizontal force P_1 be applied to the block; P_1 and W have a resultant R_1. For equilibrium, the table must exert a resultant force on the block equal and opposite to R_1 and in the same straight line; let this force be E_1, cutting BC in D. E_1 may be resolved into two forces, one Q perpendicular to BC and the other F_1 along BC. Let ϕ_1 be the angle which E_1 makes with GD. Then

$$\frac{F_1}{Q} = \frac{HG}{GD} = \tan\phi_1.$$

Now, when P_1 is zero, ϕ_1 and hence $\tan\phi_1$ will also be zero, and Q will act in the same line as W. ϕ_1 will increase as P_1 increases, and will reach a maximum value when the block is on the point of slipping. It is evident that Q will always be equal to W. Let ϕ be the value of the angle when the block slips, and let F be the corresponding value of the frictional force. Then

$$\text{Coefficient of friction} = \mu = \frac{F}{Q} = \tan\phi.$$

There will be two values of $\tan\phi$ corresponding to the static and kinetic coefficients of friction respectively. When the block is on the point of sliding, ϕ is called the **friction angle** or the **limiting angle of resistance**; when steady sliding is occurring, ϕ is lower in value, and may be termed the **angle of sliding friction.**

It is evident from Fig. 386 (*a*) that P_1 and F_1 are always equal (assuming no sliding, or sliding with steady speed), so also are W and Q These forces form couples having equal opposing moments, and so balance the block. The force Q acting at D will give rise to normal stress of a distribution as shown by the stress figure in Fig. 386 (*b*). The action is partially to relieve the pressure near the right-hand edge of the block and to increase it near the left-hand edge. With a sufficiently large value of μ, and by applying P at a large enough height above the table, the block can be made to overturn instead of sliding. The condition of overturning may also be stated by reference to Fig. 387. Here the resultant R of P and W may fall outside the base AB before sliding begins. Hence E, which must act on AB, cannot get into the same line as R, and the block will overturn. For overturning to be impossible, R must fall within AB.

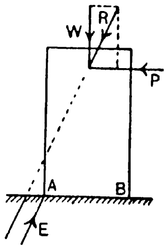


FIG. 387.—Condition that a block may overturn.

EXAMPLE. A wall of rectangular section 2 feet thick is subjected to a uniform normal pressure on one side of 50 lb. per square foot (Fig. 388). Taking the weight of material as 150 lb. per cubic foot and $\mu = 0{\cdot}7$, find whether sliding at the base is possible. For what height of wall would overturning just occur?

Consider a portion of wall one foot in length, and let H feet be the height. Then

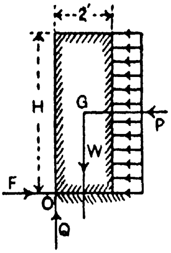


FIG. 388.—Stability of a wall.

$$W = 2H \times 150 = 300H \text{ lb. per foot run,}$$

$$\mu = \frac{F}{W};$$

$$\therefore \; F = \mu W = 0{\cdot}7 \times 300H$$
$$= \underline{210H} \text{ lb. per foot run.}$$

This result represents the maximum possible value of F.

Again, $P = \underline{50H}$ lb. per foot run.

Hence, as P will always be much less than the maximum frictional resistance possible, the wall will not slide.

When overturning is just possible, the resultant of P and W will act through O, and the moments of P and W about O will be equal. Taking moments about O, we have

$$\text{Moment of } P = 50H \times \frac{H}{2} = 25H^2 \text{ lb.-feet.}$$

$$\text{Moment of } W = 300H \times 1 = 300H \text{ lb.-feet.}$$

Equating these moments gives

$$25H^2 = 300H,$$
$$H = \underline{12} \text{ feet.}$$

Friction on inclined planes. In Fig. 389 (*a*) is shown a block of weight W lb. sliding steadily up a plane of inclination α to the horizontal, under the action of a horizontal force P lb. Draw AN perpendicular to the plane; then the angle between W and AN is equal to α. Draw AC making with AN an angle ϕ equal to the angle of sliding friction; the resultant reaction R of the plane will act in the line CA. The relation of P to W is deduced from the triangle of forces ABC.

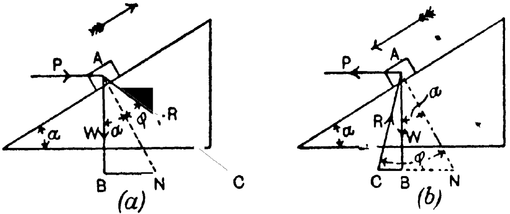


FIG. 389.—Friction on an incline; P horizontal.

$$\frac{P}{W}=\frac{BC}{AB}=\tan(\alpha+\phi),$$

$$\frac{P}{W}=\frac{\tan\alpha+\tan\phi}{1-\tan\alpha\tan\phi}\quad(\text{p. }8),$$

or

$$P=W\left(\frac{\tan\alpha+\mu}{1-\mu\tan\alpha}\right).\quad\ldots\ldots\ldots\ldots\ldots(1)$$

The case of the block sliding down is shown in Fig. 389 (*b*). Here R acts at an angle ϕ to AN, but on the other side of it.

$$\frac{P}{W}=\frac{BC}{AB}=\tan(\phi-\alpha),$$

$$\frac{P}{W}=\frac{\tan\phi-\tan\alpha}{1+\tan\alpha\tan\phi},$$

$$P=W\left(\frac{\mu-\tan\alpha}{1+\mu\tan\alpha}\right).\quad\ldots\ldots\ldots\ldots\ldots(2)$$

It will be noted in the last case, that if ϕ is less than α, the block will slide down without the necessity for the application of a force P. Rest is just possible, unaided, if α is equal to the limiting angle of resistance.

When P is applied parallel to the incline, the forces are as shown in Fig. 390 (*a*) and (*b*). For sliding up (Fig. 390 (*a*)), we have

$$\frac{P}{W}=\frac{BC}{AB}=\frac{\sin BAC}{\sin ACB}$$

$$=\frac{\sin(\alpha+\phi)}{\sin(90^\circ-\phi)}=\frac{\sin\alpha\cos\phi+\cos\alpha\sin\phi}{\cos\phi}$$

$$=\sin\alpha+\cos\alpha\tan\phi;$$

$$\therefore\ P=W(\sin\alpha+\mu\cos\alpha).\quad\ldots\ldots\ldots\ldots\ldots(3)$$

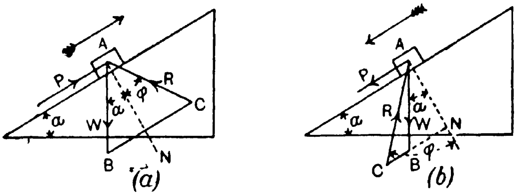


FIG. 390.—Friction on an incline ; P parallel to the incline.

For sliding down (Fig. 390 (*b*)), we have

$$\frac{P}{W}=\frac{BC}{AB}=\frac{\sin BAC}{\sin ACB}$$

$$=\frac{\sin(\phi-\alpha)}{\sin(90^\circ-\phi)}=\frac{\sin\phi\cos\alpha-\cos\phi\sin\alpha}{\cos\phi}$$

$$=\tan\phi\cos\alpha-\sin\alpha;$$

$$\therefore\ P=W(\mu\cos\alpha-\sin\alpha).\quad\ldots\ldots\ldots\ldots\ldots(4)$$

Friction of a screw. The results for a block on an inclined plane and acted on by a horizontal force may be applied to a square threaded screw. Such a screw may be regarded as an inclined plane wrapped round a cylinder. In Fig. 391 are shown two successive positions A and B of a block of weight W lb. being pushed up such an inclined plane by means of a horizontal force P lb. In the actual mechanism, the load is applied over a considerable portion of the surface of the incline (Fig. 392), and P may be assumed to act at the mean radius R inches of the screw. Let p inches be the pitch of the screw, and let one turn of the thread be developed as shown in Fig. 393.

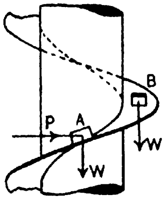


FIG. 391.—Inclined plane wrapped round a cylinder.

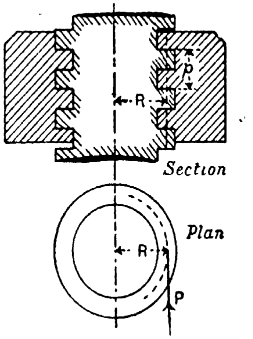


FIG. 392.—Friction of a square threaded screw

$$\tan \alpha = \frac{p}{2\pi R}$$

Using equation (1), p. 365, we have, for raising W,

$$P = W\left(\frac{\tan \alpha + \mu}{1 - \mu \tan \alpha}\right)$$

$$= W\left(\frac{\frac{p}{2\pi R} + \mu}{1 - \frac{\mu p}{2\pi R}}\right)$$

$$= W\left(\frac{p + 2\pi R\mu}{2\pi R - p\mu}\right). \quad \ldots\ldots\ldots\ldots\ldots (1)$$

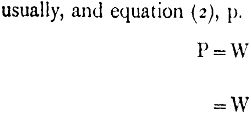


FIG. 393.—Development of one turn of a screw thread.

For lowering the load, P in Fig. 393 will be reversed in direction usually, and equation (2), p. 365, should be used:

$$P = W\left(\frac{\mu - \tan \alpha}{1 + \mu \tan \alpha}\right)$$

$$= W\left(\frac{2\pi R\mu - p}{2\pi R + p\mu}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $\mu = \tan \alpha$, or if $2\pi R\mu = p$, P will be zero, and the load will be on the point of running down unaided. Running down with continually increasing speed may occur if $\tan \alpha$ is less than μ, and may be prevented by application of a force P given by (2) above and applied in the same sense as for raising the load. If rotation of the screw is produced by means of a force Q lb. applied to a spanner at a radius L inches, the nut being fixed, we have

$$QL = PR,$$

$$Q = \frac{R}{L}P. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The above solution is applicable to the case of a screw-jack (Fig. 411), the friction of the screw alone enters into the problem.

The efficiency of such an arrangement may be calculated by considering the screw to make one revolution in raising the load. Then

$$\text{Efficiency} = \frac{\text{Work done on W}}{\text{Work done by Q}} = \frac{Wp}{Q \times 2\pi L}.$$

Also,

$$Q = \frac{R}{L}P,$$

$$\therefore \text{ Efficiency} = \frac{Wp}{\frac{R}{L}P \times 2\pi L} = \frac{W}{P} \cdot \frac{p}{2\pi R}.$$

Substituting for W/P from equation (1) above, we have

$$\text{Efficiency} = \left(\frac{2\pi R - p\mu}{p + 2\pi R\mu}\right)\frac{p}{2\pi R}. \qquad \ldots\ldots\ldots (4)$$

If n be the ratio of the mean circumference $2\pi R$ to the pitch p, so that $2\pi R = np$, equation (4) may be written:

$$\text{Efficiency} = \left(\frac{np - p\mu}{p + np\mu}\right)\frac{1}{n}$$

$$= \left(\frac{n - \mu}{1 + n\mu}\right)\frac{1}{n}. \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

EXAMPLE. In a certain square threaded screw, $n = 10$ and $\mu = 0{\cdot}125$. Find the efficiency while raising a load.

$$\text{Efficiency} = \left(\frac{10 - 0{\cdot}125}{1 + 10 \times 0{\cdot}125}\right)\frac{1}{10}$$

$$= 0{\cdot}44$$

$$= 44 \text{ per cent.}$$

In tightening a nut on a bolt (Fig. 394), not only has the moment of the friction of the screw to be considered, but also the friction

Let R_1 be the mean radius of the bearing surface in inches and W lb. be the pull on the bolt. Then

$$F = \mu W \text{ lb.}$$

$$\therefore \text{Moment of } F = \mu W R_1 \text{ lb.-inches.} \qquad \ldots\ldots(6)$$

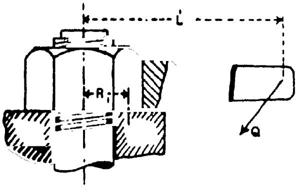


FIG. 394.—Friction of a nut.

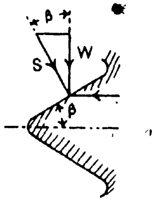


FIG. 395.—Friction of a V thread.

Let P lb. be the frictional resistance of the screw, found from equation (1), p. 366, and let R be the mean radius of the threads.

Then
$$QL = \mu W R_1 + PR. \qquad \ldots\ldots(7)$$

In **V** threaded screws (Fig. 395), the pressure between the bearing surfaces of the nut and the bolt threads is increased. If W is the load, it should be resolved into a force S perpendicular to the thread section and another horizontal force. Then

$$\frac{W}{S} = \cos\beta,$$

$$S = \frac{W}{\cos\beta} = W \sec\beta,$$

where β is half the angle of the **V**. All the results found for square threaded screws may be used for **V** threaded screws by writing $\mu \sec\beta$ instead of μ in the equations.

Friction circle for a journal. It is useful to consider the friction of a journal A resting on a loosely fitting bearing B (Fig. 396 (*a*)).

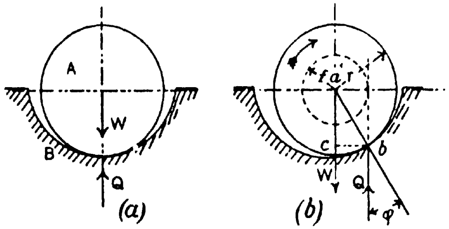


FIG. 396.—Friction of a loose bearing.

If there is no rotation, the load W on the journal will be balanced by an equal opposite reaction Q applied by the bearing. Let a couple of moment T be applied to the journal, of sufficient magnitude to produce steady rotation in the direction shown (Fig. 396 (*b*)). The journal will roll up the bearing until the place of contact is *b*, at which steady slipping will occur. The condition which fixes the position of *b* is that the vertical force Q acting at *b* must make an angle ϕ with the normal *ab*, ϕ being the angle of sliding friction. Q and W being still equal, form a couple of moment $W \times bc$, and this couple balances T, the couple applied. Hence,

$$T = W \times bc.$$

Also, $$\frac{bc}{ac} = \tan\phi = \frac{bc}{ab} \text{ very nearly};$$

$$\therefore\ bc = ab \tan\phi = r \tan\phi;$$

$$\therefore\ T = Wr\tan\phi. \qquad \ldots\ldots\ldots\ldots(1)$$

This will be in lb.-feet if r is in feet and W is in lb.

It will be noted that Q is tangential to a small circle of radius f, drawn with centre a. This circle is called the **friction circle**, and its radius is equal to *bc*. Hence,

$$\text{Radius of friction circle} = f = r\tan\phi \text{ very nearly}$$
$$= \mu r \text{ feet}, \qquad \ldots\ldots\ldots\ldots(2)$$

where $\mu = \tan\phi$, is the coefficient of friction,
$r =$ radius of journal in feet.

The same result is true for a closely fitting bearing (Fig. 397). Here R lb is the resultant reaction of the bearing, the components of which are Q, the resultant vertical reaction and F the resultant frictional force. R acts at an angle ϕ to Q for the direction of rotation as shown, or on the other side of Q for the contrary direction of rotation In either case, R is tangential to the friction circle, and gives a moment Rf lb.-feet opposing rotation.

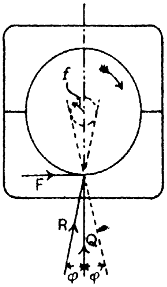


Fig. 397.—Friction circle of a bearing.

To obtain the work done, we have

$$\text{Frictional couple} = Rf \text{ lb.-feet},$$
$$\text{Work done in one revolution} = Rf \times 2\pi \text{ foot-lb}.$$

Let N = revolutions per minute.

Then $$\text{Work done per minute} = 2\pi N R f \text{ foot-lb.},$$

$$\text{H.P. wasted} = \frac{2\pi N R f}{33000}.$$

The value of R is given actually by

$$R = \sqrt{Q^2 + F^2},$$

but as the coefficient of friction and hence the frictional resistance is very small for well lubricated journals, no great error is made by taking R equal to Q, the load on the journal.

EXAMPLE 1. In a machine for raising a load W the load is suspended from a rope wound round a drum A, 8 inches in diameter, to the rope centre (Fig. 398). The axle on which the drum is fixed has journals 1·5 inches in diameter, and is rotated by a toothed wheel B, 18 inches in diameter, to which a force P is applied. Find the mechanical advantage and efficiency of the machine, taking the coefficient of friction of the bearings to be 0·1 and W to be a load of 500 lb.

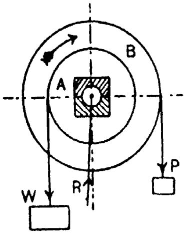


FIG. 398.—Friction of a simple machine.

Neglecting friction, and taking moments about the centre of the drum, we have

$$W \times 4 = P \times 9,$$

$$\frac{W}{P} = \frac{9}{4} = 2{\cdot}25. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$P = \frac{500 \times 4}{9} = 222{\cdot}2 \text{ lb.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Taking account of friction, and assuming that R is equal sensibly to (P + W), we have, by taking moments about the centre of the drum,

$$4W + (P + W)f = 9P.$$

Also,

$$f = \mu r$$

$$= \tfrac{1}{10} \times \tfrac{3}{4} = \tfrac{3}{40} \text{ inch.}$$

$$\therefore \; 2000 + (P + 500)\tfrac{3}{40} = 9P,$$

$$9P - \tfrac{3}{40}P = 2000 + \tfrac{1500}{40},$$

$$P = \frac{2037{\cdot}5 \times 40}{357}$$

$$= 228{\cdot}3 \text{ lb.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence, $\quad$ Mechanical advantage $= \dfrac{W}{P} = \dfrac{500}{228{\cdot}3}$

$$= \underline{2{\cdot}19}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Let the drum make one revolution. Then

Work done by $P = P \times \pi \,.\, 18$ inch-lb.

Work done on $W = W \times \pi \,.\, 8$ inch-lb.

$$\text{Efficiency} = \frac{8\pi W}{18\pi P} = \frac{8 \times 500}{18 \times 228{\cdot}3}$$

$$= 0{\cdot}97$$

$$= \underline{97} \text{ per cent.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

EXAMPLE 2. The mechanism shown in Fig. 399 consists of a crank OA fixed to a shaft having OZ for its axis of rotation. The crank is driven in the direction of rotation shown, by means of a slotted bar B; a block C may slide in the slot, and has a hole to receive the crank pin. The force P pushes during the stroke from right to left, and pulls during the return stroke. Show by drawing how the turning moment on the crank, as modified by friction, may be obtained. Give the construction for each quadrant, assuming $\mu = \tan \phi$ is the same for both block and pin.

In answering this question it is essential to remember that the force which the block gives to the crank pin must be tangential to the friction circle, and must act so as to oppose the motion of rotation of the pin.

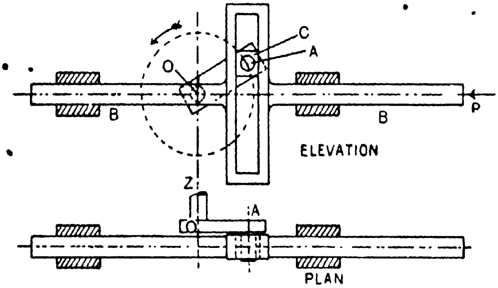


Fig. 399.—Crank and slotted-bar mechanism.

Further, the force which the slotted bar gives to the block must act at an angle ϕ to the normal, and must be applied so as to oppose the sliding motion of the block. For ordinary values of the coefficient of friction these forces, shown by R in Fig. 400, will be very nearly equal to P.

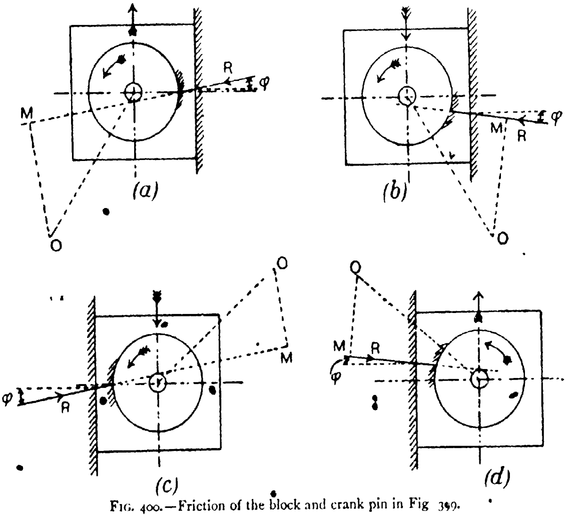


Fig. 400.—Friction of the block and crank pin in Fig 399.

The constructions required are shown in Fig. 400(*a*) to (*d*). In the first and fourth quadrants (*a*) and (*d*) the block is sliding upwards, and in the second and third quadrants (*b*) and (*c*) it is sliding downwards. In each case the turning moment is R × OM, OM being drawn perpendicular to R from O, the centre of the crank shaft.

Friction of the crank pin and the crosshead pin. Figs. 401 (*a*) and (*b*) show the application of the friction circle method to the determination of the line of thrust along a connecting rod, when account is taken of the friction at the crank pin and the crosshead pin. The diameters of the friction circles at A and B are calculated and the circles drawn. The line of thrust Q will be a common tangent to these circles for any given crank position. Draw OM perpendicular to the line of Q; the turning moment on the crank will be Q × OM. No difficulty will be experienced in choosing the line of Q if it is remembered that the frictional moments at A and B both tend to reduce the turning moment on the crank; hence the common tangent which gives the line of Q must be so drawn as to make OM a minimum. Thus, in Fig. 401 (*a*), Q touches the top of the circle at A and the bottom of that at B; in Fig. 401 (*b*), Q touches the bottom of both circles. The change in the line of Q from the top to the bottom of the circle at A takes place when the crank makes 90° with the centre line OA; in this position, the connecting rod makes its maximum angle with the centre line OA and has no angular motion for an instant, *i.e.* at this point the crosshead pin is not rubbing in its bearing. The solution for other positions is given at Q′ in Figs. 401 (*a*) and (*b*).

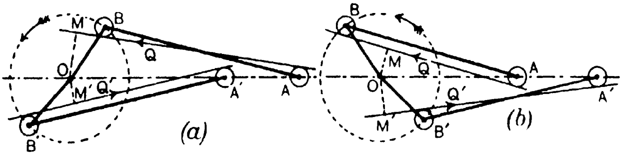


FIG. 401.—Friction of the crank pin and crosshead pin in a crank and connecting rod mechanism.

In Fig. 401 (*b*), it will be noted that, as B′ approaches the inner dead point, the line of Q′ will pass through O. In this position there is no turning moment on the crank. Further, the crank must rotate through a small angle beyond the dead point before the line of Q will pass above O. There will, therefore, be a small crank angle near each dead point in which there will be no turning moment tending to rotate the crank in the direction of rotation of the crank shaft. These angles may be determined approximately as follows: In Fig. 402, O is the crank shaft centre and A is the crosshead pin at the end of the stroke. Draw the friction circle at A; draw the lines of Q and Q′ touching the circle at A and passing through O. Draw the friction

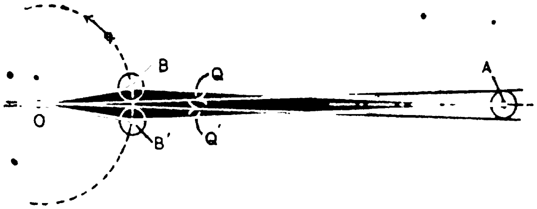


FIG. 402.—Angle of zero turning moment due to friction at the crank and crosshead pins; inner dead point.

circles at B and B′, touching the lines of Q and Q′; then BOB′ is the angle within which there is zero turning moment near the inner dead

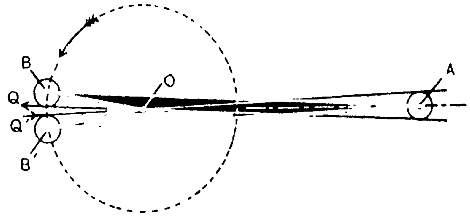


FIG. 403.—Angle of zero turning moment due to friction at the crank and crosshead pins; outer dead point.

point. The construction for the outer dead point is given in Fig. 403, and will be followed readily.

Friction of the crank-shaft bearings. The loads producing frictional resistances at the crank-shaft bearings include the weight of the shaft and its attachments, belt pulls or other forces due to the driving of machinery and a reaction owing to the thrust of the connecting rod. Considering the latter alone, and referring to Fig. 404, Q is the thrust of the connecting rod, making allowance for the friction of the crosshead pin and the crank pin as illustrated in Fig. 401 (*a*). A force Q′, equal, opposite and parallel to Q is applied by the crank-shaft bearing to the shaft, Q and Q′ together forming a couple which causes the shaft

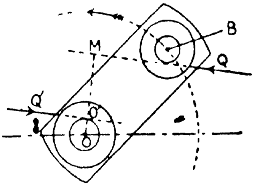


FIG. 404.—Friction of the crank-shaft bearings.

to rotate. During rotation, Q′ will be tangential to the friction circle for the shaft and is so shown in Fig. 404. Draw OM perpendicular to Q and cutting the shaft friction circle at O′. The effective turning moment will be Q × O′M, and has been diminished by an amount Q × OO′ by reason of the friction produced by Q in the shaft bearings.

Near the dead points, the effect of the friction at the crosshead pin, crank pin and crank-shaft bearings is that there will be a small angle embracing each dead point within which no force, however great, along the piston rod will cause the shaft to rotate if at rest. These

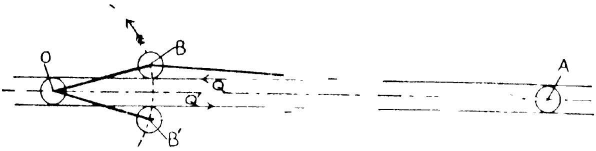


FIG. 405 — Dead angle at inner dead point

angles are called **dead angles** and are shown at BOB′ in Figs. 405 and 406. The construction is similar to that in Figs. 401 (*a*) and (*b*), with the addition of the friction circle at O for the crank-shaft bearings. The lines of the forces Q and Q′ are tangential to the friction circles at A and O, and the friction circles at B and B′ are drawn to touch the lines of the forces, produced if necessary.

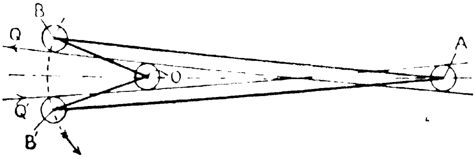


FIG. 406 - Dead angle at outer dead point.

The student will note that the dead angles so found take account only of the friction at the crank-shaft bearings produced by the thrust of the connecting rod. If at rest, the crank shaft will not commence rotation until the turning moment Q × O′M (Fig 404) is large enough to overcome the resisting moment due to the total friction at the crank-shaft bearings together with the resistances offered by any machinery to be driven.

Experiments on friction. Experiments have been described in Chapter XIV, in which the general effect of friction in the complete machine was one of the factors to be determined. The following additional experiments may be performed usefully.

EXPT. 38.—**Friction of a slider.** AB is a wooden board or flat piece of metal having its top surface brought horizontal by means of a spirit level (Fig. 407). A slider C, of wood or metal, may be drawn along AB by means of a horizontal force P applied by using a cord, pulley and scale pan. The upper surface of AB and the under surface of C should be clean and dry. Weigh the slider C and also the scale pan. R is the perpendicular reaction of the surfaces in contact, and is equal to the weight of the slider together with the load placed on it. Add loads to the scale pan, tapping AB gently after each load is applied, until the slider is drawn steadily along AB. P will be nearly equal to the weight of the scale pan together with the loads placed in it, and the kinetic friction F will have the same value. Calculate the kinetic coefficient of friction from

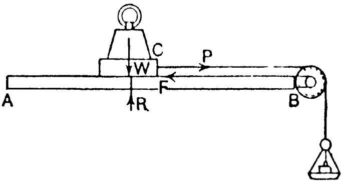


FIG. 407 Friction of a slider

$$\mu = \frac{F}{R}.$$

The experiment should be repeated with several different loads on the slider, and F and R should be tabulated for each. Plot F and R, if this gives a straight line, find the average value of μ from the graph.

Repeat the experiment, using different materials for the board and for the slider It is useful to have a set of sliders, all of the same material, but having the under sides cut away so as to give different areas of contact.

EXPT. 39.—**Determination of the angle of sliding friction.** In Fig. 408 AB is a board which may be set at different angles to the horizontal. A block C is placed on it, and the angle is varied until the block will slide steadily down after being assisted to start. Measure the angle BAD which AB makes with the horizontal; this will give the value of the angle of sliding friction. Calculate μ from

$$\mu = \tan \text{BAD}.$$

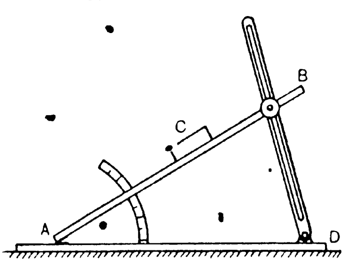


FIG. 408.—Apparatus for determining the angle of sliding friction.

Repeat the experiment using different materials.

EXPT. 40.—**Rolling friction.** In Fig. 409 is shown apparatus similar to that of Fig. 407, but having a small carriage mounted on wheels having bearings constructed to reduce friction as much as possible. The board should be levelled carefully, and the tractive effort P required to draw the carriage steadily along should be found for different loads on the carriage. It is useful to have three or four different roads for the carriage to run on; these may be of plate glass, metal, wood and rubber. The effect of the varying degrees of hardness should be contrasted by comparing the results for the different roads, and this may be done easily by plotting tractive effort and load for each road on the same sheet of squared paper.

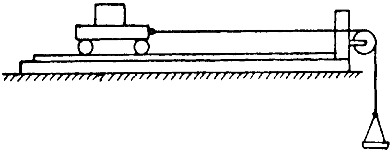


FIG. 409.—Apparatus for rolling friction.

EXPT. 41.—**Effect of speed of rubbing.** In Fig. 410, A is a wheel which may be rotated at different speeds by some source of power. B is a block which is pressed on the rim of the wheel by means of a shackle C and a load D. The block B is restrained from rotation by a cord and another load at E. The perpendicular pressure between the block and the wheel will be the weight of the block, together with those of the shackle, scale pan and load. The force of friction will be nearly equal to the combined weights of the scale pan and load at E. Hence μ may be determined for different speeds of rubbing. It will be observed that the friction is greater on starting with both wheel and block cold, and diminishes after a few seconds as the rubbing parts become warm. The experiment should be repeated with blocks of different materials.

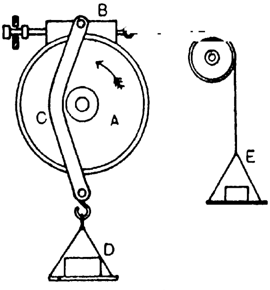


FIG. 410.—Apparatus for investigating the effect of speed of rubbing.

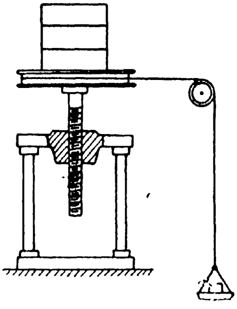


FIG. 411.—Experimental screw jack.

EXPT. 42.—**Friction of a screw.** The screw-jack shown in Fig. 411 may be experimented on in the same manner as that explained in Chapter XIV. for other types of lifting machines.

Testing of lubricants. The mechanical testing of lubricants is performed usually by feeding the lubricant into a test bearing, which may be loaded and run at varying speeds. Provision is made for measuring the torque required to rotate the shaft and also for measuring the temperature of the oil. There are many different forms of machine. One which has given useful information in the hands of Messrs. W. W. F. Pullen and W. T. Finlay at the South-Western Polytechnic* is shown in Fig. 412. A shaft AB is loaded with two equal flywheels C and D; the central enlarged portion of the shaft runs in a bearing and is lubricated by means of a loose ring G, which hangs freely on the shaft and dips into an oil bath; the ring revolves slowly as the shaft rotates. The oil leaving the bearing is spun off by collars F, F fixed to the shaft and having several sharp edges to prevent the oil travelling axially along the shaft; the oil is thus returned to the oil bath and is used again. K is a gauge tube indicating the quantity of oil in the bath. The temperature of the oil is controlled by a U tube H, through which water may be circulated. A gas flame in the space J under the bath can be used to raise the temperature of the oil. The temperature is measured by a thermometer suspended in the oil. The machine is direct driven by an electromotor arranged as shown in Fig. 413. The motor A has its bearings supported by rollers B, C and D, and is

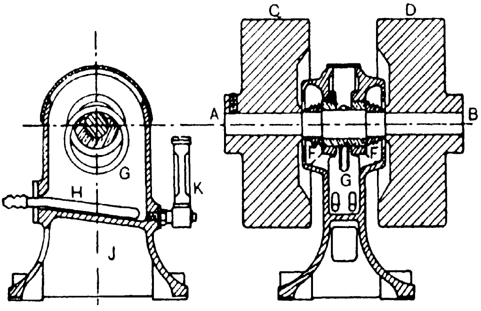


FIG. 412.—Pullen's machine for testing lubricants.

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* *Proc. Inst. Mech. Eng.*, 1909.

therefore free to rock about its axis. A balance weight is fitted at E, and a counterpoise F serves to measure the torque. The shaft runs in the direction of the arrow, and the rotor of the machine applies a torque of opposite sense to the stator; this torque is balanced by the counterpoise, and is equal to the torque required to drive the oil-testing machine. This type of machine is very useful for testing oils under steady load and under different conditions as regards speed and temperature.

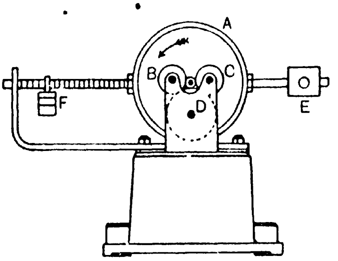

FIG. 413.—Arrangement of electromotor for driving Pullen's machine.

EXERCISES ON CHAPTER XV.

1. A shaft journal is 4 inches in diameter and has a load of 4000 lb. If the coefficient of friction is 0·06, find the torque resisting the motion. Calculate also the energy absorbed in foot-lb. per minute in overcoming friction; to what heat in B.T.U. is this energy equivalent? The shaft revolves 150 times per minute.

2. A vertical shaft is supported on a flat pivot bearing 2 inches in diameter and carries a load of 150 lb. The shaft revolves 300 times per minute. Take $\mu = 0{\cdot}03$, and calculate the moment of the frictional resistance, (*a*) assuming that the distribution of bearing pressure is uniform, (*b*) assuming that the wear is uniform. In each case calculate the horse-power absorbed by the pivot.

3. The thrust of a propeller shaft is taken by 6 collars, 12 inches diameter, the rubbing surface inner diameter being 8 inches. The shaft runs at 120 revolutions per minute. Take $\mu = 0{\cdot}05$, the bearing pressure 60 lb. per square inch of rubbing surface, and find the horse-power absorbed by the bearing.

4. A block weighing W lb. is dragged along a level table by a force P lb. acting at an angle θ to the horizontal. The coefficient of friction may be taken of constant value 0·25. Obtain the values in terms of W, (*a*) of P, (*b*) of the work done in dragging the block a distance of one foot. Give the results when θ is 0, 15, 30, 45, 60, and 75 degrees. Plot graphs showing the relation of P and θ, and also the relation of the work done by P and θ.

5. A block weighing W lb. is pushed up an incline, making an angle θ with the horizontal. The coefficient of friction has a constant value of 0·25. Find in terms of W (*a*) the values of the force P lb., parallel to the incline, (*b*) the work done in raising the block through a vertical height of one foot. Give the results for θ equal to 0, 15, 30, 45, 60, 75 and 90 degrees. Plot graphs for each case, (*a*) and (*b*).

6. Answer Question 5 if P is horizontal. What is the value of θ when P becomes infinitely great?

7. In a screw-jack the pitch of the square threaded screw is 0·5 inch and the mean diameter is 2 inches. The force exerted on the bar used in turning the screw is applied at a radius of 21 inches. Find this force if a load of 3 tons is being raised. Take $\mu = 0·2$. What is the efficiency of this machine?

8. With the screw-jack given in Question 7, find the force required at the end of the bar in order to lower the load of 3 tons.

9. Show that the horizontal force required to move a weight W up a plane whose slope is i is $W\frac{i+\mu}{1-i\mu}$, where μ is the coefficient of friction. A right- and left-hand square-threaded screw (pitch 0·25 inch, mean diameter of thread 1 inch) is used as a strainer. Find the couple required to tighten against a pull of 1000 lb. $\mu = 0·15$. (I C E.)

10. In a 1-inch Whitworth bolt and nut take the dimensions as follows: pitch, 0·125 inch; angle of the V thread, 60 degrees; mean diameter of the thread, 0·8 inch; mean radius of the bearing surface of the nut, 0·9 inch. Take the coefficient of friction to be 0·2 for both the screw and the nut. Find the force required at the end of a spanner 15 inches long in order to obtain a pull of 1000 lb. on the bolt.

11. A horizontal lever, instead of having a knife edge as a fulcrum, is pivoted on a pin 2 inches in diameter. The arms of the lever are 8 inches and 5 feet respectively. The coefficient of friction for the pin is 0·2. What load at the end of the short arm can be raised by a vertical pull of 100 lb. at the end of the long arm? (B E.)

12. The arms of a bent lever ACB are at right angles to one another; AC is 12 inches long and is horizontal, BC is 27 inches long, and B is vertically above C. The lever may turn on a fixed shaft 3 inches in diameter at C. A load of 2000 lb. is hung from A. Find what horizontal force is required at B (*a*) if A is ascending, (*b*) if A is descending. Take the coefficient of friction for the shaft to be 0·1.

13. In the mechanism shown in Fig. 399, the crank OA is 6 inches long and has anti-clockwise rotation; the crank pin at A is 2 inches in diameter and the width of the slot in the bar is 2·75 inches. Take the force P as constant and equal to 1000 lb.; find the turning moment on the crank in each of the four positions when the crank makes 45 degrees with the line of P, (*a*) neglecting friction, (*b*) taking account of friction. The coefficient of friction for all rubbing surfaces may be taken as 0·1.

14. In the crank and connecting-rod mechanism of an ordinary steam-engine, the crank and connecting-rod are 7 inches and 30 inches long respectively. The diameter of the crank pin is 3·5 inches and that of the crosshead pin is 3 inches. When the crank has travelled 45 degrees from the inner dead point the total force urging the crosshead is 3000 lb. Find the turning moment on the crank for this position, taking μ for the crank pin and crosshead pin to be 0·06. Find both angles in which there is zero turning moment on the crank.

15. Determine an expression for the work absorbed per minute in overcoming the friction of a collar bearing. State the assumptions made in deriving the formula. The thrust in a shaft is taken by 8 collars 26 inches external diameter, the diameter of the shaft between the collars being 17 inches. The thrust pressure is 60 lb. per square inch, the coefficient of friction is 0·04, and the speed of the shaft is 90 revolutions per minute. Find the horse-power absorbed by the friction of the thrust bearing. (L.U.)

CHAPTER XVI.

VELOCITY. ACCELERATION.

Velocity. The **Velocity** of a body may be defined as the rate at which the body is changing its position. The four elements which enter into a body's velocity are: (*a*) the distance travelled, (*b*) the time taken to travel this distance, (*c*) the direction in which the body is moving, (*d*) the sense along the line of direction; the sense may be described as positive or negative. A body having **uniform velocity** will travel equal distances in equal intervals of time, and the velocity may be calculated by dividing the distance by the time. In the case of **varying velocity,** the result of this calculation will be the average velocity of the body.

The units of time employed are the mean solar second, minute, or hour. The unit of distance may be the foot, mile, centimetre, metre or kilometre. Common units of velocity are the foot per second, the mile per hour, the centimetre per second, and the kilometre per hour.

Let s = distance travelled in feet,
t = time taken, in seconds,
v = the velocity.

Then $v = \frac{s}{t}$ feet per second.

This will be the velocity at any instant if the rate of travelling is uniform, and will give the average velocity if the rate is varying.

Distance-time diagrams. In Fig. 414, the distances travelled by a given body have been set off as ordinates on a time base. Thus 1A is the distance travelled during the first second of the motion, 2B is the distance travelled in the first two seconds, and so on. 6F is the total distance travelled in 6 seconds. Drawing AG, BH, CK, etc., horizontally, it is evident that BG is the distance travelled between the end of the first second and the beginning of the third, CH is the distance travelled during the third second, DK, EL and

FM are the distances travelled during the fourth, fifth and sixth seconds respectively. If all these distances were equal, the velocity would be uniform and the line OF would be straight (Fig. 415). A straight-line distance-time diagram therefore represents the case of uniform velocity.

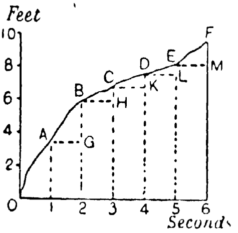

FIG. 414.—Distance-time diagram.

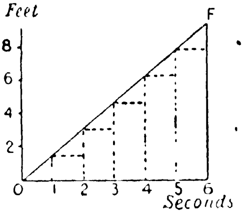

FIG. 415.—Distance-time diagram, velocity uniform.

Referring again to Fig. 414, the average velocity during the six seconds would be obtained by dividing 6F in feet by 6 seconds. The average velocity during any second such as the fourth may be calculated by dividing DK in feet by 1 second.

In Fig. 416, let AB $= s_1$ and CD $= s_2$ be the distances in feet travelled during the times t_1 and t_2 seconds respectively. Drawing AE parallel to OD, the distance travelled during the interval $t_2 - t_1$ will be CE $= s_2 - s_1$. Hence,

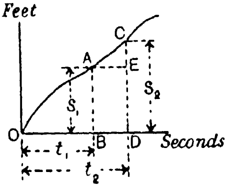

FIG. 416.

Average velocity during the interval BD

$$= \frac{s_2 - s_1}{t_2 - t_1}.$$

The actual velocity at any instant of the interval may differ somewhat from this. If the interval be made very small we may write the difference in the distances by the symbol δs and the difference in the time by δt.

$$\text{Average velocity during a small interval} = \frac{\delta s}{\delta t}.$$

Now let δt, represented by BD or AE in Fig. 416, be reduced indefinitely until finally it gives us the conception of an "instant." If dt is its value when so reduced, and if ds is the distance travelled, then, at the instant considered,

$$v = \frac{ds}{dt}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The velocity of a body at any instant may be described as the distance which would be travelled during the next second had the velocity possessed at the instant considered remained uniform.

The mathematical calculation involved in (1) above is performed by use of the rules of the differential calculus (p. 9).

EXAMPLE. Suppose the equation connecting s and t for the motion of a given body to be

$$s=\tfrac{1}{2}at^2,$$

where a is a constant. Find the velocity at any instant.

$$\begin{aligned} v=\frac{ds}{dt}&=\frac{d}{dt}(\tfrac{1}{2}at^2)\\ &=\tfrac{1}{2}a\times 2t\\ &=at. \end{aligned}$$

If the time up to the required instant be inserted in this result, the velocity at that instant will be obtained.

In dealing with a moving point in a machine the space-time diagram may be drawn by setting out the mechanism in a number of positions differing by equal intervals of time, and then measuring the distances travelled by the point in question. The average velocity over each of the intervals may be obtained very closely from the diagram.

EXAMPLE 1. A rigid bar AB, 3 1 feet in length, moves so that one end A is always in OX (Fig. 417), and the other end B is always in OY, which is perpendicular to OX. A is at first 3 feet from O and travels to O in 6 seconds with uniform velocity. Draw the space-time diagram for B.

Divide AO into six equal intervals as shown. A will traverse each interval in one second.

Velocity of A (uniform)

$=\frac{3}{6}=0{\cdot}5$-foot per sec.

Find the positions of B for each position of A; these are numbered 1′, 2′, 3′, etc. to correspond with the numbering of the positions of A. Measure B1′, B2′, B3′, etc.; these will be the distances travelled by B in 1, 2 and 3 seconds respectively.

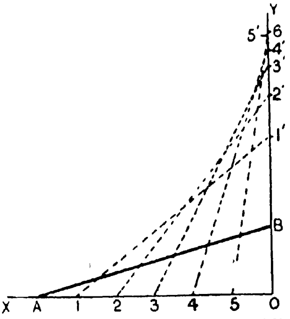

FIG. 417.—A rigid bar AB; A moves in OX; B moves in OY.

Choose suitable scales and draw the space-time diagram (Fig. 418), by setting off the distances travelled by B up to the stated times. The numbering 1′, 2′, 3′, etc. in this diagram agrees with that in Fig. 417.

EXAMPLE 2. Find from Fig. 418 the average velocity of B for each interval of time and draw a velocity-time diagram.

The average velocity during the third second may be obtained by dividing H3′ in feet by 0·5 second. It is preferable to measure 33′ and

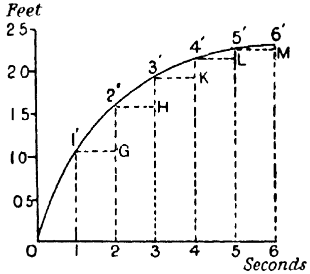

FIG 418.—Space-time diagram for the point B in Fig. 417.

22′ and take the difference as the value of H3′. The average velocity so calculated may be taken to be the actual velocity at the middle of the interval, and is set off as BC in Fig. 419, which is the velocity-time diagram. It is best to set out the quantities in a table thus:

Interval No.	Ordinate.	Distance in feet	Difference in distance, feet	Average velocity =difference—1 sec. Feet per sec.
0	0	0		
			1·06	1·06
1	11′	1·06		
			0·53	0·53
2	22′	1·59		
			0·35	0·35
3	33′	1·94		
			0 22	0·22
4	44′	2·16		
			0·11	0·11
5	55′	2·27		
			0·03	0 03
6	66′	2·30		

The last column is plotted at the middle of the intervals in Fig. 419; a fair curve through the plotted points gives the required velocity-time diagram.

A useful property of the velocity-time diagram is that its area represents the distance travelled. The distance is equal to the average velocity multiplied by the time, and the average velocity evidently will be given to scale by the average height of the diagram, while its base represents the time to scale. The area of the diagram is its average height multiplied by its base and therefore represents the distance travelled. To obtain the scale:

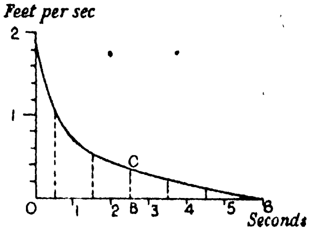

FIG. 419.—Velocity-time diagram for the point B in Fig. 417.

Let

1 inch of height represent v feet per second,

1 inch of length represent t seconds.

Then

1 square inch of area represents vt feet.

Hence, the area of the velocity-time diagram, in square inches, multiplied by vt will give the distance travelled.

Acceleration. **Acceleration means rate of change of velocity**; it is measured by dividing the change in velocity by the time in which the change is effected. The change in velocity may be either positive or negative, depending on whether the velocity is increasing or diminishing, and the acceleration will have the same sign. If the change in velocity is stated in feet per second, and if the time in which the change takes place is stated in seconds, then the units of the acceleration will be feet per second per second.

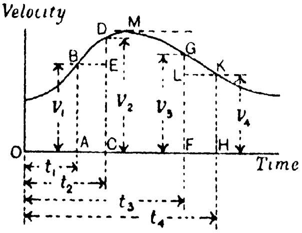

FIG. 420.—Deduction of acceleration from a velocity-time diagram.

Acceleration may be studied from the velocity-time diagram. Fig. 420 shows such a diagram in which a velocity v_1 occurs at the end of a time t_1 and a velocity v_2 at the end of t_2; these velocities are represented by AB and CD respectively. The change in velocity—an increase in this case—is DE.

$$\text{Change in velocity} = v_2 - v_1.$$

$$\text{Time in which this change is effected} = t_2 - t_1;$$

$$\therefore \text{ Acceleration during the interval AC } = \frac{v_2 - v_1}{t_2 - t_1} \quad \text{.........(1)}$$

This expression will be strictly correct if the gain in velocity is acquired uniformly throughout the interval, in which case BD would be straight. If BD is curved, then the value given by (1) will be the average acceleration over the interval. The acceleration at any instant may be calculated by diminishing $t_2 - t_1$ indefinitely, when

$$\text{Acceleration} = a = \frac{dv}{dt}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

In the interval FH (Fig. 420), the change in velocity is a decrease, shown by GL. If the acceleration at A is positive, that at F will be negative. At M, where the tangent to the curve is horizontal, there is no change in the velocity over an indefinitely small interval of time, and hence there is no acceleration.

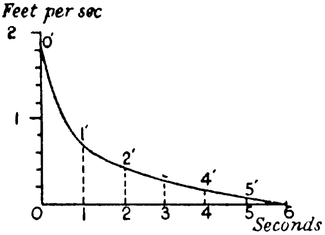

FIG. 421.—Velocity-time diagram for the point B in Fig. 417.

An acceleration-time diagram may be deduced from the velocity-time diagram by the method already applied in Example 2, p. 384, for obtaining a velocity-time diagram from a space-time diagram. The average acceleration over any interval is set out as an ordinate at the middle of the interval.

EXAMPLE. Taking the data of Example 1, p. 383, and the velocity-time diagram (Fig. 421, redrawn from Fig. 419) from Example 2, p. 384, draw an acceleration-time diagram.

The tabular form of calculation may be adopted as follows:

Interval No.	Ordinate.	Velocity, feet per sec.	Change in vel., feet per sec.	Average acceleration = change in vel − 1 sec., feet per sec. per sec.
0	00′	1·87		
			−1·20	−1·20
1	11′	0·67		
			−0·25	−0·25
2	22′	0·42		
			−0·15	−0·15
3	33′	0·27		
			−0·10	−0·10
4	44′	0·17		
			−0·09	−0·09
5	55′	0·08		
			−0·08	−0·08
6	6	0		

The last column is plotted at the middle of the intervals as shown in Fig. 422, and a fair curve is drawn through the plotted points, thus obtaining the acceleration-time diagram.

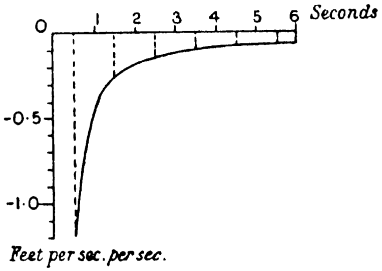

FIG. 422.—Acceleration-time diagram for the point B in Fig. 417.

If the distance travelled is given by an equation connecting s and t, the acceleration may be found by two successive differentiations. Thus: Let

$$s = ct^3,$$

where c is a constant. Then

$$v = \frac{ds}{dt} = 3ct^2,$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6ct.$$

The indices in $\frac{d^2s}{dt^2}$ simply indicate that s has been differentiated twice with respect to t (p. 12).

Equations for uniform acceleration. Reference is made to Figs. 423 and 424, the former showing the velocity-time diagram for a body starting from rest; Fig. 424 shows the diagram if the body starts with a given velocity v_1; in both cases the acceleration is uniform.

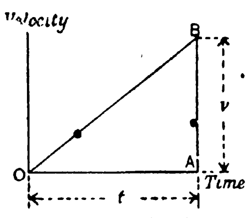

FIG. 423.—Velocity-time diagram, starting from rest.

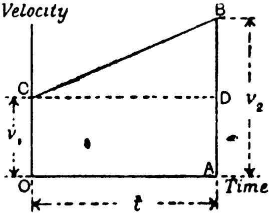

FIG. 424.—Velocity-time diagram, starting with velocity v_1.

Starting from rest (Fig. 423):

Let v = the velocity in feet per second,
t = the time in seconds taken to acquire v,
s = the distance travelled, in feet,
a = the acceleration, feet per sec. per sec.

By definition, $a = \frac{v}{t}$,

or, $$\mathbf{v = at}. \qquad \ldots \qquad (1)$$

s = the average velocity × t,

$$\therefore\ \mathbf{s = \tfrac{1}{2}vt}. \qquad \ldots \qquad (2)$$

Or, s = the area of the diagram
$= t \times \frac{1}{2}v = t \times \frac{1}{2}at$ (from (1)),

$$\therefore\ \mathbf{s = \tfrac{1}{2}at^2}. \qquad \ldots \qquad (3)$$

From (1), $t = \frac{v}{a}$.

Inserting this in (3), $s = \frac{1}{2}a\frac{v^2}{a^2} = \frac{v^2}{2a}$;

$$\therefore\ \mathbf{v^2 = 2as}. \qquad \ldots \qquad (4)$$

Starting with a velocity v_1 (Fig. 424):

Let v_1 and v_2 = the initial and final velocities respectively in feet per second,
t = the time in seconds in which v_1 increases to v_2,
s = the distance travelled, in feet,
a = the acceleration, in feet per sec. per sec.

Then $$\mathbf{v_2 - v_1 = at}. \qquad \ldots \qquad (5)$$

s = the average velocity × t

$$\therefore\ \mathbf{s = \left(\frac{v_1 + v_2}{2}\right)t} \qquad \ldots \qquad (6)$$

Or, s = the area of the diagram
= rectangle OCDA + triangle CDB
$= v_1 t + (t \times \frac{1}{2}\mathrm{DB})$.

Also, $\mathrm{BD} = at$;

$$\therefore\ \mathbf{s = v_1 t + \tfrac{1}{2}at^2}. \qquad \ldots \qquad (7)$$

From (5), $t = \frac{v_2 - v_1}{a}$.

Inserting this in (6),
$$s = \left(\frac{v_2 + v_1}{2}\right)\left(\frac{v_2 - v_1}{a}\right)$$
$$= \frac{v_2^2 - v_1^2}{2a},$$
$$\mathbf{v_2^2 - v_1^2 = 2as.} \qquad \ldots (8)$$

The case of a body falling under the action of gravity is one of nearly uniform acceleration. The acceleration would be quite constant, but for the resistance offered by the atmosphere, and for the fact that a body weighs less when at a height above the surface of the earth. The symbol g is used to denote the acceleration of a body falling freely, that is, neglecting atmospheric resistances. The value of g varies to a small extent, being about 32·088 feet per second per second at the equator and about 32·252 at the poles. The value 32·2 may be taken for all parts of the British Isles. The equations found above may be modified to suit a body falling freely, by writing g instead of a, and the height h feet instead of s.

Composition and resolution of velocities and accelerations. A given velocity is a vector quantity and may be represented in the same manner as a force by a straight line having an arrow point. Hence problems involving the resolution or composition of velocities may be solved in the same way as for forces by the application of the triangle or polygon of velocities.

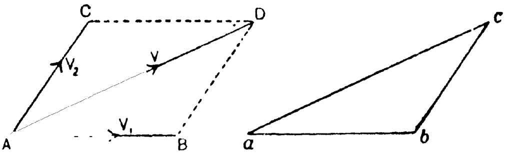

FIG. 425.—Triangle and parallelogram of velocities.

Let a point A have component velocities V_1 and V_2 in the plane of the paper (Fig. 425). The resultant velocity may be found from the triangle *abc* in which *ab* represents V_1, *bc* represents V_2 and *ac* gives the resultant velocity V which should be shown applied at A. A parallelogram of velocities, ABDC, may be used by making $AB = V_1$ and $AC = V_2$; the diagonal AD gives the resultant velocity.

Rectangular components of a given velocity V (Fig. 426) along two axes OX and OY may be calculated from
$$V_x = V \cos a,$$
$$V_y = V \sin a.$$

EXAMPLE. A body slides down an incline of 30° (Fig. 427) with a velocity of 10 feet per second. Find the horizontal and vertical components of its velocity.

$$V_h = V\cos 30° = 10 \times \frac{\sqrt{3}}{2} = 8{\cdot}66 \text{ feet per sec.}$$

$$V_v = V\sin 30° = 10 \times \tfrac{1}{2} = 5 \text{ feet per sec.}$$

It will be understood that, as acceleration has magnitude, direction and sense, this quantity can be represented also by a straight line having an arrow point. Problems involving the composition and resolution of accelerations may be solved by use of the same constructions as for velocities.

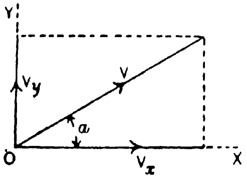

FIG. 426.—Rectangular components of a velocity.

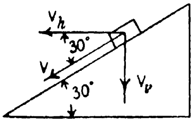

FIG. 427.

EXAMPLE. A body slides down an inclined plane with an acceleration a feet per second per second (Fig. 428). If the plane makes an angle α to the horizontal, find the component accelerations (a) normal to the plane, (b) vertical.

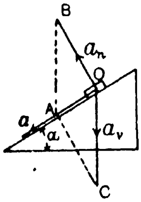

FIG. 428.

Make OA to represent a to scale and draw the parallelogram of accelerations OBAC, OB being normal to the plane and OC being vertical. The angle OBA will be equal to α. Hence,

$$\frac{OB}{OA} = \cot\alpha,$$

and Normal acceleration $= a_n = a\cot\alpha$.

Also, $$\frac{OC}{OA} = \operatorname{cosec}\alpha,$$

and Vertical acceleration $= a_v = a\operatorname{cosec}\alpha$.

The relation of a_n and a_v is given by

$$\frac{a_n}{a_v} = \frac{CA}{OC} = \cos\alpha\,;$$

$$\therefore\ a_n = a_v\cos\alpha.$$

Angular velocity. When a body is rotating about a fixed axis, the radius of any point in the body turns through a definite angle in unit time. The term **angular velocity** is given to the rate of describing angles, and may be measured in revolutions per minute or per second, or, more conveniently for the purposes of calculation, in radians per second; the symbol ω is taken usually to denote the latter.

Since there are 2π radians in a complete revolution, the connection between ω and N, the revolutions per minute, will be

$$\omega = \frac{N}{60} 2\pi = \frac{\pi N}{30} \text{ radians per second.}$$

Let a line OA (Fig. 429) have uniform speed of rotation in the plane of the paper about O as centre. The point A will have a uniform linear velocity v feet per second in the circumference of a circle; let r be the radius of the circle in feet. It is evident that the length of the arc described by A in one second will be v feet, and the angle subtended by this arc will be $\frac{v}{r}$ radians. OA turns through this angle in one second, hence its angular velocity is

$$\omega = \frac{v}{r} \text{ radians per second}$$

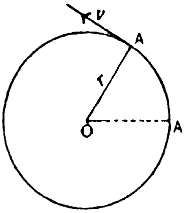

FIG. 429.—Relation of angular and linear velocities.

It will be noticed that the linear velocities of other points in the line OA will be proportional to their radii, hence such velocities will be unequal. The same numerical result will be obtained for the angular velocity by dividing the linear velocity of any point by its radius. It is obvious that, under given conditions of speed of rotation, all radii of a body turn through equal angles per second, hence only one numerical result is possible for the angular velocity.

Equations of angular motion. In uniform angular velocity equal angles are described in equal intervals of time. The total angle α described by a rotating line in a time t seconds will be, if the angular velocity ω is uniform,

$$\alpha = \omega t \text{ radians.}$$

If the angular velocity varies, the body is said to have **angular acceleration.** Angular acceleration is measured in radians per second per second and is written θ. Suppose a line to start from rest with a uniform angular acceleration θ, its angular velocity at the end of t seconds will be

$$\omega = \theta t, \text{ radians per second. } \ldots\ldots\ldots(1)$$

The average angular velocity will be $\frac{1}{2}\omega$, hence the total angle described will be

$$\alpha = \tfrac{1}{2}\omega t. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting for ω from (1) gives

$$a = \tfrac{1}{2}\theta t^2. \quad \dots\dots\dots\dots\dots (3)$$

Again, from (1), $$t = \frac{\omega}{\theta}, \quad \therefore \quad t^2 = \frac{\omega^2}{\theta^2}.$$

Substituting this value in (3), we obtain

$$a = \tfrac{1}{2}\theta \frac{\omega^2}{\theta^2} = \frac{\omega^2}{2\theta};$$

$$\therefore \quad \omega^2 = 2\theta a. \quad \dots\dots\dots\dots\dots (4)$$

It will be observed that the above results are similar to those given on p. 388 for rectilinear motion with the substitution of ω for v, and θ for a. Making these substitutions, we may obtain the corresponding equations for angular motion when the body has an initial angular velocity ω_1.

$$\omega_2 - \omega_1 = \theta t. \quad \dots\dots\dots\dots\dots (5)$$

$$a = \left(\frac{\omega_1 + \omega_2}{2}\right) t. \quad \dots\dots\dots\dots\dots (6)$$

$$a = \omega_1 t + \tfrac{1}{2}\theta t^2. \quad \dots\dots\dots\dots\dots (7)$$

$$\omega_2^2 - \omega_1^2 = 2\theta a. \quad \dots\dots\dots\dots\dots (8)$$

The relation between the linear acceleration of a point in a revolving line and the angular acceleration of the line will be given by

$$\theta = \frac{a}{r} \text{ radians per sec. per sec.,} \quad \dots\dots\dots\dots\dots (9)$$

where a = linear acceleration of A (Fig. 429) in feet per sec. per sec.,
r = radius of A in feet.

Defining the angular velocity of a rotating line as its rate of describing angles, and its angular acceleration as the rate of change of angular velocity, suppose a line to describe a small angle δa in an interval of time δt. The average angular velocity during the interval will be

$$\omega_a = \frac{\delta a}{\delta t}.$$

If δa be taken indefinitely small and written da, the time dt in which it is traversed will be our conception of an instant, and the angular velocity at this instant will be

$$\omega = \frac{da}{dt}. \quad \dots\dots\dots\dots\dots (10)$$

If the angular velocity alters by a small amount $\delta\omega$ during an interval of time δt, then

$$\text{Average angular acceleration} = \theta_a = \frac{\delta\omega}{\delta t}.$$

If these be reduced indefinitely, the result will give the angular acceleration at the instant considered, viz.,

$$\theta = \frac{d\omega}{dt} = \frac{d^2a}{dt^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The results (10) and (11) are suitable for application of the rules of the differential calculus (p. 9).

EXAMPLE 1. An engine starts from rest and acquires a speed of 300 revolutions per minute in 40 seconds from the start. What has been its angular acceleration?

$$\omega = \frac{300}{60} \cdot 2\pi = 10\pi$$

$$= 31{\cdot}41 \text{ radians per sec.}$$

$$\theta = \frac{\omega}{t} = \frac{31{\cdot}41}{40}$$

$$= \underline{0{\cdot}785} \text{ radian per sec. per sec.}$$

EXAMPLE 2. The driving wheel of a locomotive is 6 feet in diameter. Assuming no slipping between the wheel and the rail, what is the angular velocity of the wheel when the engine is running at 60 miles per hour.

$$\text{Velocity of engine} = \frac{5280 \times 60}{60 \times 60} = 88 \text{ feet per sec.}$$

As the distance travelled in one second is 88 feet, we may find the revolutions per second of the wheel by imagining 88 feet of rail to be wrapped round the circumference of the wheel.

$$\text{Number of turns} = \frac{88}{\pi d};$$

$$\therefore \text{Revolutions per sec.} = \frac{88 \times 7}{22 \times 6} = 4{\cdot}67.$$

$$\omega = 4{\cdot}67 \times 2\pi$$

$$= \underline{29{\cdot}33} \text{ radians per sec.}$$

Or the following method may be used. Referring to Fig. 430, if there is no slipping, the point A on the rim of the wheel is in contact with the rail for an instant and is therefore at rest. Hence the whole wheel is rotating about A for an instant. The angular velocity will therefore be given by

$$\omega = \frac{\text{velocity of O}}{\text{OA}}$$

$$= \frac{88}{3} = \underline{29{\cdot}33} \text{ radians per sec.}$$

EXAMPLE 3. Using the data of Example 2 and referring to Fig. 431, find the velocities of the points on the rim of the wheel marked B, C and D, supposing no slipping.

In answering this, it will be assumed that the whole wheel is rotating about A for an instant, and that the velocity of any point is proportional

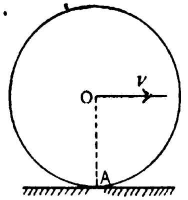

FIG. 430.—Angular velocity of a rolling wheel.

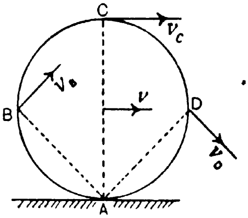

FIG. 431.—Velocities of points in a rolling wheel.

to the radius of that point from A as centre and has a direction perpendicular to that radius.

The angular velocities of AB, AC and AD are equal and are given by the angular velocity of OA in Fig. 430, viz.

$$\omega = 29{\cdot}33 \text{ radians per sec.}$$

Also,
$$v = \omega R \,;$$

$$\begin{aligned} \therefore\ v_B &= 29{\cdot}33 \times AB \\ &= 29{\cdot}33 \times 3\sqrt{2} \\ &= \underline{124{\cdot}4} \text{ feet per sec.} \\ v_C &= 29{\cdot}33 \times AC \\ &= 29{\cdot}33 \times 6 \\ &= \underline{176} \text{ feet per sec.} \\ v_D &= 29{\cdot}33 \times AD \\ &= 29{\cdot}33 \times 3\sqrt{2} \\ &= \underline{124{\cdot}4} \text{ feet per sec.} \end{aligned}$$

Angular velocity and acceleration diagrams. Diagrams showing the angle traversed, the angular velocity, and the angular acceleration, all three on bases representing time may be drawn by the same methods as have been explained on pp. 381-387 for linear velocities, etc. The angle traversed is treated in the same manner as the distance travelled and an angle-time diagram is drawn. The angular-velocity diagram is then deduced from the angle-time diagram and an angular-velocity-time diagram is drawn. The angular-acceleration-time diagram may then be deduced from the angular-velocity-time diagram.

Velocity changed in direction. Hitherto the acceleration due to changes in the magnitude of a body's velocity alone have been considered. There may be also changes effected in the direction of the velocity, and such will give rise to accelerations.

Let a point move along a straight line AB (Fig. 432 (*a*)) with a velocity v_1; on reaching the point B, let the point move along BC with a velocity v_2. To determine the change in velocity which has taken place at B, the following method may be used. Stop the point on reaching B by applying a velocity equal and opposite to v_1; this is represented by DB in the figure. The point now being at rest can be dispatched along any line with any velocity; to comply with the given conditions, give it a velocity v_2 in the line BC, represented by EB in the figure. The total change in velocity has components represented by DB and EB; hence the parallelogram BDFE gives FB = v_c as the resultant or total change in velocity.

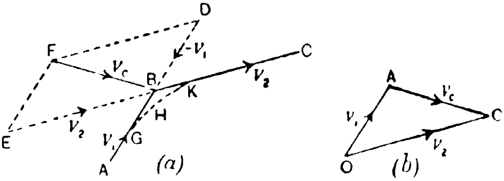

FIG. 432—Velocity changed in direction.

A convenient construction is shown in Fig. 432 (*b*). Take any point O, and draw OA and OC to represent completely v_1 and v_2 respectively. Then the change in velocity will be AC = v_c. The sense of the change in velocity may be found from the rule that it is directed *from* the end A of the initial velocity *towards* the end C of the final velocity (Fig. 432 (*b*)).

For reasons that will be apparent later, it is not possible to make a body take a sudden change in velocity; the transition from AB to BC in Fig. 432 (*a*) will take place along some curve, such as GHK. This makes no difference in the construction for finding the total change in velocity. Suppose that the body takes t seconds to pass from G to K along the curve, then this gives the time taken to effect the total change in velocity v_c. Hence,

$$\text{Resultant acceleration} = \frac{v_c}{t},$$

and has the same direction and sense as v_c.

Motion in a circle. A small body moving in the circumference of a circle with uniform velocity v is continually changing the direction of its velocity. At any point of the circumference the direction of the velocity will be along the tangent; at P_1 (Fig. 433 (*a*)) the velocity will be $v_1 = v$, and at P_2 the velocity will be $v_2 = v$. To obtain the change in velocity between P_1 and P_2, draw the triangle OAB

(Fig. 433 (*b*)). AB $= v_c$ will be the change in velocity, and is shown in Fig. 433 (*a*) passing through the point C where v_1 and v_2 intersect. It is evident that if v_c be produced it will pass through O_1, the centre of the circle, and this will be the case no matter what may be the positions chosen for P_1 and P_2. The acceleration due to v_c will also pass through O_1. The following method may be used to obtain the acceleration.

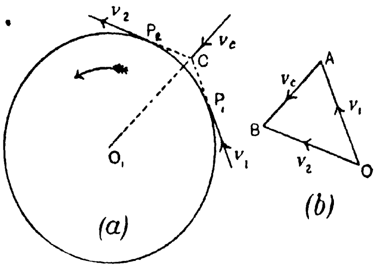

FIG. 433.—Motion in a circular path.

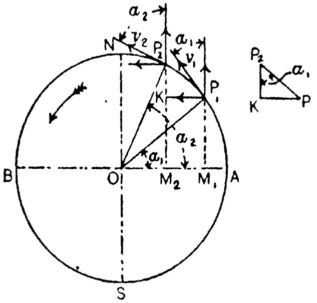

FIG. 434.—Acceleration of a point moving in the circumference of a circle.

Referring to Fig. 434, in which a point P is moving in the circumference of a circle of radius R with uniform velocity v. At P_1 the velocity v_1 is along the tangent, and its horizontal and vertical components will be $v_1 \sin a_1$ and $v_1 \cos a_1$ respectively, where a_1 is the angle OP_1 makes with the horizontal diameter AB. Similarly at P_2, the components will be $v_2 \sin a_2$ and $v_2 \cos a_2$ respectively. As $v_1 = v_2 = v$, we have

$$
\begin{aligned}
\text{Change in horizontal velocity} &= v \sin a_2 - v \sin a_1 \\
&= v(\sin a_2 - \sin a_1) \\
&= v\left(\frac{P_2M_2}{OP_2} - \frac{P_1M_1}{OP_1}\right) \\
&= \frac{v}{R}(P_2M_2 - P_1M_1) \\
&= \frac{v}{R} \cdot P_2K. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)
\end{aligned}
$$

Again, the time, t, taken to pass from P_1 to P_2 will be the time in which this change in velocity has been effected, and may be calculated from

$$P_1P_2 = vt,$$

$$t = \frac{P_1P_2}{v}. \qquad \ldots (2)$$

Hence, Horizontal acceleration of $P = \frac{\text{change in velocity}}{t}$

$$= \frac{v^2}{R} \cdot \frac{P_2K}{P_1P_2}. \qquad \ldots\ldots (3)$$

If a_1 and a_2 are very nearly equal, the angle P_1OP_2 will be very small and the arc P_1P_2 will be a straight line practically. The angle P_1P_2K will be equal to a_1. Hence,

$$\frac{P_2K}{P_1P_2} = \cos a_1.$$

$$\therefore \text{ horizontal acceleration of } P = \frac{v^2}{R} \cos a_1. \qquad \ldots\ldots (4)$$

This acceleration will be directed always towards the vertical diameter NS, as the sign of the acceleration will be the same as that of $\cos a$.

Let P be at A. Then $a = 0$, $\cos a = 1$, and the acceleration will be

$$a = \frac{v^2}{R}, \qquad \ldots\ldots (5)$$

and will be directed along AO. As any reference diameter might have been taken instead of AB, it follows that for any position of P, the acceleration towards the centre of the circle will be given by (5). The result will be in feet per second per second if

v = the velocity in feet per second,

R = the radius of the circle in feet.

The acceleration may be stated in terms of the angular velocity by writing

$$\omega = \frac{v}{R}, \quad \text{or} \quad v = \omega R.$$

From (5),

$$a = \frac{\omega^2R^2}{R} = \omega^2R. \qquad \ldots\ldots (6)$$

EXAMPLE 1. A motor car is travelling at 20 miles per hour round a curve of 600 feet radius. What is the acceleration towards the centre of the circle?

$$v = \frac{5280 \times 20}{60 \times 60} = \frac{88}{3} \text{ feet per second,}$$

$$a = \frac{v^2}{R} = \frac{88 \times 88}{3 \times 3 \times 600} = \underline{1{\cdot}434} \text{ feet per sec. per sec.}$$

EXAMPLE 2. What is the acceleration, towards the centre, of a point on the rim of a wheel 4 feet diameter and running at 300 revolutions per minute?

$$\omega = \tfrac{300}{60} \times 2\pi = 10\pi \text{ radians per second},$$

$$a = \omega^2 R = 100 \times \tfrac{22}{7} \times \tfrac{22}{7} \times 2$$

$$= \underline{1975} \text{ feet per sec. per sec.}$$

Simple harmonic motion. In Fig. 435, the point P travels in the circumference of the circle ANBS with uniform velocity v. Drawing PM perpendicular to the diameter AB, it will be noticed that M, the projection of P on AB, will vibrate in AB as P rotates. The velocity and acceleration of M at any instant will be the horizontal components of the velocity and acceleration of P, viz.

$$V = v \sin \alpha = \omega R \sin \alpha, \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

$$a = \frac{v^2}{R} \cos \alpha = \omega^2 R \cos \alpha, \quad \dots\dots\dots\dots\dots\dots(2)$$

where R is the radius of the circle.

The vibratory motion of M is called **simple harmonic motion.** One of its properties is that the acceleration is directed always towards the middle point O of the vibration. Again, since $\cos \alpha = \frac{OM}{OP}$, and is therefore proportional to OM, the acceleration is proportional to OM, the distance of M at any instant from the middle of the vibration. When M is at A, the acceleration is proportional to OA and is positive, *i.e.* directed towards the left; when M is at B, the acceleration has the same value, but is directed towards the right and is negative. M has no acceleration when at O. An acceleration diagram may be drawn by erecting ordinates AA′ and BB′, each equal to R, on the diameter AB and joining A′B′ (Fig. 436). Any

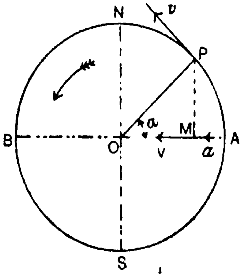

FIG. 435.—The motion of M is simple harmonic.

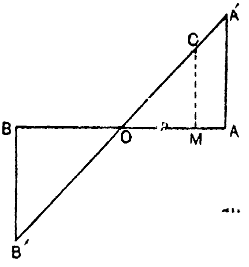

FIG. 436.—Acceleration diagram for M, on a space base.

ordinate MC will then give the acceleration for the position M. The scale of this diagram is obtained from the consideration that when P is at A (Fig. 435), $\cos\alpha = 1$ and $a = \omega^2 R$; hence the scale is such that $AA' = \omega^2 R$. As the diagram has been drawn on a distance, not a time, base, it may be called a **distance-acceleration diagram.**

The velocity of M at any instant is proportional to $\sin\alpha$. Now $\sin\alpha = \frac{PM}{OP}$ (Fig. 435), and is therefore proportional to PM; hence the velocity is proportional to PM. When M is at A the velocity is zero, and has also zero value at B. Maximum velocity is attained at O, when $V = v$. The circle in Fig. 435 is a velocity diagram on a distance base AB, as any ordinate PM will give the velocity of M at the instant considered, the scale being such that ON represents v. V is positive, *i.e.* towards the left, if PM is above AB, and negative if PM is below AB.

Velocity-time and acceleration-time diagrams may be drawn by noting that, as the velocity of P in Fig. 435 is uniform, equal angles will be described by OP in equal times. Divide the circle into twelve equal angles of 30° each, and calculate $V = v \sin\alpha$, and also $a = \frac{v^2}{R} \cos\alpha$ for each position of P. Set off a base of angles from 0° to 360° (Fig. 437), and erect ordinates having the calculated values

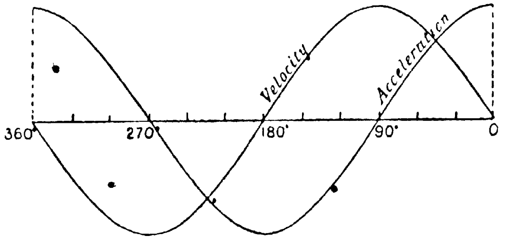

FIG. 437.—Velocity-time and acceleration-time diagrams for simple harmonic motion.

V and a. The base represents angles or time to different scales; the scale of time is such that the total length of the base line represents the time of one revolution of OP in Fig. 435.

Fig. 399 (p. 371) shows a well-known mechanism, used in pumps, which realises simple harmonic motion. The slotted bar has a sliding block, in which is bored a hole to receive the crank pin. The vertical components of the velocity and acceleration of the crank pin are thus eliminated, and the horizontal components alone are communicated to the piston rods.

The time of a complete vibration in simple harmonic motion from A to B and back again (Fig. 435) may be estimated from the fact that it will be equal to that of a complete revolution of P.

Let T = the time of one vibration in seconds.
v = the velocity of P, in feet per second.
R = the radius of the circle = the **amplitude** of the vibration, in feet.

Then
$$vT = 2\pi R,$$
$$T = 2\pi \frac{R}{v} \qquad \ldots\ldots (3)$$
$$= 2\pi \frac{R}{\omega R} = \frac{2\pi}{\omega}, \qquad \ldots\ldots (4)$$

where ω is the angular velocity of OP in radians per second.

EXAMPLE. A point is describing simple harmonic vibrations in a line 4 feet long. Its velocity at the instant of passing through the centre of the line is 12 feet per second. What is the time of a complete vibration?

$$T = \frac{2\pi R}{v},$$

where R is 2 feet and v is 12 feet per second. Hence,

$$T = \frac{2 \times 22 \times 2}{7 \times 12}$$
$$= \underline{1{\cdot}05} \text{ seconds.}$$

Change in angular velocity. A given angular velocity may be represented by means of a vector in the following manner. In Fig. 438 (*a*) is shown a wheel rotating about an axis OA with an angular velocity ω. A person situated on the right-hand side sees the wheel rotating in the clockwise direction, and may represent the angular velocity by drawing a line OA perpendicular to the plane of rotation of the wheel and on the same side of this plane as the person is situated. OA is made, to scale, of length to represent ω. A person situated on the left-hand side will see the wheel rotating in the anti-clockwise direction, and may represent the angular velocity by means of a perpendicular to the plane of rotation drawn on the opposite side of this plane. Both observers will thus agree in erecting the perpendicular on the same side of the plane of rotation. The perpendicular represents the magnitude and direction of rotation of an angular velocity in a plane perpendicular to the line, and will thus obey the same laws as a vector. Two or more component angular velocities represented in this way may be dealt with and their resultant found by means of a triangle or polygon of velocities.

In Fig. 438 (*a*) the wheel is revolving in a vertical plane; at the same time its axis is revolving in a horizontal plane as indicated by the arrows at the ends of the axis. A plan of the wheel is given in Fig. 438 (*b*); OA represents the angular velocity of the wheel at one instant, and OA′ represents its angular velocity after a short interval of time during which the wheel has turned horizontally into the position indicated by dotted lines. Since OA and OA′ represent the initial and final angular velocities respectively, it follows, by the same reasoning as for linear velocity (p. 395), that the change in

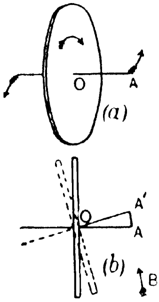

FIG. 438.—Change in angular velocity.

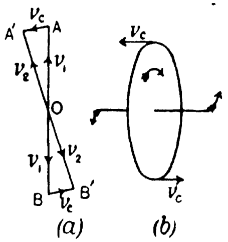

FIG. 439.—Change in angular velocity by method of linear velocities.

angular velocity is represented by AA′. The actual change in angular velocity takes place in a plane perpendicular to AA′, *i.e.* a vertical plane in the given case, and is anti-clockwise to an observer situated at B.

It may be of assistance to the student to consider the problem from the point of view of linear velocities. In Fig. 439 (*a*) is given a plan of the wheel. OA represents v_1, the initial velocity of a point on the top of the wheel; OA′ represents the final velocity v_2 of the point on the top of the wheel; AA′ represents v_c, the change in velocity of this point. In the same way BB′ represents v_c, the change in linear velocity of a point at the bottom of the wheel. Fig. 439 (*b*) shows these changes in linear velocity in their proper positions, and indicates that a change in angular velocity is taking place in a vertical plane containing the axis of the wheel.

In Fig. 440 OA represents ω, the angular velocity of the wheel. It will be evident that the successive additions of small changes in angular velocity such as that represented by AA′ will cause A to describe a complete circle. The total change in angular velocity during one rotation of the wheel axis in the horizontal plane will be the circumference of the circle, and will be given by

$$\text{Change in angular velocity} = 2\pi\omega.$$

If this result be divided by the time taken by A in describing the complete circle, *i.e.* the time in which the wheel axis makes one complete rotation in the horizontal plane, the result will give the angular acceleration. It is evident that the angular acceleration will take place in the same plane as that in which the change in angular velocity occurs, viz. a vertical plane containing the wheel axis.

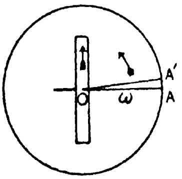

Fig. 440.—Plan of the wheel shown in Fig. 438

Relative velocity. The **relative velocity of two bodies may be defined as the velocity which an observer situated on one of them would perceive in the other.** An observer in one of two trains, moving side by side with equal velocities of the same sense, would perceive no velocity in the other and would therefore say that the relative velocity is zero. If the train carrying the observer has a velocity of 30 miles per hour, and the other, one of 35 miles per hour, he will see the other train moving past him at a rate of 5 miles per hour, which velocity he would call the relative velocity of the trains. Had the trains been moving in opposite directions, the relative velocity would be 65 miles per hour. A stream of water moving at 8 feet per second reaching a water wheel, the buckets of which are moving away from the stream at 6 feet per second, will enter the buckets with a relative velocity of 2 feet per second.

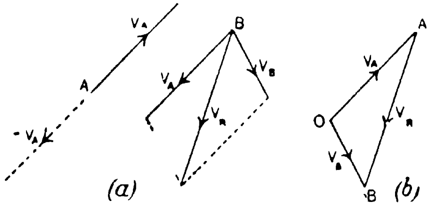

Fig. 441.—Velocity of B relative to A.

If two bodies A and B have velocities as shown at V_A and V_B (Fig. 441 (*a*)), their relative velocity may be obtained in the following manner. Stop A by giving both A and B a velocity V_A equal and opposite to that originally possessed by A; this artifice will not alter their relative velocities. B has now component velocities V_B and V_A, the resultant of which is V_R. As A is at rest, the velocity of B relative to A will be V_R.

In Fig. 442, B has been brought to rest by giving both A and B a velocity V_B equal and opposite to that originally possessed by B. The resultant velocity of A will now be V_R, and, as B is at rest, this will be the velocity of A relative to B.

It will be clear that V_R in Fig. 441 (*a*) is equal and opposite to V_R in Fig 442, showing that the velocity of B relative to A is equal and opposite to the velocity of A relative to B.

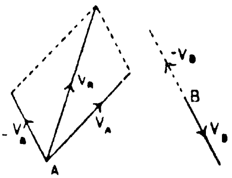

FIG. 442.—Velocity of A relative to B.

A convenient construction is given in Fig. 441 (*b*) From any point O draw OA and OB to represent respectively V_A and V_B, both being placed so that the senses are away from O. Then AB represents the relative velocity of sense from A towards B if the velocity of B relative to A is required, and of opposite sense if the velocity of A relative to B is required.

EXERCISES ON CHAPTER XVI.

1. In a crank and connecting-rod mechanism, the crank is 1 foot and the connecting rod is 4 feet in length. The line of stroke of the cross-head pin passes through the axis of the crank shaft. Find, by drawing, the distances of the crosshead from the beginning of the stroke for crank intervals of 30° throughout the revolution. Plot a distance-time diagram.

2. Use the data obtained in the solution of Question 1, and calculate the mean velocity of the crosshead for each interval. The crank rotates uniformly at 180 revolutions per minute Draw a velocity-time diagram.

3. Using the results of Question 2, calculate the mean acceleration for each interval. Plot an acceleration-time diagram.

4. Answer Questions 1, 2 and 3 for the case in which the line of stroke of the crosshead passes the axis of the crank shaft at a distance of 6 inches.

5. The distance between two stations is 1·6 miles. A locomotive, starting from one station, gives the train an acceleration of 25 miles per hour in 0·5 minute until the speed reaches 30 miles per hour. This speed is maintained until brakes are applied and the train is brought to rest at the second station under a negative acceleration of 3 feet per second per second. Find the time taken to perform the journey.

6. The distance travelled by a body is given in feet by the equation $s=0{\cdot}02t^2+3$, t being the time in seconds from the start. Find the distance travelled, the velocity and the acceleration at the end of 4 seconds, starting from rest.

7. A body, falling freely under the action of gravity, passes two points 30 feet apart vertically in 0·2 second. From what height above the higher point did it start to fall?

8. A body is thrown upwards from the foot of a cliff 40 feet high and reaches a height of 12 feet above the cliff. It finally alights on the cliff top. Find the total time of the flight and the initial velocity.

9. A body slides down a plane inclined at 10 degrees to the horizontal under the action of gravity. What is the acceleration in the direction of the motion, neglecting frictional effects? Suppose the body to start from rest, what will be the velocity after it has travelled 12 feet?

10. A boat is steered across a river 100 yards wide in such a way that, if there were no current, its line of motion would be at 90 degrees to the banks. Actually it reaches a point 40 yards further down stream, and takes 3 minutes to cross. What is the speed of the current?

11. A pistol fires a bullet with a velocity of 1000 feet per second. Suppose it to be fired by a person in a train travelling at 60 miles per hour, (*a*) forward in the line of the motion of the train, (*b*) backward along the same line, (*c*) in a line parallel to the partitions of the compartments, and calculate in each case the resultant velocity of the bullet.

12. A wheel slows from 120 to 110 revolutions per minute. What has been the change in angular velocity in radians per second? If the change took place in 2 minutes, find the angular acceleration

13. A wheel starts from rest and acquires a speed of 150 revolutions per minute in 30 seconds Find the angular acceleration and the revolutions made by the wheel while getting up speed.

14. Starting from rest, a wheel 2 feet in diameter rolls without slipping through a distance of 40 yards in 8 seconds. Find the angular acceleration and the angular velocity at the end of the given time. Plot an angular velocity-time diagram.

15. Water travels along a horizontal pipe with a uniform speed of 4 feet per second The pipe changes direction to the extent of 30 degrees. Find the change in the velocity of the water.

16. A wheel 12 inches in diameter revolves 18,000 times per minute. Find the central acceleration of a point on the rim.

17. Calculate the central acceleration of a train running at 50 miles per hour round a curve having a radius of 0·75 mile.

18. A point describes simple harmonic vibrations in a line 2 feet long. The time of one complete vibration is 0·3 second. Find the maximum velocity.

19. A wheel revolves in a vertical plane 300 times per minute. The plane keeps vertical, but rotates through an angle of 90°. Find the change in angular velocity, and show it in a diagram. If the change took place in 2·5 seconds, find the angular acceleration.

20. A carriage wheel is 4 feet in diameter and is travelling at 6 miles per hour. What is the velocity of a point at the top of the wheel relative to (*a*) a person seated in the carriage, (*b*) a person standing on the ground. Answer the same regarding a point at the bottom of the wheel.

21. A railway line A crosses another B by means of a bridge, the angle of intersection, as seen in the plan, being 30 degrees. A train on A is approaching the point of intersection with a velocity of 40 miles per hour and another train on B is receding from the intersection, on the same side of it, with a velocity of 20 miles per hour. Find the relative velocity of the trains.

22. A particle moves with simple harmonic motion; show that its time of complete oscillation is independent of the amplitude of its motion. The amplitude of the motion is 5 feet and the complete time of oscillation is 4 seconds; find the time occupied by the particle in passing between points which are distant 4 feet and 2 feet from the centre of force and are on the same side of it. (L.U.)

23. At midnight a vessel A was 40 miles due N. of a vessel B; A steamed 20 miles per hour on a S.W. course and B 12 miles per hour due W. They can exchange signals when 10 miles apart. When can they begin to signal, and how long can they continue? (I.C.E.)

CHAPTER XVII.

INERTIA.

Inertia. **Inertia** is that property of matter by virtue of which a body tends to preserve its state of rest or of uniform velocity in a straight line, and offers resistance to any change being made in the velocity possessed by it at any instant, whether the change be one of magnitude or of direction of velocity. Hence, in order to effect any such change, it will be necessary to employ force to overcome the inertia of the body. There will be no resultant force acting on any body which is travelling with uniform velocity in a straight line; in such a case the external forces, if any, applied to the body are in equilibrium. The existence of acceleration in a body implies the presence of a resultant external force, and this force must be applied in the line of, and must have the same sense as the proposed acceleration.

The estimation of the magnitude of the force required to produce a given acceleration may be obtained from an experimental law. All bodies at the same place fall freely with the same acceleration *g*. Now their weights are proportional to their masses, and as these weights are the resultant forces producing acceleration, it follows that the force required to produce a given acceleration is proportional to the mass of the body. It may also be shown by experiment that the force required to produce acceleration in a body of given mass is proportional to the acceleration. Hence, we have the law that the force required is proportional jointly to the body's mass and acceleration, and consequently will be measured by the product of the mass and the acceleration.

From the case of a body falling freely we know that a force of 1 lb. weight acting on a mass of 1 pound gives an acceleration of *g* feet per second per second. It follows that the algebraic statement of the above law will be

$$P = \frac{ma}{g} \text{ lb. weight, } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where m = the mass of the body in pounds,
a = its acceleration in feet per sec. per sec.

The result of the calculation by use of equation (1) will vary to a small extent depending on the value of g at the particular place An **absolute unit of force** may be employed which does not vary, and is defined as the force required to give unit acceleration to a body having unit mass. The British absolute unit of force is the **poundal,** and is the force that would give an acceleration of one foot per second per second to a body free to move and having a mass of one pound The metric absolute unit is the **dyne**; a force of one dyne acting on a body free to move and of mass one gram would produce an acceleration of one centimetre per second per second. Using these units, equation (1) becomes

$$F = ma, \text{ in absolute units, } \ldots\ldots\ldots (2)$$

the result being in poundals, or dynes, respectively if

m = the mass of the body in pounds, or grams,
a = its acceleration in feet, or centimetres per sec. per sec.

The weight of a body expressed in absolute units will be given by

$$W = mg. \ldots\ldots\ldots\ldots (3)$$

Also, a force stated in poundals or dynes may be converted into lb. weight or grams weight by dividing by the proper value of g, which may be taken as 32·2 feet per second per second in the British system, or as 981 centimetres per second per second in the metric system, for all parts of the British Isles.

In using the above equations, it must be understood clearly that each side of the equation represents a force. the left hand side represents the resultant force applied to the body from the outside; the right-hand side represents the force due to the collective resistance of all the particles of the body to any change being made in the velocity. The whole equation expresses the equality of these forces. The student would do well to recall again the fact that a force cannot act alone, there must always be an equal opposite force, and if the latter is not wholly supplied by some resistance given by an outside agency such as friction, etc, it must be supplied in part by the inertia of the body. Equality of the forces is an invariable law.

EXAMPLE 1. A train has a mass of 200 tons. If frictional resistances amount to 12 lb. weight per ton, what steady pull must the locomotive exert in order to increase the speed on a level road from 20 to 40 miles per hour, the change to take place in $1\frac{1}{2}$ minutes?

Let T = pull required, in lb. weight units.
F = total frictional resistance, lb. weight.
P = resultant force producing acceleration, lb. weight.

Then $$P = T - F = \frac{ma}{g}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, $$F = 200 \times 12 = 2400 \text{ lb. weight.}$$

$$\text{Initial velocity} = \frac{20 \times 5280}{60 \times 60} = \frac{88}{3} \text{ feet per sec.}$$

$$\text{Final velocity} = \frac{176}{3} \text{ feet per sec.}$$

$$\text{Acceleration} = a = \left(\frac{176}{3} - \frac{88}{3}\right) \div 90$$

$$= \frac{88}{270} \text{ foot per sec. per sec.}$$

Substituting these values in (1) gives

$$T - 2400 = \frac{200 \times 2240 \times 88}{32 \cdot 2 \times 270}.$$

$$T = 4535 + 2400$$

$$= \underline{6935} \text{ lb. weight.}$$

EXAMPLE 2. The mass of a train is 250 tons and frictional resistances amount to 11 lb. weight per ton. The speed on reaching the top of an incline of 1 in 80 is 30 miles per hour, and the train runs down with steam shut off. If the incline is is 0·5 mile long, what will be the speed at the bottom?

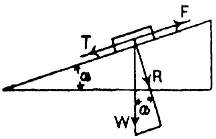

FIG. 443.

Referring to Fig. 443, the weight of the train, W, may be resolved into two forces T and R respectively, parallel and perpendicular to the incline. Let α be the angle made by the incline with the horizontal. Then

$$T = W \sin \alpha = W \tan \alpha, \text{ very nearly,}$$

$$= 250 \times 2240 \times \tfrac{1}{80}$$

$$= 7000 \text{ lb. weight.}$$

Also, $$\text{Friction} = F = 250 \times 11 = 2750 \text{ lb. weight.}$$

$$P = T - F$$

$$= 7000 - 2750 = 4250 \text{ lb weight.}$$

Also, $$P = \frac{ma}{g};$$

$$\therefore\ a = \frac{Pg}{m} = \frac{4250 \times 32 \cdot 2}{250 \times 2240}$$

$$= 0 \cdot 244 \text{ foot per sec. per sec.}$$

Again, $$\text{Initial velocity} = v_1 = \frac{30 \times 5280}{60 \times 60} = 44 \text{ feet per sec.}$$

And $$v_2^2 - v_1^2 = 2as \text{ (p. 389)};$$

$$\therefore\ v_2^2 - (44 \times 44) = 2 \times 0{\cdot}244 \times \frac{5280}{2},$$

$$v_2^2 = 1290 + 1936 = 3226,$$

$$v = 56{\cdot}8 \text{ feet per sec}$$

$$= \underline{38{\cdot}8} \text{ miles per hour.}$$

Kinetic energy. In Fig. 444 is shown a body of mass *m* pounds able to move freely. Let the body be at rest at A and let a force P lb.-weight be applied, in consequence of which the body moves with continually increasing velocity to B, a horizontal distance of *s* feet. Work will be done by P against the resistance due to the inertia of the body.

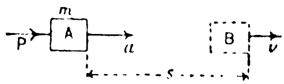

FIG. 444.—Kinetic energy of a body

Work done by P = P*s* foot-lb.

As there has been no external resistances of any kind, it follows that the whole of the work done by P will be stored in the body at B in the form of kinetic energy. Let *v* feet per second be the velocity at B, and let *a* be the acceleration in feet per second per second.

Then $$P = \frac{ma}{g} \text{ lb. weight.}$$

Also, $$s = \frac{v^2}{2a} \text{ feet (p. 388).}$$

Hence, Work done by $$P = \frac{ma}{g} \cdot \frac{v^2}{2a}, \text{ or,}$$

$$\textbf{Kinetic energy of body} = \frac{\mathbf{mv^2}}{\mathbf{2g}} \textbf{ foot-lb.} \quad \ldots \quad (1)$$

Note that the velocity is squared in this result, hence its sign, positive or negative, is immaterial. The interpretation of this is that kinetic energy is not a directed, or vector, quantity, and a body moving in any direction will have kinetic energy which may be calculated by use of the expression found above. The kinetic energy may be expressed in absolute units by omitting *g*.

$$\textbf{Kinetic energy} = \frac{\mathbf{mv^2}}{\mathbf{2}} \textbf{ foot-poundals.} \quad \ldots \quad (2)$$

EXAMPLE 1. A railway truck of mass 20 tons moving at 6 feet per second comes into collision with buffer stops and is brought to rest in a distance of 9 inches. What has been the average resistance of the buffers?

$$\text{Kinetic energy} = \frac{mv^2}{2g} = \frac{20 \times 6 \times 6}{64{\cdot}4} = 11{\cdot}18 \text{ foot-tons.}$$

Let P = the average resistance in tons weight.

Then, Work done against $P = P \times \frac{9}{12}$ foot-tons.

Hence,
$$P \times \frac{9}{12} = 11 \cdot 18,$$
$$P = \frac{11 \cdot 18 \times 12}{9}$$
$$= \underline{14 \cdot 9} \text{ tons weight.}$$

Average forces calculated in this manner are described sometimes as **space-average forces.**

EXAMPLE 2. A vessel of mass 10,000 tons and having a speed of 30 feet per second is slowed to 10 feet per second in travelling a distance of 3000 feet. Calculate the average resistance to the motion.

Here we have

$$\text{Change in kinetic energy} = \frac{mv_1^2}{2g} - \frac{mv_2^2}{2g},$$
$$= \frac{m}{2g}(v_1^2 - v_2^2)$$
$$= \frac{10,000}{64 \cdot 4}(900 - 100)$$
$$= 124,100 \text{ foot-tons.}$$

Let P = the average resistance in tons weight.

Then, Work done against P = P × 3000 foot-tons.

Hence,
$$3000\ P = 124,100,$$
$$P = \underline{41 \cdot 37} \text{ tons weight.}$$

Momentum. The **momentum** of a body in motion is measured by the product of its mass and velocity. The units will be stated by giving the units of mass and velocity employed, thus, if the pound and the foot-second units are employed, then

Momentum = mv pound-foot-seconds.

Suppose a body of mass m pounds, free to move, to be acted on by a force P lb. weight during a time t seconds, and that the body is at rest at first. An acceleration a feet per second per second will be produced, such that
$$P = \frac{ma}{g} \text{ lb. weight.} \quad \ldots\ldots\ldots \quad (1)$$

Since P acts for a time t seconds, the velocity of the body at the end of this time will be

$$v = at \text{ feet per second (p. 388);}$$
$$\therefore\ a = \frac{v}{t} \text{ feet per second per second.}$$

And from (1), by substitution,

$$P = \frac{mv}{gt} \text{ lb. weight} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Now mv is the momentum possessed by the body at the end of the time t seconds, consequently $\frac{mv}{t}$ will be the momentum it acquires each second, *i.e.* the rate of change of momentum. Hence, **the force in lb. weight generating momentum will be numerically equal to the rate of change of momentum in pound-foot-second units divided by g.**

Or, we may write $P = \frac{mv}{t}$ **poundals,** $\ldots\ldots\ldots\ldots\ldots (3)$

showing that the force in absolute units is equal to the rate of change of momentum.

Suppose equal forces P, P, to act during the same interval of time on two bodies A and B, free to move and initially at rest. Let the masses be m_A and m_B respectively, and let v_A and v_B be the velocities acquired at the end of the time t.

From (2) above, $P = \frac{m_A v_A}{gt}$, for the body A,

$P = \frac{m_B v_B}{gt}$, for the body B;

$$\therefore \frac{m_A v_A}{gt} = \frac{m_B v_B}{gt}, \text{ or,}$$

$$m_A v_A = m_B v_B$$

It may be stated therefore that **equal forces, acting during the same time, generate equal momenta irrespective of the masses of the bodies.**

Impulsive forces. Supposing a body in motion to possess a momentum mv, which is abstracted by the body encountering a uniform resistance P. If this is accomplished in t seconds, then

$$P = \frac{mv}{gt}.$$

It will be noticed that if t becomes very small, P will become very large, and is then said to be **impulsive.** If P be not uniform, its average value may be found from the above equation. In the case of impulsive action, P is called the **average force of the blow.**

Change of momentum. Momentum depends on the velocity of a body, and, since velocity has direction, momentum will also be a directed quality and so can be represented by a vector. Momentum differs in this respect from kinetic energy which depends on v^2.

Change of momentum must be estimated always by taking the change in the body's velocity, paying attention to both magnitude and direction. Having found the magnitude and direction of the change in momentum, the force required may be calculated and will act in the same line of direction.

EXAMPLE 1. A locomotive picks up a supply of water from a long trough laid between the rails (Fig. 445) while travelling at 40 miles per hour. Suppose the speed to remain unaltered, what additional resistance is offered if 5 tons of water be picked up in 50 seconds?

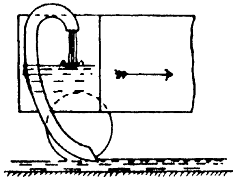

FIG. 445.—Locomotive picking up water

The water in the trough has no momentum; after it is picked up it has the same velocity as the train, hence

$$P=\frac{mv}{gt}=\frac{5\times40\times5280}{32{\cdot}2\times50\times60\times60}$$

$$=\underline{0{\cdot}182}\text{ ton weight.}$$

EXAMPLE 2. A gun discharges 350 bullets per minute, each of mass 0·025 pound, with a velocity of 2000 feet per second. Neglecting the mass of the powder gases, find the backward force on the gun.

$$\text{Mass of bullets ejected per second}=\frac{0{\cdot}025\times350}{60}=0{\cdot}146\text{ pound.}$$

$$\text{Momentum generated per second}=0{\cdot}146\times2000\text{ pound-foot-sec.}$$

$$\text{Force required to eject the bullets}=\frac{0{\cdot}146\times2000}{32{\cdot}2}$$

$$=\underline{9{\cdot}07}\text{ lb. weight.}$$

It is evident that the backward force acting on the gun will be equal to the force required to eject the bullets, viz. 9·07 lb. weight.

EXAMPLE 3. A hammer head of mass 2 pounds and having a velocity of 24 feet per second is brought to rest in 0·005 second. Find the average force of the blow.

$$P=\frac{mv}{gt}=\frac{2\times24}{32{\cdot}2\times0{\cdot}005}$$

$$=\underline{298}\text{ lb. weight.}$$

Average forces calculated in this manner are described sometimes as **time-average forces.**

Centre of mass. It will be understood that every particle in a body offers resistance, due to its inertia, to any attempted change in its velocity. In Fig. 446 is shown a body travelling in a straight line towards the left and having an acceleration *a*. There being no rotation of the body, every particle will experience the same acceleration a. Calling the masses of the particles m_1, m_2, etc., the resultant resistance will be

$$R = m_1a + m_2a + m_3a + \text{etc.}$$

This force will act through the centre C of the parallel forces m_1a, m_2a, etc (p. 48), a centre which is called the **centre of mass** of the body. It may be assumed for all practical purposes that the centre of mass and the centre of gravity of a body coincide.

Let C be the centre of mass of a body (Fig. 447), and let R acting through C be the resultant inertia resistance. The resultant external force F producing acceleration must clearly act in the same straight line as R if there is to be no rotation of the body. Hence, we have the principle that if **the external forces acting on a body free to move are to produce no rotation, their resultant must pass through the centre of mass of the body.** The truth of this may be tested easily by laying a pencil on the table and flicking it with the finger nail. An impulse applied near the end will cause the pencil to fly off rotating as it goes; an impulse applied through the centre will produce no rotation.

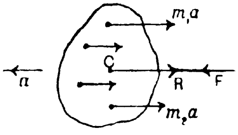

FIG. 446.—Centre of mass of a body.

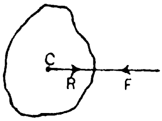

FIG. 447.

Rotational inertia. In Fig. 448 is shown a body which is capable of turning freely about an axis OZ perpendicular to the plane of the paper. In order to produce rotation, without tendency to displace or translate the body, a couple must be applied. Let the two forces P, P, form a couple, one of the forces being applied at the axis, and let the forces rotate with the body so that a constant moment is exerted. The forces being in lb. weight and the arm D being in feet, the moment will be

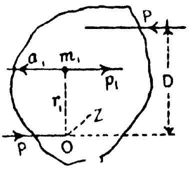

FIG. 448.—Rotational inertia.

$$T = PD \text{ lb.-feet.} \quad \ldots\ldots(1)$$

As the body is free to rotate, the only resistance which will be opposed to the couple must be due to the inertia of the body causing it to endeavour to rotate with uniform angular velocity. For inertia resistance to be possible there must be angular acceleration, consequently each particle of the body will have a linear acceleration in the direction of its path of motion.

Considering one such particle m_1 pound at radius r_1 feet; its linear acceleration will be a_1 feet per sec. per sec., and the resistance which the particle will offer is

$$p_1 = \frac{m_1 a_1}{g} \text{ lb. weight.}$$

Now, $a_1 = \theta r_1$,

where θ is the angular acceleration in radians per sec. per sec. (p. 392).

$$\therefore \ p_1 = \frac{m\theta r_1}{g}.$$

To obtain the moment of this resistance, multiply p_1 by r_1 giving

$$\text{Moment of resistance of particle} = p_1 r_1 = \frac{m_1 \theta r_1^2}{g}$$

$$= \frac{\theta}{g} . m_1 r_1^2. \qquad \ldots\ldots\ldots \quad (2)$$

Now had any other particle been chosen, a similar expression for its moment of resistance would result. Hence

Total moment of resistance due to inertia of body

$$= \frac{\theta}{g} \Sigma m r^2, \quad \ldots\ldots\ldots \quad (3)$$

the summation being taken throughout the body. The quantity Σmr^2 may be called the second moment of mass, or more commonly, the **moment of inertia of the body**, written I. Using a suffix OZ to indicate the axis about which moments must be taken, (3) becomes

$$\text{Total moment of resistance} = \frac{\theta}{g} \mathrm{I}_{OZ} \quad \ldots\ldots\ldots \quad (4)$$

Clearly this moment must be equal to the moment of the couple applied to the body. Hence equating (1) and (4) we have

$$\mathrm{T} = \mathrm{PD} = \frac{\mathrm{I}_{oz}\theta}{g}. \quad \ldots\ldots\ldots \quad (5)$$

If the couple is measured in absolute units, say L poundal-feet, (5) becomes

$$\mathrm{L} = \mathrm{I}_{oz}\theta. \quad \ldots\ldots\ldots \quad (6)$$

The analogy between this equation and the corresponding one for rectilinear motion may be noted; viz.

$$\mathrm{F} = ma. \quad \ldots\ldots\ldots \quad (7)$$

In (7) a force appears on the left hand side, and in (6) the moment of a force; in (7) the product of mass and linear acceleration are on the right hand side, and in (6) the product of second moment of mass or moment of inertia and angular acceleration.

The following common cases of moments of inertia may be noted

Moments of Inertia.

The results are all in pound (foot)^2 units if the mass M is taken in pounds and the linear dimensions in feet.

I. A slender uniform rod.

(*a*) Axis OX parallel to rod and at a distance D from it (Fig. 449).

$$I_{OX} = MD^2.$$

(*b*) Axis OX perpendicular to rod through one end (Fig. 450).

$$I_{OX} = \frac{ML^2}{3}.$$

(*c*) Axis OX perpendicular to rod through its centre of gravity (Fig. 451).

$$I_{OX} = \frac{ML^2}{12}.$$

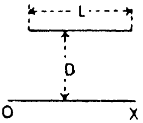

FIG. 449.

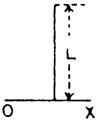

FIG. 450

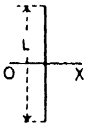

FIG. 451.

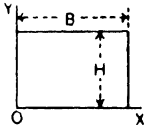

FIG. 452

II. A thin uniform rectangular plate.

(*a*) Axis OX coinciding with a long edge (Fig. 452).

$$I_{OX} = \frac{MH^2}{3}.$$

(*b*) Axis OY coinciding with a short edge (Fig. 452).

$$I_{OY} = \frac{MB^2}{3}.$$

(*c*) Axis GX through centre of gravity and parallel to long edge (Fig. 453).

$$I_{GX} = \frac{MH^2}{12}.$$

(*d*) Axis GY through centre of gravity and parallel to short edge (Fig. 453).

$$I_{GY} = \frac{MB^2}{12}.$$

(*e*) Axis OZ through one corner and perpendicular to plate (Fig. 454).

$$I_{OZ} = \frac{M(H^2+B^2)}{3}.$$

(*f*) Axis GZ through the centre of gravity and perpendicular to plate (Fig. 454).

$$I_{GZ} = \frac{M(H^2+B^2)}{12}.$$

III. **A thick uniform plate.**

(*a*) Axis OY coinciding with one edge (Fig. 455).

$$I_{OY} = \frac{M(B^2+T^2)}{3}.$$

(*b*) Axis GZ parallel to OY and passing through the centre of gravity (Fig. 455).

$$I_{GZ} = \frac{M(B^2+T^2)}{12}.$$

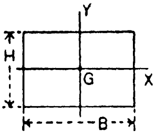

Fig. 453

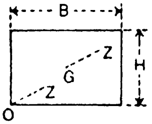

Fig. 454

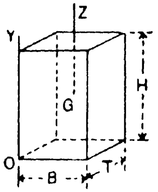

Fig. 455

IV. **A thin circular plate.**

(*a*) Axis OX forming any diameter of the plate (Fig. 456).

$$I_{OX} = \frac{MR^2}{4}.$$

(*b*) Axis TV forming any tangent to the plate (Fig 456).

$$I_{TV} = \frac{5MR^2}{4}.$$

(*c*) Axis OZ passing through the centre and perpendicular to the plate (Fig. 456).

$$I_{OZ} = \frac{MR^2}{2}.$$

(*d*) Axis TZ touching the circumference and perpendicular to the plate (Fig. 456).

$$I_{TZ} = \frac{3MR^2}{2}.$$

V. **A thin circular plate having a concentric hole.**

(*a*) Axis OX forming any diameter of the plate (Fig. 457).

$$I_{OX}=\frac{M(R_1^2+R_2^2)}{4}.$$

(*b*) Axis OZ passing through the centre and perpendicular to the plate (Fig. 457).

$$I_{OZ}=\frac{M(R_1^2+R_2^2)}{2}.$$

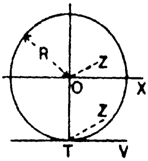

FIG. 456.

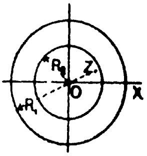

FIG. 457.

VI. **A solid cylinder.**

Axis OX coinciding with axis of cylinder.

$$I_{OX}=\frac{MR^2}{2}.$$

VII. **A hollow concentric cylinder.**

Axis OX coinciding with axis of cylinder.

$$I_{OX}=\frac{M(R_1^2+R_2^2)}{2}.$$

VIII. **A solid sphere.**

Axis OX forming any diameter.

$$I_{OX}=\frac{2MR^2}{5}.$$

The following rules are useful in calculating moments of inertia.

(*a*) Given I_{OX} and I_{OY} for a thin uniform plate, to find I_{OZ}, OZ being perpendicular to the plane containing OX and OY:

$$I_{OZ}=I_{OX}+I_{OY}.$$

(*b*) Given I_{GX} for a thin uniform plate, GX being an axis passing through the centre of gravity, to find I_{OX}, OX being parallel to GX at a distance D: $I_{OX}=I_{GX}+MD^2$.

(*c*) Routh's rule: If a body is symmetrical about three axes which are mutually perpendicular, the moment of inertia about one axis is equal to the mass of the body multiplied by the sum of the squares of the other two semi-axes and divided by 3, 4, or 5 according as the body is rectangular, elliptical or ellipsoidal.

EXAMPLE 1. A rectangular plate, as shown in Fig. 455, is symmetrical about GZ and other two axes passing through G, and parallel to B and T respectively.

$$I_{GZ}=\frac{M\{(\frac{1}{2}B)^2+(\frac{1}{2}T)^2\}}{3}=\frac{M(B^2+T^2)}{12}.$$

EXAMPLE 2. A solid cylinder (special case of an *elliptical* body) is symmetrical about the axis of the cylinder OX and about other two axes forming diameters at 90° and passing through the centre of gravity of the cylinder. Hence,

$$I_{OX}=\frac{M(R^2+R^2)}{4}=\frac{MR^2}{2}.$$

EXAMPLE 3. A solid sphere is symmetrical about any three diameters which are mutually perpendicular. Hence, about one diameter, OX

$$I_{OX}=\frac{M(R^2+R^2)}{5}=\frac{2MR^2}{5}$$

The **radius of gyration** of a body is defined as a quantity k such that, if its square be multiplied by the mass of the body, the result gives the moment of inertia of the body about a given axis. Taking the case of a solid cylinder as an example, the moment of inertia about OZ, the axis of the cylinder, is

$$I_{OZ}=\frac{MR^2}{2}.$$

Let $I_{OZ}=Mk^2.$

Then $k^2=\frac{R^2}{2}$, or $k=\frac{R}{\sqrt{2}}$,

which gives the value of the radius of gyration for this particular axis.

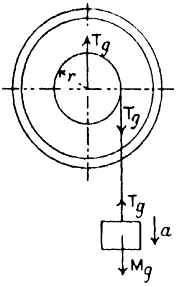

FIG. 458.—An experimental flywheel

EXAMPLE 1. A flywheel has a moment of inertia of 8000 in pound and foot units, and is brought from rest to a speed of 180 revolutions per minute in 25 seconds. What average couple must have acted?

$$\text{Final angular velocity}=\omega=\frac{180}{60}\times 2\pi=6\pi \text{ radians per sec.}$$

$$\text{Angular acceleration}=\theta=\frac{\omega}{t}=\frac{6\pi}{25} \text{ radians per sec. per sec.}$$

$$T=\frac{I\theta}{g}=\frac{8000\times 6\pi}{25\times 32\cdot 2}=\underline{187} \text{ lb.-feet.}$$

EXAMPLE 2. In a laboratory experiment, a flywheel of mass 100 pounds and radius of gyration 1·25 feet (Fig. 458) is mounted so that it may be rotated by a falling weight attached to a cord wrapped round the wheel axle. Neglect friction and find what will be the accelerations if a body of 10 pound weight is attached to the cord and if the radius of the axle is 2 inches.

Let M = mass hung on, in pounds.
Mg = its weight, in absolute units.
Tg = pull in cord, in absolute units.
r = radius of axle, in feet
I = moment of inertia of wheel
$= 100 \times 1.25 \times 1.25 = 156.2$ pound and foot units.
a = the linear acceleration of M, in feet per sec. per sec.
θ = the angular acceleration of the wheel, in radians per sec. per sec.

Then, considering M, we have

Resultant force producing acceleration $= Mg - Tg = Ma$ .... (1)

Considering the wheel, we have

Couple producing acceleration $= Tgr = I\theta$ . ... (2)

Also, $\theta = \frac{a}{r}$ . . . .(3)

These three equations will enable the solutions to be obtained. Thus:

From (2) and (3), $Tgr = I\frac{a}{r}$;

$Tg = I\frac{a}{r^2}$. ....(4)

Substitute this value in (1), giving

$$Mg - I\frac{a}{r^2} = Ma,$$

$$Mg = \left(M + \frac{I}{r^2}\right)a;$$

$$\therefore a = \frac{Mg}{M + \frac{I}{r^2}} = \frac{10 \times 32.2}{10 + (156.2 \times 6 \times 6)}$$

$$= \underline{0.0572} \text{ feet per sec. per sec.}$$

From (3), $\theta = \frac{0.0572}{r} = 0.0572 \times 6$

$= \underline{0.343}$ radian per sec. per sec.

Kinetic energy of rotation. In Fig. 459 is shown a body rotating with uniform angular velocity ω about an axis OZ perpendicular to the plane of the paper. Considering one of its particles m_1, the linear velocity of which is v_1, we have

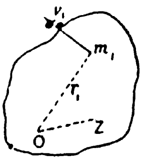

FIG. 459.—Kinetic energy of rotation.

Kinetic energy of particle $= \frac{m_1 v_1^2}{2g}$.

Now, $v_1 = \omega r_1$,

$v_1^2 = \omega^2 r_1^2$.

Hence, Kinetic energy of particle $= \frac{\omega^2}{2g} \cdot m_1 r_1^2$. ....(1)

A similar expression would result for any other particle, hence

Total kinetic energy of body $= \frac{\omega^2}{2g} \Sigma m r^2$, or,

Kinetic energy $= \frac{\omega^2}{2g} \mathrm{I}_{oz}$. .... (2)

In using this equation with ω in radians per second, g should be in feet per second per second and I in pound-(foot)^2 units to obtain the result in foot-lb. The corresponding equation for foot-poundals would be

Kinetic energy $= \frac{\omega^2}{2} \cdot \mathrm{I}_{oz}$. ....(3)

EXAMPLE 1. A flywheel has a mass of 5000 pound and a radius of gyration of 4 feet. Find its kinetic energy at 150 revolutions per minute.

$\omega = \frac{150}{60} \cdot 2\pi = 5\pi$ radians per sec per sec.

$\mathrm{I} = \mathrm{M}k^2 = 5000 \times 16 = 80{,}000$ pound-(foot)^2.

$$\text{Kinetic energy} = \frac{\omega^2}{2g}\mathrm{I} = \frac{25 \times \pi^2 \times 80{,}000}{64 \cdot 4}$$

$$= \underline{306{,}500} \text{ foot-lb.}$$

EXAMPLE 2. The above flywheel slows from 150 to 148 revolutions per minute. Find the energy which has been abstracted

$$\text{Change in kinetic energy} = \frac{\omega_1^2 \mathrm{I}}{2g} - \frac{\omega_2^2 \mathrm{I}}{2g} = \frac{\mathrm{I}}{2g}(\omega_1^2 - \omega_2^2).$$

Also, $\omega_1 = 5\pi,$

$\omega_2 = \frac{148}{60} \cdot 2\pi = 4 \cdot 933\pi$;

$$\therefore \text{ Energy abstracted} = \frac{\mathrm{I}}{2g}(\omega_1 - \omega_2)(\omega_1 + \omega_2)$$

$$= \frac{80{,}000 \cdot \pi^2}{64 \cdot 4} \times 0 \cdot 067 \times 9 \cdot 933$$

$$= \underline{8160} \text{ foot-lb.}$$

Energy of a rolling wheel. The total kinetic energy of a wheel rolling along a road will be made up of kinetic energy of rotation together with kinetic energy of translation.

Let ω = the angular velocity, radians per sec.

v = the linear velocity in feet per sec. of the carriage to which the wheel is attached (this will also be the velocity of the centre of the wheel).

M = the mass of the wheel in pounds.

k = its radius of gyration, in feet, with reference to the axle.

Then, Kinetic energy of rotation $= \frac{\omega^2 I}{2g} = \frac{\omega^2 M k^2}{2g}$ foot-lb.

Kinetic energy of translation $= \frac{Mv^2}{2g}$ foot-lb.

$$\text{Total kinetic energy} = \frac{\omega^2 I}{2g} + \frac{Mv^2}{2g}. \quad \ldots\ldots\ldots\ldots(1)$$

Again, if there be no slipping between the wheel and the road, we have (p. 393)

$$\omega = \frac{v}{R}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where R is the radius of the wheel in feet.

Substituting in (1), we obtain, for perfect rolling:

$$\text{Total kinetic energy} = \frac{v^2 M k^2}{2gR^2} + \frac{Mv^2}{2g}$$

$$= \frac{Mv^2}{2g}\left(\frac{k^2}{R^2} + 1\right) \text{ foot-lb.} \ldots\ldots\ldots(3)$$

Energy of a wheel rolling down an inclined plane. The principle of the conservation of energy may be applied to the case of a body rolling down an inclined plane (Fig. 460). In rolling from A to B, the body descends through a vertical height H feet; hence, if M is the mass in pounds,

$$\text{Work done by gravity} = MgH \text{ foot poundals.} \quad \ldots\ldots\ldots(1)$$

Assuming that none of this is wasted, the total kinetic energy at B will be equal to the same quantity. The energy at B is made up of translational kinetic energy owing to the linear velocity v feet per second of the mass centre and of rotational kinetic energy owing to the angular velocity ω radians per second.

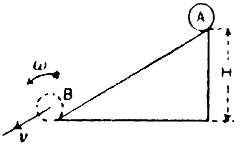

FIG. 460.—Energy of a wheel rolling down an incline

$$\text{Hence, Total energy at B} = \left(\frac{Mv^2}{2} + \frac{\omega^2}{2} I_{oz}\right) \text{ foot-poundals,} \quad \ldots\ldots(2)$$

I_{oz} being the moment of inertia in pound and feet units about the axis of rotation passing through the mass centre of the body. Equating (1) and (2), we have

$$MgH = \frac{Mv^2}{2} + \frac{\omega^2}{2} I_{oz}.$$

Writing Mk^2 for I_{oz}, this will give

$$MgH = \frac{Mv^2}{2} + \frac{\omega^2}{2} Mk^2,$$

or

$$gH = \tfrac{1}{2}(v^2 + \omega^2 k^2). \quad \ldots\ldots\ldots\ldots\ldots(3)$$

If there is no slipping between the wheel and the plane, we have

$$\omega = \frac{v}{R},$$

where R is the radius of the body in feet.

Hence,
$$gH = \frac{1}{2}\left(v^2 + \frac{v^2}{R^2}k^2\right),$$

or
$$v^2 = \frac{2gH}{1 + \frac{k^2}{R^2}},$$

$$v = \sqrt{\frac{2gH}{1 + \frac{k^2}{R^2}}} \quad \ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Motion of a wheel rolling down an incline. The following way of regarding the same problem should be studied. Fig. 461 shows a body rolling down a plane inclined at an angle α to the horizontal. The weight Mg may be resolved into two forces respectively parallel to and at right angles to the plane; these will be M$g \sin \alpha$ and M$g \cos \alpha$. The normal reaction Q of the plane will be equal to M$g \cos \alpha$. If there is no friction, all these forces act through the mass centre O, and there will be no rotation, *i.e.* the body will slip down the plane without any rolling. Suppose that a maximum frictional force F may act between the plane and the body and that μ is the coefficient of friction, then

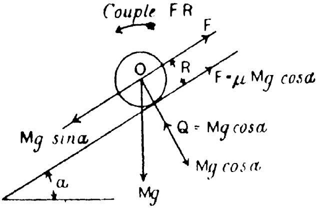

FIG. 461.—Forces acting on a wheel rolling down an incline.

$$F = \mu Mg \cos \alpha.$$

To investigate the effect of F, transfer it to the mass centre O as shown and apply an anti-clockwise couple of magnitude FR. Then

$$\begin{aligned} P &= \text{Resultant force at O acting down the plane} \\ &= Mg \sin \alpha - F \\ &= Mg \sin \alpha - \mu Mg \cos \alpha. \end{aligned}$$

Let a be the linear acceleration of O down the plane. Then

$$P = Ma,$$

or

$$Mg(\sin\alpha - \mu\cos\alpha) = Ma,$$

$$\therefore\ a = g(\sin\alpha - \mu\cos\alpha) \qquad (5)$$

Also, owing to the couple FR, an angular acceleration θ will be produced, to be obtained from

$$\theta = \frac{FR}{I_{Oz}} = \frac{\mu Mg\cos\alpha\ R}{Mk^2}$$

$$= g\frac{\mu R\cos\alpha}{k^2} \quad \ldots\ldots \quad (6)$$

If the rolling is perfect, *i.e.* no slip, we have

$$\theta = \frac{a}{R}.$$

Hence, from (5) and (6),

$$\frac{g}{R}(\sin\alpha - \mu\cos\alpha) = g\frac{\mu R\cos\alpha}{k^2},$$

$$\sin\alpha - \mu\cos\alpha = \frac{\mu R^2\cos\alpha}{k^2},$$

$$\sin\alpha = \mu\cos\alpha\left(\frac{R^2}{k^2} + 1\right);$$

$$\therefore\ \mu = \frac{\tan\alpha}{\frac{R^2}{k^2} + 1}. \quad \ldots \quad (7)$$

This expresses the minimum value of the coefficient of friction consistent with perfect rolling. Assuming that the rolling is perfect, the value of the linear acceleration a may be calculated as follows:

From (6),

$$\mu\cos\alpha = \theta\frac{k^2}{gR} = \frac{a}{R}\cdot\frac{k^2}{gR}$$

$$= \frac{a}{g}\cdot\frac{k^2}{R^2}. \quad \ldots\ldots \quad (8)$$

Substituting this value in (5) gives

$$a = g\left(\sin\alpha - \frac{a}{g}\cdot\frac{k^2}{R^2}\right)$$

$$= g\sin\alpha - a\frac{k^2}{R^2},$$

$$\therefore\ a = \frac{g\sin\alpha}{1 + \frac{k^2}{R^2}}. \quad \ldots\ldots \quad (9)$$

Suppose that the body starts from rest at A (Fig. 462) and rolls to B. The linear velocity v of the mass centre at B may be calculated thus,

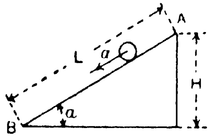

FIG. 462.

$$v^2 = 2aL \text{ (p. 388).}$$

Also, $\frac{H}{L} = \sin\alpha$; or, $L = \frac{H}{\sin\alpha}$;

$$\therefore \; v^2 = 2a\frac{H}{\sin\alpha}.$$

Inserting the value of a from (9), we have

$$v^2 = \frac{2g\sin\alpha}{1 + \frac{k^2}{R^2}} \cdot \frac{H}{\sin\alpha} = \frac{2gH}{1 + \frac{k^2}{R^2}},$$

$$v = \sqrt{\frac{2gH}{1 + \frac{k^2}{R^2}}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Comparison of (4) with (10) will show that the same result has been obtained by both methods.

EXAMPLE. In a laboratory experiment, a small steel ball was allowed to roll down a plane of length 6 feet and inclination 1° 40′. The average time taken (six experiments) was 4·25 seconds. Compare the experimental and calculated accelerations of the ball.

To obtain the experimental acceleration, we have

$$s = \tfrac{1}{2}a_1t^2,$$

where s is the length of the incline and t is the time of descent. Hence,

$$a_1 = \frac{2s}{t^2} = \frac{2 \times 6}{4{\cdot}25 \times 4{\cdot}25}$$

$$= \underline{0{\cdot}664} \text{ feet per sec. per sec.}$$

To calculate the acceleration, take equation (9), p. 423.

$$a_2 = \frac{g\sin\alpha}{1 + \frac{k^2}{R^2}}.$$

For a ball, $I = \tfrac{2}{5}MR^2$ (p. 417);

$$\therefore \; k^2 = \tfrac{2}{5}R^2;$$

$$\therefore \; \frac{k^2}{R^2} = \frac{2}{5}$$

Hence, $a_2 = \frac{g\sin\alpha}{1 + \frac{2}{5}} = \tfrac{5}{7}g\sin\alpha$

$$= \tfrac{5}{7} \times 32{\cdot}2 \times 0{\cdot}02908$$

$$= \underline{0{\cdot}669} \text{ feet per sec. per sec.}$$

The experimental and calculated accelerations differ by about three-quarters of one per cent.; the agreement is good.

Centrifugal force. It has been shown (p. 397) that, when a small body moves in the circumference of a circle of radius R feet with uniform velocity *v* feet per second (Fig. 463), there is a constant acceleration towards the centre of the circle given by

$$a = \frac{v^2}{R} \text{ feet per sec. per sec.}$$

To produce this acceleration requires the application of a uniform force P, also continually directed towards the centre of the circle, and given by

$$P = \frac{ma}{g}, \text{ or,}$$

$$P = \frac{mv^2}{gR} \text{ lb. weight} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This force overcomes the inertia of the body, which would otherwise pursue a straight line path, and may be called the **central force.**

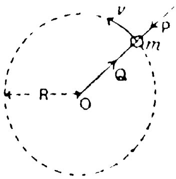

FIG. 463.—Central and centrifugal forces.

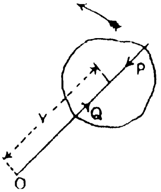

FIG. 464.—Resultant centrifugal force.

It is resisted by an equal and opposite force Q (Fig. 463), produced by the inertia of the body. Q is called the **centrifugal force.**

Expressed in terms of the angular velocity ω radians per second,

$$P = Q = \frac{m\omega^2R^2}{gR}, \text{ or,}$$

$$P = \frac{\omega^2}{g}.\text{mR lb. weight} \quad . \quad . \quad . \quad . \quad (2)$$

A large body rotating about an axis may be considered as being made of a large number of small bodies; for each of these, ω^2/g will be the same, hence the total central force will be

$$P = \frac{\omega^2}{g}(m_1R_1 + m_2R_2 + m_3R_3 + \text{etc.}).$$

The quantity inside the brackets would have the same numerical value if the whole mass were concentrated at the centre of mass.

Let M = mass of whole body, in pounds,
Y = radius of the centre of mass, in feet (Fig. 464).

Then $$P - Q = \frac{\omega^2}{g} MY \text{ lb. weight} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$= \omega^2 MY \text{ poundals.} \quad \ldots\ldots\ldots\ldots \quad (3')$$

It follows from this result that if a body rotates about an axis passing through its centre of mass (in which case $Y = 0$), there will be no resultant pull on the axis due to centrifugal action. There may be a disturbance set up if the body is not symmetrical about an axis at right angles to the axis of rotation, and passing through the centre of mass. For example, in Fig. 465 is shown a rod rotating about an

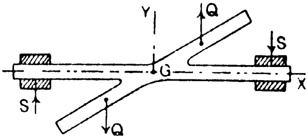

FIG. 465.—An unsymmetrical load produces rocking couple.

axis GX, G being the centre of mass. The rod is not symmetrical about GY, hence, considering the halves separately, there will be centrifugal forces as shown by Q, Q, forming a couple tending to bring the rod into the axis GY. If this tendency is to be balanced, forces S, S, forming an equal opposite couple must be applied by the bearings. These forces will, of course, rotate with the rod and produce what is called a **rocking couple.** In Fig. 466 is shown a

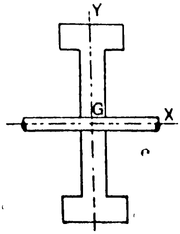

FIG. 466.—A balanced symmetrical body.

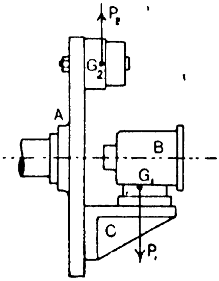

FIG. 467.—Balancing a piece of work in a lathe.

body symmetrical about GY and consequently having neither rocking couple nor resultant centrifugal force.

In Fig. 467 is shown the face plate of a turning lathe with a piece of work B attached to the face plate by means of an angle plate C.

To the other side of the face plate is attached a **balance weight** which is adjusted until there is no tendency to rotate the spindle of the lathe from any position of rest, *i.e.* the centre of gravity of the whole falls on the axis of rotation. This is called **static balancing** and will serve very well for low speeds. It will be observed, however, that the bodies attached to the face plate are not symmetrical. G_1 and G_2, the centres of gravity of the work and of the balance weight, are not in the same vertical line, hence the centrifugal forces P_1 and P_2, being equal, form a rocking couple which will set up troublesome vibrations if the speed be increased. The effect may be reduced by having the balance weight further from the face of the plate.

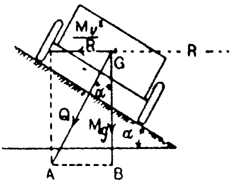

FIG. 468. Section of a banked motor track.

In Fig 468 is shown a motor car travelling in a curved path. To prevent side slipping, the road is banked up to such an extent that the resultant Q of the centrifugal force and the weight falls perpendicularly to the road surface.

Let M = mass of car, in pounds.

v = its speed, in feet per sec.

R = radius of curve, in feet.

Then, Weight of car = Mg poundals.

$$\text{Centrifugal force} = \frac{Mv^2}{R} \text{ poundals.}$$

Also, ABG is the triangle of forces. Hence,

$$\frac{\text{Centrifugal force}}{\text{Weight of car}} = \frac{Mv^2}{RMg} = \frac{v^2}{gR} = \frac{AB}{BG}.$$

$$\text{Now,} \qquad \frac{AB}{BG} = \tan \alpha,$$

and α is also the angle which the section of the road surface makes with the horizontal; hence,

$$\tan \alpha = \frac{v^2}{gR}.$$

Railway tracks are also banked up in a similar manner; the superelevation of the outer rail prevents the flanges of the outer wheels grinding against the rail in rounding a curve.

EXERCISES ON CHAPTER XVII.

1. A body of mass 200 pounds has an acceleration of 150 feet per second per second at a given instant. Calculate the resistance due to the inertia of the body.

2. A resultant force of 1220 dynes acts on a body of mass 1·25 grams. Calculate the acceleration in cm per sec. per sec.

3. A train has a mass of 250 tons, and starts with an acceleration of 1·1 feet per second per second. Frictional resistances amount to 11 lb. weight per ton. Find the pull which the locomotive must exert

4. A body slides down a plane inclined at 20 degrees to the horizontal. The coefficient of friction is 0·1, find the acceleration and the time taken to travel the first 20 feet

5. A load of 10 pounds is attached to a cord which exerts a steady upward pull less than 10 lb weight Starting from rest, the load is found to descend 6 feet vertically in 4 seconds Find the pull in the cord

6. A shot has a mass of 20 pounds and a speed of 1500 feet per second. Find its kinetic energy in foot-tons. Supposing an obstacle to be encountered and that the shot is brought to rest in a distance of 12 feet, what is the average resistance?

7. Calculate the momentum of the shot given in Question 6 Suppose that the shot had been brought to rest in 0·02 second, and calculate the average resistance.

8. A man stands on a small truck mounted on wheels which are practically frictionless. If the man jumps off at the rear end, what will happen to the truck? Take the masses of the man and the truck to be 150 and 200 pounds respectively, and assume that the man is travelling at 8 feet per second immediately he has left the truck.

9. A jet of water delivers 50 pounds of water per second with a velocity of 35 feet per second The jet strikes a plate which is fixed with its plane at 90 degrees to the jet Find the pressure on the plate.

10. Suppose in Question 9 that the plate had been curved in such a manner that the jet slides on to it and has the direction of its velocity on leaving the plate inclined at 90 degrees to its original direction Find the change in velocity, and hence find the pressure on the plate.

11. A wheel has a moment of inertia of 10,000 in pound and feet units, and is brought from rest to 200 revolutions per minute in 25 seconds. Calculate what steady couple must have acted on it

12. An iron plate 4 feet high, 2 feet wide and 2 inches thick is hinged at a vertical edge Calculate its moment of inertia about the axis of the hinges. Take the density of iron to be 480 pounds per cubic foot.

13. Find the moment of inertia about the axis of rotation of a hollow shaft 20 inches external and 8 inches internal diameter by 60 feet long. Take the density as given in Question 12.

14. A solid ball of cast iron is 12 inches in diameter; density of metal 450 pounds per cubic foot. Find the moment of inertia about an axis of which touches the surface of the ball.

15. Referring to Question 5. The upper part of the cord is wrapped round a drum 6 inches diameter measured to the cord centre, and a flywheel is attached to the same shaft as the drum. Find the moment of inertia of the flywheel.

16. A solid disc of cast iron is 4 feet in diameter and 6 inches thick, and rotates about an axis at 90 degrees to its plane and passing through its centre. For the density, see Question 14. Speed 150 revolutions per minute. Find the radius of gyration and the kinetic energy of the disc.

17. If the disc given in Question 16 slows to 140 revolutions per minute, how much energy will be given up?

18. Suppose that the disc given in Question 16 were to roll without slip down an incline of 1 in 10, what would be the linear acceleration of its centre?

19. A blade of a small steam turbine has a mass of 0·05 pound and revolves in the circumference of a circle 8 inches in diameter 24,000 times per minute. Find the centrifugal force.

20. An oval track for motor cycles has a minimum radius of 80 yards, and has to be banked to suit a maximum speed of 65 miles per hour. Find the slope of the cross section at the places where the minimum radii occur.

21. A tramcar weighs 12 tons complete. Each of the axles with its wheels, etc., weighs 0·5 ton, and has a radius of gyration of 1 foot. The diameter of the wheel tread is 3 feet, and the car is travelling at 12 miles per hour. Find (*a*) the energy of translation of the car; (*b*) the energy of rotation of the two axles; (*c*) the total kinetic energy of the vehicle. (B.E.)

22. Prove the formula for the acceleration of a point moving with uniform speed in a circle. Find in direction and magnitude the force required to compel a body weighing 10 lb. to move in a curved path, the radius of curvature at the point considered being 20 feet, the velocity of the body 40 feet per second, and the acceleration in its path 48 feet per second per second. (I.C.E.)

23. A motor car, whose resistance to motion on the level is supposed to be the same at all speeds, has been running steadily on the level at 20 miles per hour; it now gets into a rise of 1 in 12. What is the maximum length of this rising road which may be traversed by the car without changing gear? (B.E.)

24. A train weighing 300 tons, travelling at 60 miles per hour down a slope of 1 in 110, with steam shut off, has the brakes applied and stops in 450 yards. Find the space-average of the retarding force in tons exerted by the brakes; if the time that elapses between the putting on of the brakes and the moment of stopping is 36 seconds, find the time-average of the retarding force in tons. (I.C.E.)

CHAPTER XVIII.

INERTIA - CONTINUED

Angular momentum. The **angular momentum** or **moment of momentum** of a particle may be defined by reference to Fig 469. A particle of mass m pounds revolving in a circle of r feet in radius has a linear velocity of v feet per second at any instant in the direction of the tangent. Hence its linear momentum at any instant will be

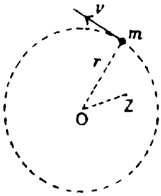

FIG. 469 Angular momentum of a body.

$$\text{Linear momentum of particle} = mv,$$

and

$$v = \omega r,$$

$$\therefore \text{ Linear momentum of particle} = \omega m r. \qquad (1)$$

The moment of this about OZ (Fig 469) may be obtained by multiplying by r, the result being called the moment of momentum, or angular momentum of the particle

$$\text{Angular momentum of particle} = \omega m r^2 \qquad (2)$$

A body having many particles would have a similar expression for each. Hence,

$$\textbf{Angular momentum of a body} = \omega \Sigma m r^2$$
$$= \omega I_{oz} \qquad \ldots\ldots (3)$$

Consider now a body free to rotate about a fixed axis, and, starting from rest, acted on by a constant couple T lb.-feet. The constant angular acceleration being θ, we have, as in equation (5) p. 414,

$$T = \frac{I_{oz}\theta}{g}.$$

Let T act during a time t seconds, then the angular velocity ω at the end of this time will be

$$\omega = \theta t, \quad \text{or,} \quad \theta = \frac{\omega}{t}.$$

Hence,

$$T = \frac{\omega I_{oz}}{gt} \text{ lb.-feet,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

or

$$L = \frac{\omega I_{oz}}{t} \text{ poundals-feet,} \ldots\ldots\ldots\ldots (5)$$

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to the action of a horizontal
frame, and produced by
ce offered by the stand to
spindle FG. In Fig. 475 an
heel is shown, rotating in the
Oa represents the angular ve
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ning the wheel axis, and produ
the altered angular velocity of
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Schlick's anti-rolling gyrost

method used by Schlick fo

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Hence the angular velocity of the
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l, and the wheel spindle which
will revolve clockwise when viewed

pound-feet
ia of the wheel about its axis, pound
of the wheel about its axis, radians
of the wheel axis in the horizontal
second.
axis makes a complete revolution in
seconds.
ion of the axis, change in angular
402.)

Also, $L = \theta I = \frac{2\pi\omega_1}{t} I$. (p. 414.)

Also, $\omega_2 t = 2\pi$; $\therefore t = \frac{2\pi}{\omega_2}$;

$$\therefore L = \frac{2\pi\omega_1\omega_2}{2\pi} I = \omega_1\omega_2 I \quad \ldots(1)$$

In Figs. 470 and 471,

$$L = Mg \times CO,$$

$$Mg \times CO = \omega_1\omega_2 I,$$

$$\omega_2 = \frac{Mg \times CO}{\omega_1 I} \quad \ldots\ldots(2)$$

here CO is the horizontal distance
etween the centre of gravity of the
heel and the suspending cord.

Gyrostatic action in motor cars.

nary
In Fig. 472 is shown a motor car
travelling round a curve; when
e engine flywheel has a clockwise angular
nt the angular velocity of the flywheel
sition shown, and let OB represent the

ow, ωI_{OZ} is the angular momen
hence $\omega I_{OZ}/t$ will be the gain
We may therefore state that **the to rotate about a fixed axis is nume angular momentum divided by** g; or, poundal feet. This statement sho linear momentum given on p. 411.

It will be evident that the appli plane of rotation of the body; sh rectangular components of the cou component which is in the plane equation (4)

III.

Gyrostatic action. In Fig. 470 is by means of a long cord attached at C

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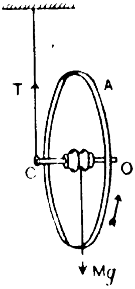

FIG. 470. A cycle wheel showing gyrostatic action.

If the wheel be at rest, it cannot n particle ωmr^2
assistance, but, if set revolving, d have a similar e
m taining its plane of revolut **f a body** $\omega \Sigma mr^2$
however, that the wheel spindle s $= \omega I_{OZ}$
a horizontal plane; the vertical pl ate about a fix
of course, be always perpendicular stant couple T
owing to the action of the couple , we have, as in equ
pull of the cord, and Mg, the weig
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occur in the plane containing the couple he angular v

In Fig. 471 is shown a plan of the wh
wise when viewed from the right hand

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momentum, and therefore no couple is required. A couple is required in every case where the new plane of revolution is inclined to the initial plane.

In Fig 474 is shown a common form of gyrostat by use of which useful information regarding the behaviour of gyrostats may be obtained. The revolving wheel A rotates on a spindle BC, the bearings of which at B and C are formed in a ring which has freedom to rotate about an axis DE. The ring has bearings at D and E in another semicircular ring DFE, which has freedom to rotate about the vertical axis FG The spindle FG is dropped into a vertical hole in a heavy stand The effect of a weight W hung from C will be to cause the axis BC to rotate in a horizontal plane, accompanied, of course, by the whole frame, the direction of this rotation, as viewed from above, will be either clockwise or anticlockwise, depending on the direction of rotation of the wheel.

It will be noted that the original vertical plane of rotation gradually becomes inclined to the vertical as the motion goes on. This is owing to the action of a horizontal couple acting on the frame, and produced by the frictional resistance offered by the stand to the rotation of the spindle FG. In Fig 475 an elevation of the wheel is shown, rotating in the vertical plane OA. *Oa* represents the angular velocity of the wheel; *ab* represents the change in angular velocity in a given interval of time in the horizontal plane containing the wheel axis, and produced by the frictional couple. *Ob* is the altered angular velocity of the wheel at the end of the interval. The wheel will now be rotating in the plane OA′, perpendicular to *Ob* and inclined at an angle AOA′ to the vertical.

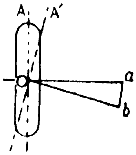

FIG. 475

While the wheel is rotating, if the free motion of the semicircular ring DFE be impeded by application of a finger, it will be noted that the wheel turns over. This effect is precisely the same as the effect of the frictional couple, only it is more marked, as the horizontal couple produced by the finger is larger than that produced by the friction. If DFE be held forcibly from rotating, the wheel will assume a horizontal plane of rotation instantly. In fact, the wheel is only capable of exerting a couple which will equilibrate the couple applied by means of W, provided its motion in azimuth is allowed to take place freely.

Schlick's anti-rolling gyrostat. Fig. 476 illustrates in outline the method used by Schlick for reducing the rolling of a ship among waves. The view is a cross section of the ship; A is a heavy wheel revolving in a horizontal plane about the axis BC. The frame in which the wheel rotates can rock about a horizontal axis DE, which works in bearings secured to the ship's frames. DE is perpendicular to the direction of length of the ship. When the ship rolls, the axis DE is forced out of the horizontal, and the axis BC will be inclined by either B or C coming out of the paper. A couple, applied by vertical forces acting at D and E, is required to give the wheel frame this motion, and an equal opposite couple acts on the ship, tending to give to it a motion opposite to the rolling motion. In consequence of this reaction on the ship, the rolling effect is made much smaller. Freedom of motion about DE must be provided, otherwise the wheel is incapable, as has been shown above, of offering any resistance to rolling.

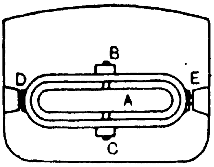

FIG. 476.—Schlick's anti rolling gyrostat.

There are many other applications of the principle of the gyrostat, such as in the gyro-compass used on board ships, in the Brennan monorail cars, and in steering torpedoes.

Simple harmonic vibrations. It has been shown (p. 398) that a body, in describing simple harmonic vibrations, possesses at any instant an acceleration directed towards the centre of the vibration, and proportional to the distance of the body from the centre of the vibration. A force will be necessary in order to produce this acceleration, and the force will evidently follow the same law as the acceleration, *i.e.* it will be constantly directed towards the centre of the vibration, and will be proportional to the distance of the body from the centre.

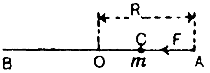

FIG 477.—Simple harmonic vibrations.

In Fig. 477 a body of mass m pounds vibrates with simple harmonic motion in the line AB. Let v feet per second be the velocity when the body is passing through the centre O, then the accelerations at A and B will be

$$a_1 = \frac{v^2}{R} \text{ feet per second per second,}$$

R being the length of OA in feet (p. 397).

Let F_1 be the force in poundals required at A and B, then

$$F_1 = \frac{mv^2}{R} \text{ poundals. } \dots\dots\dots\dots\dots\dots (1)$$

Supposing the body to be situated at C, its acceleration a may be found from

$$\frac{a}{a_1} = \frac{OC}{OA} = \frac{OC}{R},$$

$$\therefore\ a = a_1 \frac{OC}{R}.$$

Also, the force F required to produce the acceleration may be found from

$$\frac{F}{F_1} = \frac{OC}{OA} = \frac{OC}{R},$$

$$\therefore\ F = F_1 \frac{OC}{R} = \frac{mv^2}{R} \frac{OC}{R}$$

$$= \frac{mv^2}{R^2} \cdot OC \text{ poundals} \quad \ldots\ldots\ldots\ldots (2)$$

Suppose OC to be one foot, and that μ represents the value of the force required when the body is at this distance from O, then

$$\mu = \frac{mv^2}{R^2} \text{ poundals} \quad \ldots\ldots\ldots (3)$$

The time of one complete vibration from A to B and back to A is given by (p. 400)

$$T = 2\pi \frac{R}{v}.$$

From (3),

$$\frac{R^2}{v^2} = \frac{m}{\mu},$$

$$\therefore\ \frac{R}{v} = \sqrt{\frac{m}{\mu}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Substitution of this value gives

$$\mathbf{T = 2\pi\sqrt{\frac{m}{\mu}}\ \text{seconds.}} \quad \ldots\ldots\ldots\ldots (5)$$

EXAMPLE. A body of mass 2 pounds executes simple harmonic vibrations. When at a distance of 3 inches from the centre of the vibration, a force of 0·4 lb. weight is acting on it. Find the time of vibration.

The force required at a distance of one foot from the centre will be four times that required at 3 inches. Hence,

$$\mu = 0{\cdot}4 \times 4 = 1{\cdot}6 \text{ lb. weight}$$

$$= 1{\cdot}6\,g \text{ poundals.}$$

Hence,

$$T = 2\pi\sqrt{\frac{m}{\mu}} = \frac{2 \times 22}{7}\sqrt{\frac{2}{1{\cdot}6 \times 32{\cdot}2}}$$

$$= \underline{1{\cdot}238} \text{ seconds.}$$

Simple harmonic torsional oscillations. A body will execute **simple harmonic torsional oscillations** if it is under the influence of a couple which varies as the angle described by the body from the mean position, the couple having a sense of rotation always tending to restore the body to the mean position. Thus, a body secured to the lower end of a vertical wire, the upper end of which is fixed rigidly, will hang, when at rest, in a position which may be described as the mean position. As has been explained on pp. 255 and 295, if the wire be twisted by rotation of the body, it will exert a couple which will be proportional to the angle of twist; this couple will be constantly endeavouring to restore the body to the mean position, hence the body will describe simple harmonic torsional oscillations. The time of vibration may be deduced by analogy from equation (5), p. 437, showing the time of simple harmonic rectilinear vibrations; the moment of inertia of the body about the wire axis must be substituted for m, and the couple acting at unit angle (one radian) from the mean position must be substituted for μ. Thus

Let M = the mass of the body, in pounds.

k = its radius of gyration about the axis of vibration, in feet.

$I = Mk^2$ = the moment of inertia about the same axis, in pound and foot units.

λ = the couple acting at one radian displacement from the mean position, in poundal-feet.

T = the time elapsing between successive passages of the body through the same position.

Then,

$$\mathbf{T = 2\pi\sqrt{\frac{I}{\lambda}}}$$

$$\mathbf{= 2\pi\sqrt{\frac{Mk^2}{\lambda}}\ \text{seconds.}}$$

EXAMPLE. A flywheel having a mass of 1000 pounds and a radius of gyration of 2 feet, is fixed to the end of a shaft 4 feet long and 3 inches in diameter. It has been found from a separate calculation that the shaft has an angle of twist of 0·0005 radian when a torque of 1000 lb.-inches is applied. Find the time of a free torsional oscillation. Take $g = 32{\cdot}2$.

The term "free" indicates that the frictional effects of the bearings and of the atmosphere are to be disregarded.

$$I = Mk^2$$
$$= 1000 \times 2 \times 2 = 4000 \text{ pound and foot units.}$$

The angle of twist is proportional to the torque, and if this were true up to one radian, we have

$$\frac{\text{Torque at one radian}}{1000} = \frac{1}{0{\cdot}0005},$$

$$\therefore \text{ Torque at one radian} = \frac{1000}{0{\cdot}0005}$$

$$= 2{,}000{,}000 \text{ lb.-inches;}$$

$$\lambda = \frac{2{,}000{,}000 \times g}{12}$$

$$= 5{,}367{,}000 \text{ poundal-feet.}$$

Hence,

$$T = 2\pi\sqrt{\frac{I}{\lambda}}$$

$$= \frac{44}{7}\sqrt{\frac{4000}{5{,}367{,}000}}$$

$$= \underline{0{\cdot}1715} \text{ second.}$$

Suppose n to be the number of torsional oscillations per minute; then

$$n = \frac{60}{0{\cdot}1715}$$

$$= \underline{350}.$$

If this shaft were driven by means of an engine connected to a crank fixed to the shaft at the end remote from the flywheel, and if the shaft were to have a speed of 350 revolutions per minute, the engine would be delivering impulses to the shaft which would keep time with the free oscillations of the shaft. In these circumstances, the angle of oscillation would rapidly increase in magnitude. As the stress in the shaft is proportional to the angle of twist, a very large stress would be produced and the shaft would be in danger of breaking. A somewhat higher or lower speed of revolution is necessary in order to avoid these effects; in no case should the impulses given to the shaft synchronise with the free torsional oscillations.

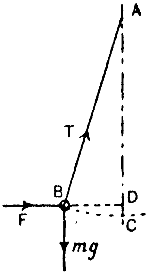

FIG. 478.—A simple pendulum.

The simple pendulum. A simple pendulum may be realised by suspending a small heavy body at the end of a very light thread and allowing it to vibrate through small angles under the action of gravity. In Fig. 478 the body at B is under the action of its weight mg and the pull T of the thread. The resultant of these forces is F, a force which is urging the body towards the vertical. The triangle of forces will be ABD, and we have

$$\frac{F}{mg}=\frac{BD}{AD},$$

or

$$F=mg\frac{BD}{AD}.$$

Now, if the angle BAD is kept very small, AC and AD will be very nearly equal. Let L be the length of the thread in feet; then

$$F=mg\frac{BD}{AC}=\frac{mg}{L}\cdot BD. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Hence we may say that F is proportional to BD. For very small angles of swing BD and BC coincide practically, therefore the body will execute simple harmonic vibrations under the action of a force F which varies as the distance from the vertical through A. To obtain the value of μ, the force at unit distance, make BD equal to one foot in (1); then

$$\mu=\frac{mg}{L} \text{ poundals.}$$

Now,

$$T=2\pi\sqrt{\frac{m}{\mu}} \quad \text{(p. 437)}$$

$$=2\pi\sqrt{\frac{m}{\frac{mg}{L}}}$$

$$=2\pi\sqrt{\frac{L}{g}}. \quad \ldots\ldots\ldots\ldots(2)$$

EXAMPLE. Find the time of vibration of a simple pendulum of length 4 feet at a place where g is 32 feet per second per second.

$$T=\frac{2\times 22}{7}\sqrt{\frac{4}{32}}$$

$$=\underline{2\cdot 222} \text{ seconds.}$$

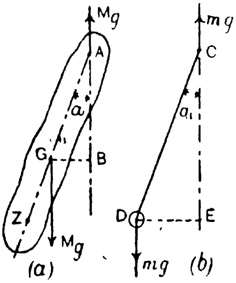

FIG. 479.—A compound pendulum and an equivalent simple pendulum.

The compound pendulum. Any body vibrating about an axis under the action of gravity and having dimensions which do not comply with those required for a simple pendulum may be called a compound pendulum. In Fig. 479 (*a*) is shown a compound pendulum consisting of a body vibrating about A. G is the centre of mass of the body, and the line AG makes an angle α with the vertical passing through A in the position under consideration. In Fig. 479 (*b*) is shown a simple pendulum CD vibrating about C: at the instant considered CD makes the same angle α with the vertical passing through C. Both pendulums will execute small vibrations in the same time provided that their angular accelerations in the given positions are equal.

Considering the compound pendulum,

Let M = its mass, in pounds.

Y = the distance AG, in feet.

$I_A = Mk_A^2$ = its moment of inertia about A, in pound and foot units.

θ_1 = the angular acceleration in radians per sec. per sec. in the given position

Then
$$\theta_1 = \frac{\text{couple applied}}{I_A} = \frac{Mg \times GB}{Mk_A^2}$$
$$= \frac{g \times GA \sin\alpha}{k_A^2} \qquad \ldots \qquad (1)$$

Considering now the simple pendulum,

Let m = its mass, in pounds.

L = its length, in feet.

$I_C = mL^2$ = its moment of inertia about C, in pound and foot units.

θ_2 = its angular acceleration in radians per sec. per sec. in the given position.

Then
$$\theta_2 = \frac{\text{couple applied}}{I_C}$$
$$= \frac{mg \times DE}{mL^2} = \frac{g \times DC \sin\alpha}{L^2}$$
$$= \frac{g \sin\alpha}{L} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2)$$

To comply with the required conditions, we have

$$\theta_1 = \theta_2;$$
$$\therefore \frac{g \times GA \sin\alpha}{k_A^2} = \frac{g \sin\alpha}{L},$$

or
$$\frac{Y}{k_A^2} = \frac{1}{L};$$
$$\therefore L = \frac{k_A^2}{Y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

The length L of the corresponding simple pendulum may be calculated from this result, and hence the time of vibration of both pendulums may be found. If AG be produced to Z (Fig. 479 (*a*)), making AZ equal to L, the point so found is called the centre of oscillation. The **centre of oscillation** may be defined as the point at which the whole mass of a compound pendulum may be concentrated without thereby altering the time of vibration.

Centre of percussion. If a body is capable of rotating freely about a fixed axis, it will be found that, in general, a blow delivered to the body will produce an impulse on the axis. There is, however, one point in the body at which a blow will produce no impulse on the axis; this point is called the **centre of percussion**.

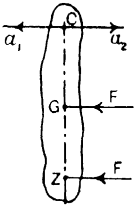

FIG. 480.—Centre of percussion.

In Fig. 480, C is the axis about which the body may turn freely and G is the centre of mass. Let an impulse F be delivered to the body at a point Z. The effects of F may be examined by transferring F to the centre of mass, applying at the same time a clockwise couple of moment F × GZ. The force F acting at G will produce pure translation, and if the mass of the body is M pounds, every point in it will have an acceleration a_1 feet per second per second, found from

$$F = Ma_1,$$

or

$$a_1 = \frac{F}{M}. \qquad (1)$$

In particular, C will have this acceleration a_1 towards the left.

Further, the couple F × GZ will produce a clockwise angular acceleration θ, found from

$$\theta = \frac{F \times GZ}{I_G},$$

where I_G is the moment of inertia of the body about an axis passing through G and parallel to the axis at C. Writing Mk_G^2 for this moment of inertia, we have

$$\theta = \frac{F \times GZ}{Mk_G^2}.$$

As a consequence of this angular acceleration, C will have a linear acceleration a_2 feet per second per second towards the right, to be found from

$$a_2 = \theta \times CG$$

$$= \frac{F \times GZ \times CG}{Mk_G^2}. \qquad \ldots\ldots(2)$$

If there is to be no impulse on the axis at C, there must be equality of a_1 and a_2. Hence,

$$\frac{F}{M} = \frac{F \times GZ \times CG}{Mk_G^2},$$

or

$$1 = \frac{GZ \times CG}{k_G^2},$$

$$\therefore\ k_G^2 = GZ \times CG. \qquad (3)$$

Also,

$$I_C = I_G + M\,CG^2, \quad (p.\ 417.)$$

or

$$Mk_C^2 = Mk_G^2 + M\,.\,CG^2.$$

$$\therefore\ k_G^2 = k_C^2 - CG^2. \qquad (4)$$

Also,

$$GZ = CZ - CG. \qquad (5)$$

Substituting these values in (3) gives

$$k_C^2 - CG^2 = (CZ - CG)CG,$$

or

$$k_C^2 = CZ \times CG;$$

$$\therefore\ CZ = \frac{k_C^2}{CG}. \qquad (6)$$

Comparison of this result with that found for the position of the centre of oscillation (p. 441, equation (3)), indicates that the centre of percussion of a body coincides with the centre of oscillation.

Reduction of a given body to an equivalent dynamical system. It is often convenient to substitute for a given body two separate bodies connected by means of an imaginary rigid rod, and arranged in such a way that the substituted bodies behave under the action of any force or forces in exactly the same manner as the given body. In Fig. 481 (*a*) is shown a body of mass M, and having its centre of mass at G; Fig. 481 (*b*) shows an equivalent system, consisting of two bodies at A′ and B′, having masses m_1 and m_2 respectively, and having their centre of mass at G′. A and B in Fig. 481 (*a*) correspond with A′ and B′. The conditions of equivalence may be stated as follows:

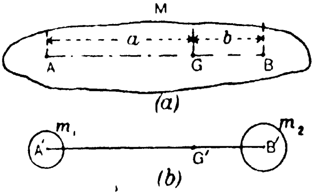

FIG. 481.—Equivalent dynamical system.

(1) The mass M must be equal to the sum of m_1 and m_2, and the points G and G′ must divide AB and A′B′ respectively in the same proportion. A force applied at G or at G′ will then produce pure

translation with equal accelerations in the given body or in the substituted system. Hence,

$$m_1 + m_2 = M, \quad \ldots \ldots \ldots (1)$$

$$m_1 \times AG = m_2 \times BG,$$

or

$$m_1 a = m_2 b. \quad \ldots \ldots \ldots (2)$$

(2) The moment of inertia of the given body about any axis passing through its centre of mass must be the same as the moment of inertia of the substituted system about a similarly situated axis. This condition ensures that the given body and the substituted system shall possess equal angular accelerations when acted on by equal couples. Hence,

$$m_1 a^2 + m_2 b^2 = Mk_G^2. \quad \ldots \ldots \ldots (3)$$

These equations may be reduced as follows:

From (2), $m_1 = \dfrac{b}{a} m_2$, $m_2 = \dfrac{a}{b} m_1$.

Substituting these in (1) gives.

$$m_1 + \frac{a}{b} m_1 = M;$$

$$\therefore \; m_1 = \frac{M}{1 + \dfrac{a}{b}} = \frac{Mb}{a+b}. \quad \ldots \ldots \ldots (4)$$

Also,

$$\frac{b}{a} m_2 + m_2 = M;$$

$$\therefore \; m_2 = \frac{M}{1 + \dfrac{b}{a}} = \frac{Ma}{a+b}. \quad \ldots \ldots \ldots (5)$$

Inserting these values in (3), we have

$$\frac{Mba^2}{a+b} + \frac{Mab^2}{a+b} = Mk_G^2,$$

or

$$\frac{a^2b + ab^2}{a+b} = k_G^2,$$

$$\frac{ab(a+b)}{a+b} = k_G^2;$$

$$\therefore \; ab = k_G^2. \quad \ldots \ldots \ldots (6)$$

The required equivalent system may thus be obtained by first selecting *a*. *b* may then be calculated from (6), having first determined the value of k_G^2. m_1 and m_2 may now be calculated from (4) and (5).

Example. A connecting rod (Fig. 482) 4 feet long has its centre of mass G at 2·8 feet from the small end. The mass of the rod is 200 pounds, and an equivalent system is required in which one of the two masses is to be situated at the small end. k_G^2 is 2 in foot units. Find the system.

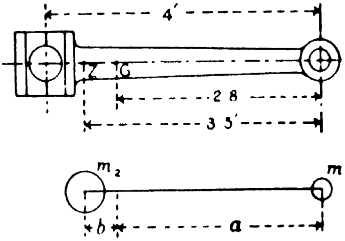

Fig. 482.—Equivalent dynamical system for a connecting rod.

Here a is 2·8 feet; hence, from (6),

$$ab = k_G^2 = 2,$$

$$b = \frac{2}{2\cdot8} = \underline{0\cdot714} \text{ foot.}$$

From (4),

$$m_1 = \frac{Mb}{a+b}$$

$$= \frac{200 \times 0\cdot714}{2\cdot8 + 0\cdot714} = \frac{200 \times 0\cdot714}{3\cdot514}$$

$$= \underline{40\cdot6} \text{ pounds.}$$

From (5),

$$m_2 = \frac{Ma}{a+b}$$

$$= \frac{200 \times 2\cdot8}{3\cdot514}$$

$$= \underline{159\cdot4} \text{ pounds.}$$

If it is desired to give the body shown in Fig. 481 (a) any assigned motion, the forces required may be obtained as follows: Find the linear accelerations at A and B, both in direction and magnitude; let these be a_1 and a_2 respectively. In Fig. 481 (b), showing the equivalent system, m_1 and m_2 will have accelerations a_1 and a_2 respectively, and forces will be required acting in the lines of these accelerations, and given by

$$P_1 = m_1a_1,$$
$$P_2 = m_2a_2,$$

both in absolute units. The same forces applied at A and B respectively in Fig. 481 (a) will give the proposed motion to the body.

EXPT. 43. The law F = *ma* may be verified roughly by means of the apparatus illustrated in Fig. 483. A and B are two similar scale pans connected to a fine cord C which passes over two aluminium pulleys D and E. A cord C_1, of the same kind as C, is attached to the bottom of each pan, and compensates for the extra weight of cord on the B side of the pulleys. A fall of about 10 feet should be arranged for the scale pans.

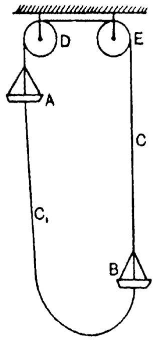

FIG. 483 Apparatus for verifying the law F = *ma*.

Place equal masses in the scale pans, and find by trial what additional mass placed in A will cause it to descend with uniform velocity when given a start. Any additional mass placed in A will now give A an acceleration downwards and B an equal acceleration upwards. Let A have a total fall of H feet, and make several experiments without changing the masses, noting the time in seconds for each descent by means of a stop-watch. Take the average time *t* seconds and calculate the acceleration a_1 from

$$H = \tfrac{1}{2}a_1 t^2,$$

$$a_1 = \frac{2H}{t^2} \text{ feet per sec per sec.} \quad \ldots (1)$$

The acceleration should also be calculated as follows.

Let M_s = mass of each scale pan, pounds

M_w = the equal masses added, pounds.

M_ϵ = the additional mass in A required to secure uniform velocity, pounds.

M_a = mass added to A for the purpose of producing acceleration, pounds.

The total mass to which acceleration has been given, neglecting the cord and pulleys, is

$$\begin{aligned} M &= 2M_s + 2M_w + M_\epsilon + M_a, \\ &= 2(M_s + M_w) + M_\epsilon + M_a. \end{aligned}$$

The force F which has produced the acceleration is the weight of M_a, *i.e.* $M_a g$ in poundals. Hence,

$$F = Ma_2,$$

or

$$M_a g = \{2(M_s + M_w) + M_\epsilon + M_a\}a_2,$$

$$a_2 = \frac{M_a g}{2(M_s + M_w) + M_\epsilon + M_a} \text{ feet per sec. per sec.} \ldots \ldots (2)$$

This value should agree fairly well with that experimentally found and given by a_1 in equation (1).

Repeat the experiment two or three times with different masses.

EXPT. 44.—**To find the moment of inertia of a small flywheel by the method of a falling load.** The apparatus used consists of a small flywheel (Fig. 458, p. 418) having a drum on its shaft and capable of being rotated by means of a cord wrapped round the drum, and having a scale pan containing a load attached to its end. The cord is attached to the drum in such a manner that it drops off when the scale pan reaches the floor.

Allow the scale pan to descend slowly through a measured height, and note the number of revolutions made by the wheel during this operation. Wind up the scale pan to the same height, place a load in it, then allow the wheel to start unaided, at the same moment starting a stop watch. Stop the watch at the instant the scale pan reaches the floor, and note the time of descent. Allow the wheel to go on revolving until friction brings it to rest, and note the total number of revolutions which it makes from start to stop.

Let m_1 = the mass of the scale pan, in pounds.
m_2 = the mass placed in the scale pan, in pounds.
$M = m_1 + m_2$ = the total falling mass, in pounds.
H = the height of fall of the scale pan, in feet.
t = the time of fall, in seconds.
N_1 = number of revolutions made by the wheel during the fall.
N_2 = the total number of revolutions from start to stop.

The total work done by gravity will be MgH foot poundals, and, up to the instant that the scale pan is on the point of touching the floor, this work has been expended as follows: (*a*) in giving kinetic energy to the falling mass M; (*b*) in overcoming frictional resistances; (*c*) in giving kinetic energy to the wheel. If v be the velocity of M when the scale pan arrives at the floor, the average velocity of descent will be $\frac{1}{2}v$ feet per second. Hence,

$$\tfrac{1}{2}vt = H,$$

$$\therefore v = \frac{2H}{t} \text{ feet per second.}$$

$$\therefore \text{ Kinetic energy acquired by } M = \frac{Mv^2}{2} = \frac{M}{2} \cdot \frac{4H^2}{t^2}$$

$$= \frac{2MH^2}{t^2} \text{ foot poundals.}$$

The difference between MgH and the kinetic energy acquired by the falling mass M represents the energy reaching the drum, and is expended in overcoming friction and in giving kinetic energy to the wheel.

$$\text{Energy reaching the drum} = MgH - \frac{2MH^2}{t^2}$$

$$= MH\left(g - \frac{2H}{t^2}\right) \text{ foot-poundals.}$$

Ultimately, the whole of this energy is dissipated in overcoming frictional resistances throughout the entire motion of the wheel, *i.e.* in N_2 revolutions. Assuming that the frictional waste per revolution is constant, we have

$$\text{Energy wasted per revolution} = MH\left(g - \frac{2H}{t^2}\right) \div N_2,$$

$$\text{Energy wasted while M is falling} = MH\left(g - \frac{2H}{t^2}\right)\frac{N_1}{N_2} \text{ foot-poundals.}$$

Let E = the kinetic energy possessed by the wheel at the instant the scale pan reaches the floor.

Then
$$E = MH\left(g - \frac{2H}{t^2}\right) - MH\left(g - \frac{2H}{t^2}\right)\frac{N_1}{N_2}$$
$$= MH\left(g - \frac{2H}{t^2}\right)\left(1 - \frac{N_1}{N_2}\right) \text{ foot-poundals}$$
$$= MH\left(1 - \frac{2H}{gt^2}\right)\left(1 - \frac{N_1}{N_2}\right) \text{ foot-lb.}$$

The angular velocity of the wheel at the instant the scale pan reaches the floor may be calculated as follows:

Revolutions described in t seconds $= N_1$
$= \text{average revs. per sec.} \times t$;

$\therefore$ Average revolutions per sec. $= \frac{N_1}{t}$.

And, maximum revolutions per sec. $= \frac{2N_1}{t}$;

$$\therefore \text{ Maximum angular velocity of wheel} = \omega = \frac{2N_1}{t} \cdot 2\pi$$
$$= \frac{4\pi N_1}{t} \text{ radians per sec.}$$

Now,

$$\text{Maximum kinetic energy of the wheel} = \frac{\omega^2}{2g} I = E \text{ foot-lb.};$$

$$\therefore I = \frac{2Eg}{\omega^2}$$
$$= \frac{2MHg\left(1 - \frac{2H}{gt^2}\right)\left(1 - \frac{N_1}{N_2}\right)t^2}{16\pi^2 N_1^2}$$
$$= \frac{MH}{8\pi^2 N_1^2}(gt^2 - 2H)\left(1 - \frac{N_1}{N_2}\right) \text{ pound and foot units.}$$

The experiment should be repeated several times with different masses m_2 and with different heights of fall H; the values of I should be calculated for each experiment and the mean value taken.

EXPT. 45 — **To find the centre of oscillation, or the centre of percussion, of a given body.** A connecting rod has been selected as a useful example (Fig. 484). The rod AB is suspended from a knife edge consisting of a square bar of tool steel CD, passing through the hole in the small end and resting on V blocks at E and E. The rod can vibrate now in the same plane as that in which it will vibrate when built into the engine. GH is a simple pendulum consisting of a small heavy bob and a light cord. Cause both rod and simple pendulum to execute small vibrations, starting both together at the end of a swing. Adjust the length of GH until both vibrate in the same time. Measure GH and mark a point on the connecting rod at this length from its axis of vibration. This will give the centre of oscillation or percussion when the rod is vibrating about the upper edge of the tool steel bar.

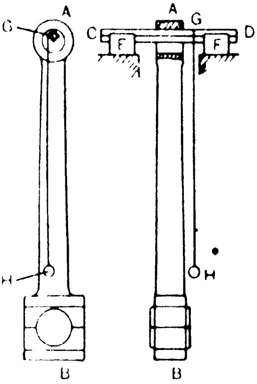

FIG. 484. Centre of oscillation by experiment.

EXPT. 46. Take a uniform bar of metal about 3 feet long and of section about 1 inch by ½ inch. Referring to p. 443, it will be seen that the centre of percussion Z for this bar will be at a distance from C given by

$$CZ = \frac{k_C^2}{CG}.$$

Let L be the length of the bar. Then

$$CG = \tfrac{1}{2}L,$$

$$k_C^2 = \tfrac{1}{3}L^2,$$

$$\therefore CZ = \tfrac{2}{3}L.$$

Mark clearly the position of Z on the bar, allow the bar to hang vertically, using a finger and thumb at C. Use another short bar and strike the bar sharply at different points. The absence of any jar on the fingers when the bar is struck at Z will be observed readily, and gives confirmation of the calculated position of Z.

EXPT. 47. — **To find the radius of gyration of a given body about an axis passing through its centre of mass.** In Fig. 485 is shown a flywheel arranged in the same manner as the connecting rod in Fig. 484. Find the length of the corresponding simple pendulum as directed previously, being careful to cause the flywheel to vibrate in the same plane as that in which it will rotate subsequently. Measure BK, the distance from the axis of vibration to the centre of mass of the wheel. Weigh the wheel in order to estimate its mass.

Taking equation (3), p. 441, and applying it to the present case, we have

$$L = \frac{k_B^2}{Y},$$

where $L = GH$, in feet;

$Y = BK$, in feet;

$k_B =$ the radius of gyration about B, in feet.

Hence, $k_B^2 = LY$.

Now, $I_K = I_B - MY^2$,

or $Mk_K^2 = Mk_B^2 - MY^2$;

$$\therefore\ k_K^2 = k_B^2 - Y^2 = LY - Y^2;$$

$$\therefore\ k_K = \sqrt{Y(L - Y)} \text{ feet.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

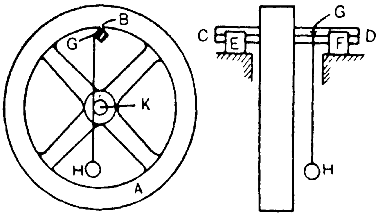

FIG. 485.—Radius of gyration of a flywheel by experiment.

The moment of inertia of the wheel about its axis of rotation will be

$$I_K = Mk_K^2$$
$$= MY(L - Y) \text{ pound and foot units.} \quad . \ (2)$$

This experiment therefore provides a means of finding the data required for estimating the kinetic energy and the rotational inertia of a given flywheel.

EXPT. 48.—**To find the velocity acquired by a wheel in rolling down an incline.** In Fig. 486 is shown a long incline AB consisting of two angle bars with a gap between them. The angle bars are pivoted to a bracket at A, and a prop at F enables the angle of inclination to be altered.

The wheel D has a spindle projecting on each side of the wheel, and has a collar E on each side secured by a nut to the spindle. The collars are coned slightly for the purpose of keeping the wheel centrally in the gap as it rolls down and to prevent the wheel from rubbing on the angles. A fixed stop is fitted at C. The object of the arrangement is to increase the time taken in rolling down the incline.

First determine the square of the radius of gyration of the wheel and its attachments about the axis of the spindle, by the method explained in the last experiment. Let this be k^2 in foot units.

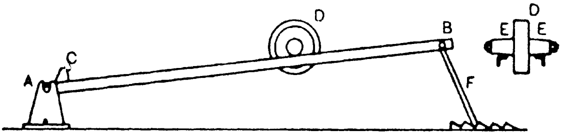

FIG. 486.—Apparatus for investigating the motion of a wheel rolling down an incline.

Set the incline to a suitable angle by means of the prop. Measure the difference in level between the centre of the wheel spindle when in the starting position and when in the stopping position; let this be H feet. Measure also the distance travelled, parallel to the incline, by the wheel centre; let this be L feet. Let the wheel start unaided, and note the time taken in rolling down; let this be t seconds. Let the linear velocity of the wheel centre at the instant of arriving at the bottom be v feet per second. Then

$$\begin{aligned} L &= \text{the average velocity} \times t \\ &= \tfrac{1}{2}vt; \end{aligned}$$

$$\therefore\ v = \frac{2L}{t} \text{ feet per second.} \qquad \ldots\ldots \quad (1)$$

Taking equation (4), p. 422, and writing r instead of R, where r is the mean radius of the collars E in feet,

$$v = \sqrt{\frac{2gH}{1 + \dfrac{k^2}{r^2}}} \text{ feet per second.} \quad \ldots\ldots \quad (2)$$

(1) and (2) are expressions for the velocity found by entirely independent methods, and the results obtained from them should agree. Give the results for v by both methods; repeat the experiment, using different angles of inclination and collars having a different diameter.

EXERCISES ON CHAPTER XVIII.

1. A wheel has a moment of inertia of 24,000 in pound and foot units, and runs at 90 revolutions per minute. Find its moment of momentum. Suppose that the speed changes to 88 revolutions per minute in 0·5 second, what couple must have acted?

2. A wheel has a moment of inertia of 20 in pound and foot units, and has a speed of 90 revolutions per minute; the plane of revolution is vertical. The wheel is mounted so that its axis is capable of turning in a horizontal plane (*i.e.* in azimuth). The axis is found to have an angular velocity of $\frac{\pi}{10}$ radian per second in azimuth. Calculate the couple acting. Show the couple and the directions of both angular velocities clearly in a diagram.

3. A cycle wheel has a mass of 5 pounds and its radius of gyration is one foot. It is suspended as shown in Fig. 470, the distance between the suspending cord and the mass centre being 2·5 inches. The wheel is spun and revolves with its plane vertical 120 times per minute Find the angular velocity in azimuth.

4. A body having a mass of 12 pounds vibrates in a straight line 18 inches long with simple harmonic motion. The time of one complete vibration is 0·25 second. Find what force must act on it at the end of each stroke and the velocity at the middle of the stroke

5. A small wheel having a moment of inertia of 0·4 in pound and foot units has its plane horizontal and is attached firmly at its centre to a vertical steel wire, the top end of which is fixed to a rigid bracket The wheel can execute torsional oscillations under the control of the wire The wire is 0·06 inch in diameter and 36 inches long, and its modulus of rigidity is known to be 11,000,000 lb per square inch Find the time of one complete oscillation

6. A thin disc 24 inches in diameter can execute small vibrations under the influence of gravity about a horizontal axis at 90 degrees to the plane of the disc and bisecting a radius Find the length of the equivalent simple pendulum and the time of one complete vibration.

7 A thin uniform steel rod 3 feet long hangs freely from its top end. Find the centre of percussion.

8 A uniform bar of mild steel, section 2 inches by 1 inch, 4 feet long, has masses of 4 and 2 pounds attached at distances of 1 foot and 3·5 feet respectively from one end Take the density of the bar to be 0·28 pound per cubic inch. Find the mass centre and the moment of inertia about an axis at 90 degrees to the flat face of the bar and passing through the mass centre.

9. Take the system given in Question 8 and reduce it to an equivalent dynamic system having a mass situated at the end of the bar adjacent to the given 2 pound mass

10 Take the equivalent dynamic system found in answer to Question 9. A force of 100 lb weight is applied at 90 degrees to the bar (*a*) at the mass centre, (*b*) at 3 inches from the mass centre. Find, in each case, the translational acceleration of the mass centre and the angular acceleration, if any, of the bar.

11. Explain what is meant by moment of momentum Calculate the moment of momentum of a body weighing 300 lb. rotating at 1250 revolutions per minute, the radius of gyration of the body about the axis of rotation being 1·7 foot. What property is measured by rate of change of moment of momentum? (I.C.E.)

12. A body of 40 pounds hangs from a spiral spring, which it elongates 2·5 inches. The body is then pulled down a short distance and let go. Determine the number of complete oscillations the body will make per minute, assuming that the weight of the spring is 20 lb. (B.E.)

13 A body weighing 161 lb has a simple harmonic motion, the total length of one swing being 2 feet, the periodic time is 1 second. Make a diagram showing its velocity and another showing its acceleration at every point of its path What force is giving to the body this motion? What is its greatest value? (B E.)

14. A heavy circular disc is supported on a shaft 3 inches in diameter, carried on roller bearings; a cord is wrapped round the shaft. It is found by experiment that a weight of 6 lb suspended from this cord is just sufficient to overcome the friction of the roller bearings and maintain a uniform speed of rotation of the disc When a weight of 30 lb is suspended from the cord, it is found that this weight descends vertically 14 feet in 2 seconds of time Determine the moment of inertia of the disc in pound-foot^2 units Neglect the inertia of shaft and cord, and assume that the speed of rotation of the disc increases at a uniform rate in the second experiment. (B E.)

15 Obtain the magnitude and position of the single force which when applied perpendicularly to the axis of a uniform bar (48 inches long, weighing 200 lb) will give it a translational acceleration of 40 feet per second per second and a rotational acceleration of 10 radians per second per second (I C.E.)

16. In a hoisting gear a load of 300 lb is attached to a rope wound round a drum, the diameter to the centre of the rope being 4 feet. A brake drum is attached to the rope drum and fitted with a band brake The combined weight of the two drums is 720 lb, and the radius of gyration of the two together is 20 inches The weight starts from rest and attains a speed of 10 feet per second The brake is then applied and the speed is maintained constant until the load reaches 20 feet from the bottom, when the brake is tightened so as to give uniform retardation until the load comes to rest The total descent is 100 feet, find the time taken for the descent and the tension in the rope during slowing (I.C.E.)

17. Show that the natural period of vertical oscillation of a load supported by a spring is the same as the period of a simple pendulum whose length is equal to the static deflection of the spring due to the load. When a carriage underframe and body are mounted on the springs, these are observed to deflect 1½ inch Calculate the time of a vertical oscillation. (I.C.E.)

18. Show that a body having plane motion may be represented by two masses supposed concentrated at points. A rocking lever (mass 600 pounds) has a radius of gyration about its centre of gravity of 18 inches, and the centre of gravity is distant 6 inches from the axis round which the lever rocks Find the magnitude of the equivalent masses if one is supposed to be concentrated at the axis, and find also the distance of the other mass from the axis. Find the torque required to give the lever an acceleration of 10 radians per second per second (L.U.)

19 The revolving parts of a motor car engine rotate clockwise when looked at from the front of the car, and have a moment of inertia of 400 in pound and foot units The car is being steered in a circular path of 400 feet radius at 12 miles per hour, and the engine runs at 800 revolutions per minute. (*a*) What are the effects on the steering and driving axles due to gyroscopic action? The distance between these axles is 8 feet. (*b*) Suppose the car to be turned and driven in the reverse direction over the same curve at the same speed, what will be the effects on the axles? (L.U.)

20. A hollow circular cylinder, of mass M, can rotate freely about an external generator, which is horizontal. Its cross-section consists of concentric circles of radii 3 and 5 feet. Show that its moment of inertia about the fixed generator is 42 M units, and find the least angular velocity with which the cylinder must be started when it is in equilibrium, so that it may just make a complete revolution. (L.U.)

CHAPTER XIX.

LINK MECHANISMS.

Link mechanisms. **Links** are used for transmitting motion from one point to another in a mechanism. In any complete mechanism containing links, usually each part is constrained so as to move always over the same path in the same definite manner; the whole may then be defined as a **kinematic chain.** The slider-crank-chain is a well known example of complete restraint (Fig. 487), here the crank CB revolves about an axis C and forms one link in the chain, the connecting rod AB is connected to the crank at B, hence this end of the rod revolves about the centre C; its other end A is constrained by the sliding block D and slotted frame E so as to move always in a straight line.

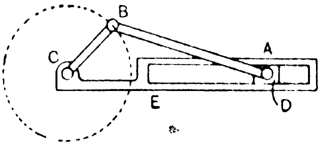

Fig. 487. Slider crank chain.

Fig. 488 (a) shows a case of incomplete restraint, there are two cranks AB and CD, capable of rotation about A and C respectively, and connected by two links BE and DE, jointed at E. It is impossible to make any calculations in a case such as this. Complete restraint may be secured by having a block at the joint E and guiding it to move in a definite line (Fig. 488 (b)), the addition of another crank GF and a connecting rod FE will secure definite motion for every

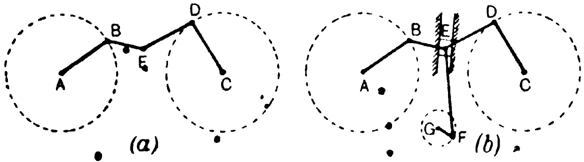

Fig. 488.—Examples of incompletely and completely restrained mechanisms.

part of the mechanism In cases of complete restraint, problems regarding the path, velocity and acceleration of any point may be solved, and calculations made regarding the effects of inertia in producing stresses in the parts and in modifying the forces given to the mechanism by outside agencies.

The path of any point in a mechanism is found best by drawing the mechanism in several different positions and marking in each the position of the point under consideration; a fair curve may be drawn through these points and will give the desired path. The path of a point D in the connecting rod of a slider-crank-chain is shown in Fig. 489 as an illustration of the method A simple method of obtaining velocity and acceleration diagrams has been given in Chapter XVI.; some special methods will now be examined.

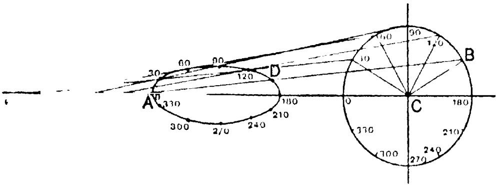

FIG. 489 - Path of a point in a connecting rod

Velocity of any point in a rotating body. In Fig 490 is shown a body rotating about an axis at C which is perpendicular to the plane of the paper. The direction of the velocity of any point, such as A or B, will be perpendicular to the radius. To calculate the velocity of B, if the velocity of A is given, let the body make one revolution, then

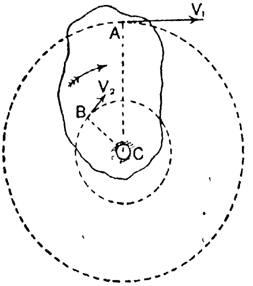

FIG. 490 - Velocities of points in a rotating body

Distance travelled by $A = 2\pi . CA$.

Distance travelled by $B = 2\pi . CB$.

As these distances are travelled in the same time, we have

$$\frac{V_2}{V_1} = \frac{2\pi . CB}{2\pi . CA}$$

$$V_2 = V_1 \frac{CB}{CA}.$$

This result shows that **the velocities of different points in a body having motion of rotation only are proportional to the radii.**

Possible velocities in a link. Let AB be a rigid rod or link (Fig. 491), and let A have a velocity V_A at a given instant. V_A will have components $V_A \cos \alpha$ and $V_A \sin \alpha$ along and perpendicular to the rod respectively. Let B have a velocity V_B at the same instant, the components of V_B in the same directions will be $V_B \cos \beta$ and $V_B \sin \beta$. As the rod is rigid, *i.e.* cannot bend or alter its length, it follows that $V_B \cos \beta$ and $V_A \cos \alpha$ must be equal, otherwise the rod is becoming shorter or longer. The other component of the velocity of B may be of any magnitude and of either sense along BY. The result may be expressed by saying that **the velocity of B relative to A, or of A relative to B must be perpendicular to the line AB**.

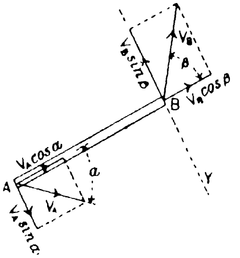

FIG. 491.—Possible velocities of the ends of a link.

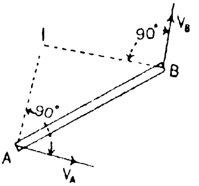

FIG. 492.—Instantaneous centre of a link.

Instantaneous centre. The relations of the velocities of the ends of the rod AB may be examined by the following method, which is suitable for graphical solutions. Reference is made to Fig. 492. Here the velocity of A is along V_A, but for an instant it might be imagined that A is rotating about any centre in AI which is perpendicular to V_A; this will not alter the direction of the velocity, which will still be along V_A. In the same way, we may imagine that B is rotating about any centre in BI for an instant, BI being perpendicular to V_B. Hence I, the point of intersection of AI and BI may be looked upon as a centre about which both A and B are rotating for an instant, and is called the **instantaneous centre**. We have, therefore,

$$\frac{V_A}{V_B} = \frac{AI}{BI}.$$

The application of this method to a **crank and connecting rod** is shown in Fig. 493 (*a*). Given the velocity of B, equal to V_B, to find the velocity of A draw AI perpendicular to V_A, *i.e.* to AC, and also produce CB, which is perpendicular to V_B, to cut AI in I. Then

$$\frac{V_A}{V_B}=\frac{IA}{IB}.$$

A more convenient construction is to produce, if necessary, the line of the connecting rod AB to cut CN in Z. The triangles IAB and CBZ are similar. Hence,

$$\frac{CZ}{CB}=\frac{IA}{IB}=\frac{V_A}{V_B},$$

or

$$CZ=\frac{CB}{V_B}.V_A.$$

If the crank is rotating with uniform angular velocity, V_B will be constant and CZ may be taken to represent the velocity of A to a scale in which V_B is represented by CB, the length of the crank. It is evident that V_A is zero when the crank pin is at either L or R;

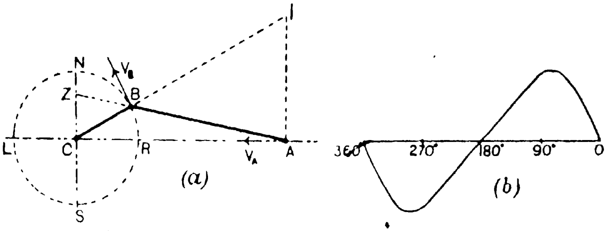

Fig. 493.—Velocity diagram for the point A, deduced by the instantaneous centre method.

also, when the crank is at 90° to LR, Z coincides with N, and CZ will be equal to the crank, therefore V_A and V_B will be equal. Fig. 493 (*b*) shows a velocity time curve for A, drawn by setting off the values of CZ on a base of equal crank angles.

Four-bar chain. Fig 494 shows an example of a **double crank and connecting rod.** Two cranks, one AB, revolving about A, and another CD, revolving about C, are connected by a link BD; the frame forms the fourth bar of the chain. For the position shown, I is the instantaneous centre, obtained by producing BA and CD. As before

$$\frac{V_D}{V_B}=\frac{ID}{IB}.$$

If V_B is given, V_D may be found from this construction, and the angular velocity of CD may be calculated from

$$\text{Angular velocity of CD}=\frac{V_D}{CD}.$$

In Fig. 495 is shown another pair of cranks AB revolving about A, and CD revolving about C, BD is the connecting link, and I is its instantaneous centre. As before

$$\frac{V_B}{V_D} = \frac{IB}{ID} \quad \ldots \quad (1)$$

Let ω_1 = the angular velocity of AB,

ω_2 = the angular velocity of CD.

Then $V_B = \omega_1 . AB$,

$V_D = \omega_2 . CD$.

Hence,
$$\frac{\omega_1 . AB}{\omega_2 . CD} = \frac{V_B}{V_D} = \frac{IB}{ID}$$

or
$$\frac{\omega_1}{\omega_2} = \frac{IB}{ID} \cdot \frac{CD}{AB} \quad \ldots \quad (2)$$

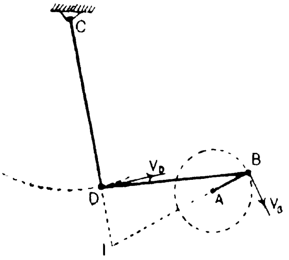

FIG. 494 —A four-bar chain.

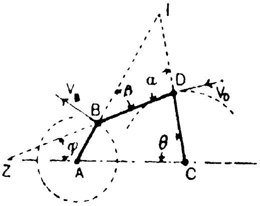

FIG. 495 —Angular velocities in a four bar chain.

Produce DB and CA to meet in Z, and mark the angles α, β, θ and ϕ as shown; then, in the triangle ZAB,

$$\frac{ZA}{AB} = \frac{\sin\beta}{\sin\phi},$$

and, in the triangle ZCD,

$$\frac{ZC}{ZD} = \frac{\sin(180 - \alpha)}{\sin\theta} = \frac{\sin\alpha}{\sin\theta}.$$

Hence,
$$\frac{AB}{ZD} \cdot \frac{ZC}{ZA} = \frac{\sin\alpha}{\sin\theta} \cdot \frac{\sin\phi}{\sin\beta}$$

$$\therefore \frac{ZC}{ZA} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\phi}{\sin\theta} \cdot \frac{ZD}{AB}.$$

Now, in the triangle IBD, $\sin\alpha/\sin\beta = IB/ID$; and in the triangle ZCD, $\sin\phi/\sin\theta = CD/ZD$. Hence,

$$\frac{ZC}{ZA} = \frac{IB}{ID} \cdot \frac{CD}{ZD} \cdot \frac{ZD}{AB}$$

$$= \frac{IB}{ID} \cdot \frac{CD}{AB} \quad \ldots \quad \ldots(3)$$

Therefore, from (2) and (3),

$$\frac{\omega_1}{\omega_2} = \frac{ZC}{ZA} \quad \text{.......................................(4)}$$

The result shows that **the angular velocities of AB and CD are inversely proportional to the segments in which CA is divided by DB produced.**

Wheel and racks. As a further example of the use of the instantaneous centre, Fig. 496 (*a*) shows a wheel between two racks. If the wheel is moving towards the left with a velocity V_C, and if the rack AD is fixed, then A will be the instantaneous centre of the wheel. Hence,

$$\frac{V_B}{V_C} = \frac{BA}{CA} = 2;$$

$$\therefore\ V_B = 2V_C,$$

showing that the velocity of the top rack is twice that of the centre of the wheel.

If the racks are moving as shown in Fig. 496 (*b*), then I may be found from the given values of V_A and V_B; thus

$$\frac{V_A}{V_B} = \frac{IA}{IB}.$$

Having found the position of I, the velocity V_C of the centre of the wheel may be calculated from

$$\frac{V_C}{V_A} = \frac{IC}{IA}.$$

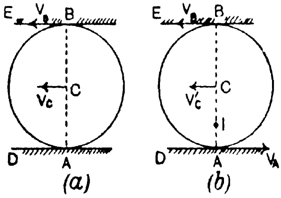

FIG. 496.—Wheel and racks.

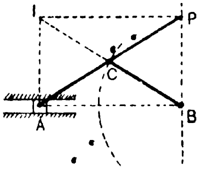

FIG. 497.—Scott-Russell parallel motion.

Parallel motions. By the term **parallel motion** is meant an arrangement for constraining a point to move in a straight line. In the **Scott-Russell parallel motion** (Fig. 497), a link AP has one end A guided so as to move in the straight line AB. Another link BC is pivoted at B, and is connected by a pin to the centre C of AP. AC = CP = BC, hence P, B and A will always lie on a semicircle which has AP for diameter. The angle ABP will always be 90°, and hence P will move in a straight vertical line passing through B.

It will be noted that the instantaneous centre for AP is at the intersection of BC produced and AI drawn perpendicular to AB. Further, from the geometry of the figure, P will lie always on a horizontal line drawn from I, and will, therefore, be moving vertically in any position of the mechanism. This confirms the result already noted.

In practice it is often convenient to guide A as shown in Fig. 498. The short arc in which A now moves interferes with the straight line motion of P to a small extent only. This modification of the Scott-Russell parallel motion is used sometimes in indicators for guiding the pencil in a straight line. The arrangement permits of P having a magnified copy of the motion of the piston G. The instantaneous centre I for AP is the point of intersection of DA and BC when produced, and IP, drawn horizontally through I, gives the position of P on the link AP. Joining AB, it will be seen that the triangles ABC and ICP are nearly similar. Also AC and IC are nearly equal for all practicable positions of the mechanism. Hence,

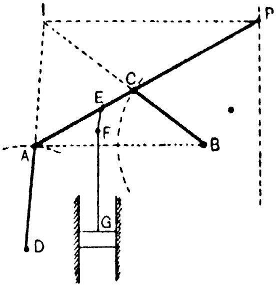

Fig. 498. Parallel motion for an indicator.

$$\frac{IC}{CP} = \frac{BC}{AC},$$

or

$$\frac{AC}{CP} = \frac{BC}{AC},$$

$$\therefore \quad CP = \frac{AC^2}{BC},$$

a result which enables CP to be calculated when AC and BC are given.

In the **Watt parallel motion** (Fig. 499), two equal links, AB and DC are pivoted at A and D respectively, and connected by a third link BC. It is evident that movement of the mechanism will cause B and C to deviate to the left and right respectively; hence P, the centre of BC, will move in a straight vertical line for a considerable distance. If the movement of the mechanism continues, P will describe a curve resembling a rough figure eight (the lemniscate).

In Fig. 500 is shown a Watt parallel motion in which AB and DC are not equal. I will be the point of intersection of AB and DC,

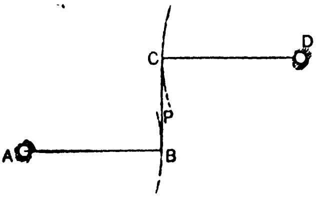

FIG. 499.—Watt parallel motion; equal arms.

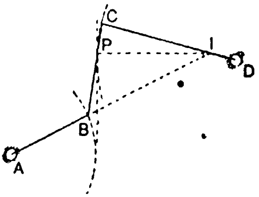

FIG. 500.—Watt parallel motion; unequal arms.

and P may be found by drawing IP horizontally from I. From the geometry of the figure, it may be shown that

$$BP : PC = CD : AB.$$

In Fig. 501 is shown the arrangement of Watt's parallel motion used in beam engines. AB and DC are equal, and P, the centre of BC, moves in a straight vertical line. DC is extended to E, CE being equal to DC, and bars EF and FB are added so as to form a parallelogram CEFB. EF will then be double of CP, and F, P and D will lie in a straight line always. FD will be double of PD, consequently, if P is moving in a straight vertical line, so also will F. In the engine, F and P serve to guide the ends of the low pressure and high pressure piston rods respectively.

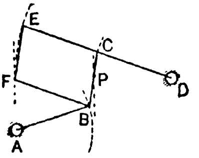

FIG. 501.—Watt parallel motion applied to a beam engine.

Inertia effects in a mechanism. In investigating problems regarding the forces or turning moments which may be delivered by a machine, it is often necessary to consider the effects of the inertia of the parts of the machine. The following case of **a slotted bar mechanism** (Fig. 502) giving simple harmonic motion to a piston A should be studied. Frictional effects have been considered already partly (p. 371), and are disregarded here.

Fig. 502 (*a*) is a diagram showing the effective pressure on the piston throughout the stroke; any ordinate such as p_1 gives the difference in pressure on the two sides of the piston at the moment considered. Hence, the net force P urging the piston towards the left is

$$P = p_1 \frac{\pi d^2}{4} \text{ lb. weight,}$$

where d is the diameter of the cylinder in inches and p_1 is the pressure in pounds per square inch.

But for the inertia of the piston, piston rod, and slotted bar, the whole of this force would be transmitted to the crank pin. These parts will all have equal accelerations in this mechanism.

Let M = mass of reciprocating parts, pounds.
a = their acceleration, feet per sec. per sec.

Then the force required to overcome inertia will be

$$F = \frac{Ma}{g} \text{ lb. weight.}$$

The force Q actually reaching the crank pin in the position considered will be given by

$$Q = P - F$$
$$= p_1 \frac{\pi d^2}{4} - \frac{Ma}{g}.$$

The acceleration a may be found for any position by the method explained on p. 398. Fig. 502 (b) is a diagram in which the ordinates p_2, etc., have been calculated from

$$p_2 = \frac{Ma}{g} \div \frac{\pi d^2}{4}.$$

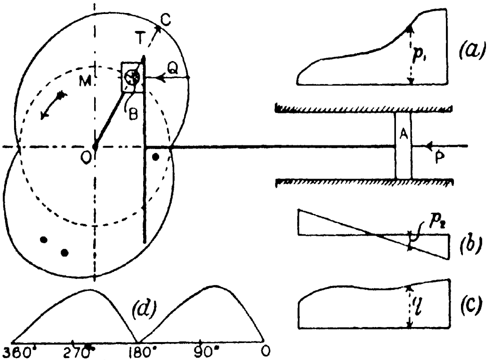

FIG. 502.—Diagrams for a slotted bar mechanism, taking account of inertia.

These ordinates will then represent the forces required to overcome inertia per square inch of piston area. The scales used in Fig. 502 (b) are the same as for Fig. 502 (a), hence a combined diagram (Fig. 502 (c)) may be drawn by simply adding the ordinates algebraically, the result showing q, the force per square inch of piston area which is transmitted to the crank pin.

The **turning moment** on the crank pin will be

$$T = Q \times OM,$$

where OM is perpendicular to the line of Q. **A polar turning-moment diagram** may be drawn by producing the crank OB and making BC equal to T to a convenient scale. This being done for a number of crank angles, a fair curve drawn through the ends will give the required diagram. Or a turning-moment diagram may be drawn as in Fig. 502 (*d*) by using a base of equal crank angles and setting off the values of T at the chosen angles.

Locomotive side rod. In Fig. 503, A and C are the centres of two driving wheels of a locomotive, the equal cranks AB and CD are connected by the side rod BD. The velocities V_B and V_D for the given position may be found by taking I_1 and I_2 as the instantaneous centres of the wheels, assuming that there is no slipping between the wheels and the rails. V_A and V_C will be equal to the velocity of the locomotive. Hence,

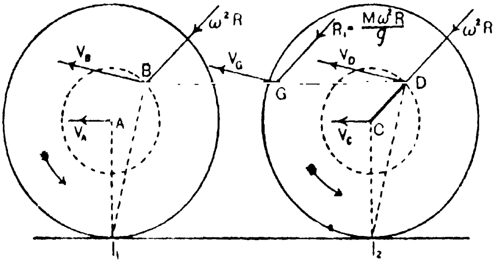

FIG. 503. Motion of a locomotive side rod.

$$\frac{V_B}{V_A} = \frac{I_1B}{I_1A} \quad \text{or} \quad V_B = \frac{I_1B}{I_1A} V_A.$$

Also,

$$V_D = \frac{I_2D}{I_2C} V_C = V_B.$$

The velocities of B and D being equal in all respects, it follows that the velocity of any point in the side rod will be equal to that of B or D; thus V_G is equal to V_B or V_D.

Assuming that the speed of the locomotive is constant, and that the consequent angular velocity of each wheel is ω radians per second, the accelerations of B and D will be unaltered if we imagine that the wheels rotate with an angular velocity ω, and that their centres remain fixed in position. B and D will therefore have accelerations directed towards A and C respectively, and of amount ω^2R feet per second per second, R being the crank radius in feet. These accelerations are equal in all respects; hence the acceleration of any point in the rod will have an equal value and will have the same direction.

Since the side rod is always moving parallel to the rail and has no angular motion, the resultant force required to give it its motion must act through its centre of mass, and must be in the same line as the acceleration of the mass centre. If the rod is uniform, the centre of mass G will bisect BD; let M be the mass of the rod in pounds, then the resultant force R_1 required to overcome the inertia of the rod will be

$$R_1 = \frac{M\omega^2R}{g} \text{ lb. weight.}$$

Obviously R_1 is the resultant of two equal and parallel forces, one acting at each crank pin.

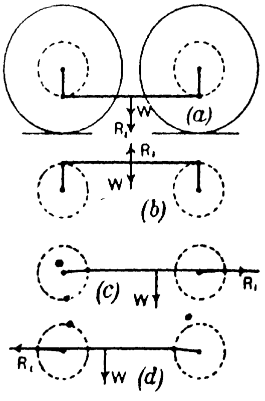

FIG. 504.—Inertia effects in a locomotive side rod.

The force R_1, reversed in sense, gives the effect of the inertia resistance of the rod on the wheel bearings at A and C. It is evident that there will be a lifting effort when the side rod is in its highest position (Fig. 504 (b)), and an additional pressure on the rails when the rod is in its lowest position (Fig. 504 (a)). R_1 acts towards the right (Fig. 504 (c)), or towards the left (Fig. 504 (d)), when the cranks are horizontal.

Fig. 504 also indicates the effect of R_1 in producing a transverse load on the rod. For a uniform rod, R_1 is the resultant of an inertia load which has a uniform distribution per unit length of the rod, and in this respect resembles the weight W of the rod. As will be seen by inspection of Figs. 504 (a) and (b), R_1 and W conspire when the rod is in the lowest position, and are opposed when the rod is in its highest position. The maximum bending effect on the rod will, therefore, occur in the position shown in Fig. 504 (*a*). If W is the weight of the rod in lb., the total uniformly distributed load producing bending moment will be

$$\text{Total distributed load} = R_1 + W$$

$$= \frac{M\omega^2 R}{g} + W \text{ lb. weight.} \quad ..$$

The calculation of the maximum bending moment and stresses produced by this may be performed by the methods explained in Chapter VII.

Crank and connecting rod. The inertia of the moving parts in the crank and connecting rod mechanism produces effects similar to those in the slotted bar mechanism (p. 462), but the problem is somewhat more complicated owing to the oblique action of the connecting rod. In the slotted bar mechanism, the piston has simple harmonic vibrations, and hence has equal accelerations when at equal distances from the centre of the stroke; the connecting rod causes the accelerations to be unequal to an extent which is more marked if the connecting rod is short. A very long rod produces nearly equal accelerations, a rod of infinite length would give simple harmonic motion; hence the name **infinite connecting rod mechanism** sometimes given to the slotted bar arrangement. Further, the piston, piston rod and crosshead have straight-line motion, and hence are dealt with easily, while the connecting rod has one end moving in a straight line and the other end in a circle. For simplicity, it is customary to treat the rod in two parts, a fraction, say one-half, of its mass being assumed to be concentrated at the centre of the crank pin and rotating with it, while the remainder of the mass is assumed to move in a straight line with the crosshead. The mass of the reciprocating parts will then include the piston, piston rod, crosshead and the assigned part of the connecting rod, and this mass will require forces in order to overcome its inertia.

The **acceleration diagram** may be drawn by the method described for another mechanism on p. 386. The work may be made more accurate by first drawing a velocity-time diagram for the piston by the instantaneous centre method (p. 458); then the average accelerations over equal intervals may be calculated, and the results set off at the centres of the intervals. Or **Klein's construction** may be used as follows to obtain an acceleration diagram direct on a base representing the stroke.

Klein's construction. In Fig. 505 is shown a crank CB of radius

R feet and a connecting rod AB in a given position. On AB as diameter describe a circle; produce AB, if necessary, to cut NS in Z; describe another circle with centre B and radius BZ, cutting

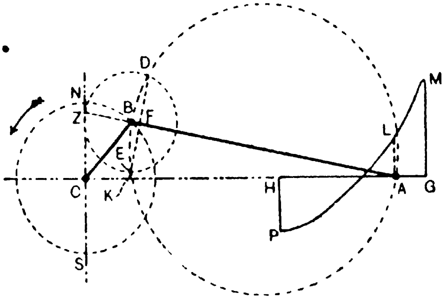

FIG. 505 – Klein's construction for the acceleration of the piston.

the first circle in D and E. Join DE, cutting AB in F and AC in K, producing DE if necessary. Then, as will be proved later, KC represents the acceleration of the piston to a scale in which the central acceleration of B, viz. $\frac{v^2}{R}$ is represented by BC. It is assumed usually that B is moving with uniform velocity v feet per second.

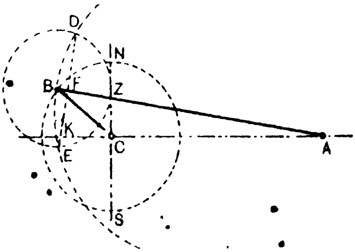

FIG. 506 – Klein's construction, crank in second quadrant.

The construction should be repeated for crank angles differing by 30°; KC should be measured for each position, and the results set off as at AL on a base GH, which represents the stroke of the piston. The acceleration diagram is obtained by drawing a fair curve through the ordinates, and is shown at GMLPH. Fig. 506 shows the construction when the crank is in the second quadrant.

and in Fig. 507 is given the construction when the crank is on the dead points B and B'.

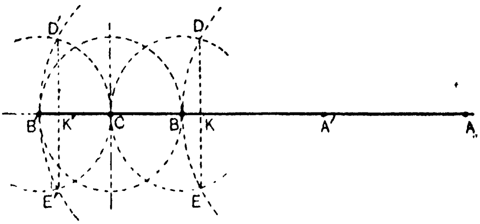

FIG. 507.—Klein's construction, crank at dead points.

Accelerations at ends of the stroke. The accelerations of the piston when the crank is at the dead points may be found also by the following method. In Fig. 508 the crank is shown at a very small angle from the dead point; imagine that the connecting rod is so guided that it is moving parallel to the line of the stroke, *i.e.* BA' is parallel to AC. Every part of the connecting rod will have the same acceleration as B, viz. $\frac{v^2}{R}$ towards the left. Now, the connecting rod is actually moving in such a manner that one end, B, has a velocity v at right angles to the rod; to bring A' into the centre line AC, give A' a velocity v as shown. Owing to this, A will have an acceleration $\frac{v^2}{L}$ towards the left; hence total acceleration of A will be

$$a=\frac{v^2}{R}+\frac{v^2}{L}=\frac{v^2}{R}\left(1+\frac{R}{L}\right) \text{ feet per sec. per sec.}, \quad \ldots\ldots \quad (1)$$

when $v=$ the velocity of the crank pin, feet per sec.;
$R=$ the radius of the crank, in feet;
$L=$ the length of the connecting rod, in feet.

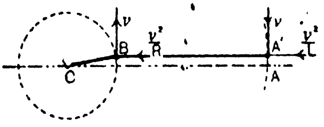

FIG. 508.—Acceleration of the piston at the inner dead point.

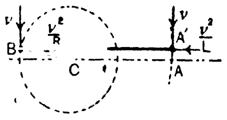

FIG. 500.—Acceleration of the piston at the outer dead point.

At the outer dead centre (Fig. 509) a similar method may be used, but now $\frac{v^2}{R}$ is towards the right and $\frac{v^2}{L}$ is towards the left. Hence the resultant acceleration of A will be towards the right, and will be given by

$$a = \frac{v^2}{R} - \frac{v^2}{L} = \frac{v^2}{R}\left(1 - \frac{R}{L}\right) \text{ feet per sec. per sec.} \quad \ldots \quad (2)$$

These results, (1) and (2), are of service in making preliminary calculations of the accelerations of the piston when at the ends of the stroke.

The effective force Q acting on the crosshead in the line of the stroke may be estimated now.

Let d = the diameter of the cylinder, in inches.

p_1 = the effective pressure on the piston at a given position, lb. per square inch.

M = the mass of the reciprocating parts, including the assigned part of the connecting rod, pounds.

a_1 = their acceleration in the given position.

Then $$Q = p_1 \frac{\pi d^2}{4} - \frac{M a_1}{g}, \text{ lb. weight} \quad \ldots \quad (3)$$

Turning moment. The **turning moment** produced by Q may be calculated as follows, reference being made to Fig. 510 and friction being neglected.

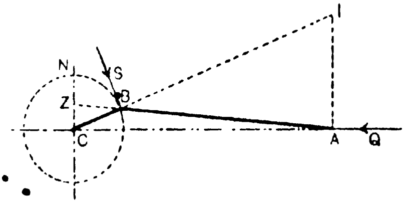

FIG. 510.—Turning moment on the crank.

I is the instantaneous centre for the given position, from which it appears that rotation of the rod round I is produced by Q and resisted by the crank pin with a force S. Hence,

$$Q \times IA = S \times IB,$$

$$S = Q\frac{IA}{IB} \quad \ldots \quad (4)$$

Produce AB to cut CN in Z, then the triangles ABI and BZC are similar. Hence,

$$\frac{IA}{IB} = \frac{CZ}{BC} = \frac{CZ}{R}.$$

Substitution in (4) gives

$$S = Q\frac{CZ}{R} \quad \ldots \quad (5)$$

S is the reaction of the crank pin, and, if reversed, will be the **crank effort** given by the connecting rod to the crank pin. Hence,

$$\text{Turning moment} = \text{T} = \text{S} \times \text{R}$$

$$= \text{Q}\frac{\text{CZ}}{\text{R}}\,.\,\text{R}$$

$$= \text{Q}\,.\,\text{CZ}. \quad \ldots \qquad \ldots(6)$$

This result will be in lb.-feet if Q is in lb.-weight units and CZ is measured in feet to the same scale as that used in drawing the mechanism.

With the alterations and additions noted above, the method of obtaining a turning moment diagram used on p. 463 may be employed.

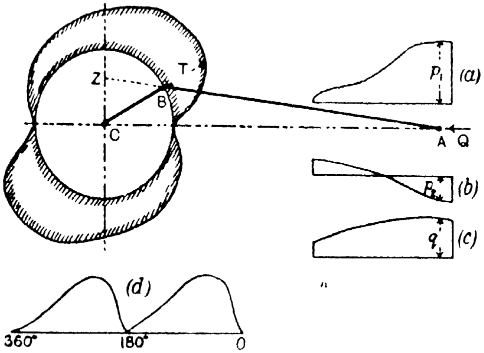

Fig. 511.—Diagrams for a slider-crank-chain, taking account of inertia.

The various diagrams required are shown in Fig. 511, and will be followed readily.

General effects of inertia. The student will observe that the general effect of the inertia of the moving parts is to produce a more uniform turning moment on the crank. During the early part of the stroke, the gaseous pressure on the piston is high, but is absorbed partly in accelerating the moving parts, hence the turning moment is smaller; later in the stroke, the gaseous pressure is low, but the moving parts are losing velocity now, and their inertia assists the gaseous pressure in making the turning moment larger.

Greater uniformity in the turning moment may be obtained by having two or more cylinders with pistons operating on separate cranks. If there are two cylinders, the cranks are placed generally at 90° to each other; in the case of three cylinders, the cranks are generally at 120°; with a larger number of cranks, the precise crank angles cannot be stated, as other considerations are involved. Usually an attempt is made in such cases to produce a self-balanced machine, *i.e.* one in which the inertia effects balance one another without producing disturbances in the frame or foundation.

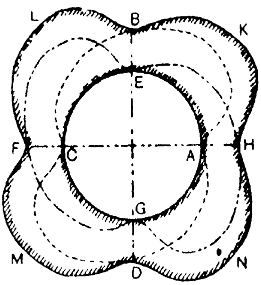

FIG. 512.—Turning moment diagram for two cranks at 90°

Turning moment diagrams are given in Fig. 512 for two cylinders similar to the case illustrated in Fig. 511. The cranks are at 90°, and the turning moment diagrams for each crank separately are shown by ABCDA and EFGHE, these are displaced relatively to each other by 90°. Summing the corresponding ordinates, the combined turning moment diagram is HKBLFMDNH. Greater uniformity has been obtained, and there is no point where the turning point is zero.

Further points regarding the motion of the connecting rod. It has been explained that, for positions near the dead points, the motion of the connecting rod may be assumed to be compounded of a motion of translation together with another motion of rotation round the crank pin (p. 468). The same assumptions may be made when the rod is in any other position (Fig. 513). The first of these

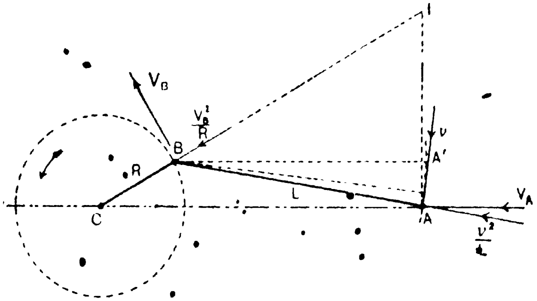

FIG. 513.—Analysis of the motion of a connecting rod

motions would cause the rod always to move parallel to the centre line AC, and it would occupy the position A′B when the crank is at CB; the latter motion produces the effect of rotating the rod into its proper position AB. Owing to the first of these component motions, **all points in the rod will possess the same velocity and acceleration as the crank pin B**; in this respect, the motion is precisely the same as that of the side rod of a locomotive (p. 464). The acceleration thus produced at A will be $\frac{V_B^2}{R}$, and may be represented by the length of the crank BC. Hence,

$$\frac{V_B^2}{R} = BC,$$

or

$$V_B^2 = BC \times R = BC^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The point A will possess other accelerations owing to the component motion of rotation of the rod about B; in consequence of this angular motion, A will have a variable velocity v in a direction at right angles to the rod. The value of v will depend on the position of the crank, and hence will be undergoing change in most positions of the mechanism. Owing to this, there will be an **acceleration of A in the line of v, i.e. at right angles to the connecting rod.** Further, A will possess the ordinary **central acceleration towards B**, of magnitude given by $\frac{v^2}{L}$. Hence in all A possesses **three component accelerations, and the resultant of these must have a direction coinciding with that of AC.**

To find an expression for v, reference is made to Fig. 513, showing I, the instantaneous centre of the rod. The angular velocity of the rod will be $\frac{V_B}{IB}$, and will be given also by $\frac{v}{L}$. Hence,

$$\frac{v}{L} = \frac{V_B}{IB},$$

$$v = \frac{V_B}{IB} \cdot L. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Also,

$$\text{Central acceleration of A towards B} = \frac{v^2}{L} = \frac{1}{L} \cdot \frac{V_B^2}{IB^2} L^2$$

$$= \frac{L}{IB^2} \cdot V_B^2 \quad \ldots\ldots\ldots \quad (3)$$

Referring to Fig. 514, showing Klein's construction together with at 90° to each other; in the case of three cylinders, the cranks are generally at 120°; with a larger number of cranks, the precise crank angles cannot be stated, as other considerations are involved. Usually an attempt is made in such cases to produce a self-balanced machine, *i.e.* one in which the inertia effects balance one another without producing disturbances in the frame or foundation.

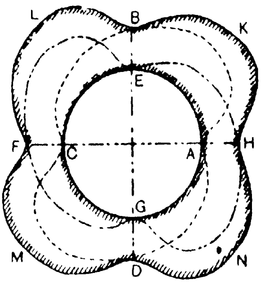

FIG. 512.—Turning moment diagram for two cranks at 90°

Turning moment diagrams are given in Fig. 512 for two cylinders similar to the case illustrated in Fig. 511. The cranks are at 90°, and the turning moment diagrams for each crank separately are shown by ABCDA and EFGHE, these are displaced relatively to each other by 90°. Summing the corresponding ordinates, the combined turning moment diagram is HKBLFMDNH. Greater uniformity has been obtained, and there is no point where the turning point is zero.

Further points regarding the motion of the connecting rod. It has been explained that, for positions near the dead points, the motion of the connecting rod may be assumed to be compounded of a motion of translation together with another motion of rotation round the crank pin (p. 468). The same assumptions may be made when the rod is in any other position (Fig. 513). The first of these

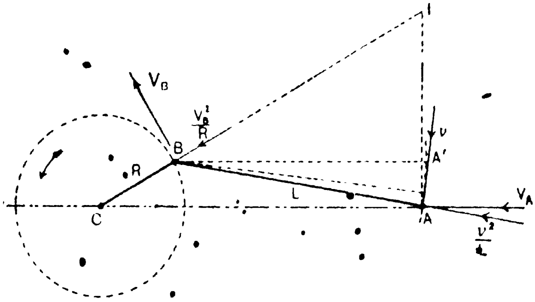

FIG. 513.—Analysis of the motion of a connecting rod

represented by KF, and **the closing line KC gives the resultant acceleration of A along AC.** It will be noted that this result proves the truth of Klein's construction.

Since KF gives the linear acceleration of A in the direction at 90° to that of the rod, it follows that the angular acceleration of the rod will be given by

$$\text{Angular acceleration of the connecting rod} = \frac{KF}{L} \quad \ldots\ldots(6)$$

Acceleration image of the connecting rod. In Fig. 515 KF and FB have been copied from Fig. 514. The resultant of these accelerations will be KB, the closing line of the triangle of accelerations FBK. The acceleration of A along AC may be taken to be the resultant of the accelerations KB and BC, and is represented by the closing line of the triangle of accelerations KBC. Consider any other point in the connecting rod, such as G. Its velocity at 90° to the rod, and hence its accelerations, owing to the rod rotating about B, will be simply proportional to BG, that is,

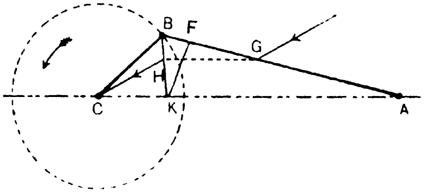

FIG. 515 Acceleration image of a connecting rod

acceleration of G : acceleration of A = BG : BA.

Draw GH parallel to AC, and cutting KB in H; then

Acceleration of G : acceleration of A = BH : BK.

Now KB represents the resultant of the two component accelerations of A which are respectively along and at 90° to AB; hence HB will represent the resultant of the similar components of G. The component acceleration of G, owing to B rotating about C, remains of unaltered value $\frac{V^2}{R}$, and is represented by BC. Hence the resultant acceleration of G will be the closing line HC of the triangle of accelerations HBC. **The resultant acceleration of any other point in the rod may be found in a similar manner by drawing a line from the point parallel to AC to cut KB, and joining the point so found on KB to C.** On account of this property of KB it is called usually **the acceleration image** of the connecting rod.

Resultant force required to give acceleration to the connecting rod. In Fig. 516, let G be the mass centre of the connecting rod.

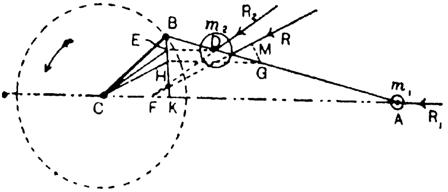

FIG. 516—Resultant force required to overcome the inertia of the connecting rod.

The acceleration of G is HC, and might be produced by a force R acting at G in a line parallel to HC. The magnitude of R, if the rod has a mass M pounds, will be

$$R = \frac{M \cdot HC}{g} \text{ lb weight. } \quad \ldots\ldots \quad (1)$$

This force would not produce any angular acceleration in the connecting rod on account of its line of action passing through the mass centre of the rod. In order to obtain the actual motion of the rod, which includes angular acceleration in most positions, R will require to be shifted from G, thereby giving a couple which will produce the required angular acceleration. A convenient way is to use an equivalent dynamic system by substituting two masses, m_1 and m_2 pounds (Fig. 517), for the actual mass of the rod. One of these, m_1, may be situated at the centre of the crosshead pin A, at a distance a from G; the other mass, m_2, will be at a distance b on the other side of G. For this arrangement to be equivalent to the actual rod, the following conditions must be complied with (p. 444):

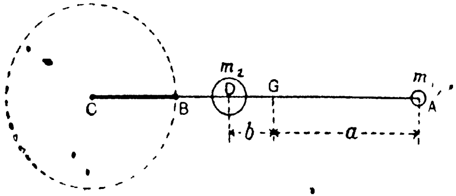

FIG. 517—A dynamical system equivalent to the connecting rod

$$m_1 + m_2 = M \quad \ldots\ldots \quad (2)$$

$$m_1 a = m_2 b \quad \ldots\ldots \quad (3)$$

$$m_1 a^2 + m_2 b^2 = M k_G^2 \quad \ldots\ldots \quad (4)$$

k_G is the radius of gyration of the connecting rod about an axis passing through G and parallel to the crank shaft. The solution of these equations gives

$$ab = k_G^2, \quad \ldots \ldots \qquad (5)$$

or

$$b = \frac{k_G^2}{a}. \quad \ldots \ldots \qquad (6)$$

From this result b may be calculated when a and k_G are known; the masses m_1 and m_2 may be determined then from equations (2) and (3).

Reference may be made now to Fig. 516, which shows the crank and connecting rod, the latter being represented by the equivalent masses m_1 and m_2. To accelerate m_1 requires a force R_1 acting in the line of the acceleration of A, viz. AC. To accelerate m_2 requires a force R_2 acting in the line of the acceleration of D; this line may be found by drawing DE parallel to AC and cutting BK in E; the acceleration of D will be represented then by EC. R_2 will be parallel to EC, and cuts the line of R_1 produced in F. Hence the resultant of R_1 and R_2, which will be the resultant force R required to accelerate the rod, must pass through F. The line of R will be parallel to the acceleration of the mass centre G, viz. HC, and the magnitude of R will be given by equation (1) (p. 475). The couple giving angular acceleration to the connecting rod will be R × GM, GM being the perpendicular from G to the line of R.

Reactions on the engine frame produced by the inertia of the connecting rod. In Fig. 518, R is the resultant force required to overcome the inertia of the connecting rod in the given position. R is actually the resultant of two forces, one of which, P, is applied by the guide bar to the pin at A; the other force, Q, is applied to the rod at B by the crank pin. If the friction of the slipper be neglected, P will act at right angles to AC, and its line will intersect

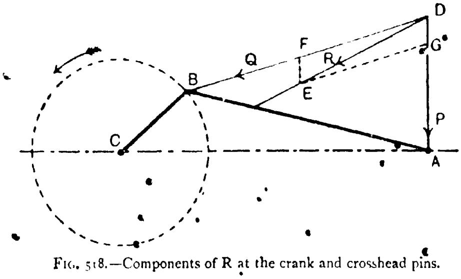

FIG. 518.—Components of R at the crank and crosshead pins.

the line of R at D; hence Q also acts through D. The magnitudes of P and Q may be determined by drawing the parallelogram of forces DGEF, in which DE is made equal to R, and P and Q will be represented by DG and DF respectively. The reaction on the guide bar at A will be obtained by reversing the sense of P (Fig 519); in the same diagram, the reaction on the crank pin is shown by reversing the sense of Q.

The force Q in Fig. 519 is equivalent to an equal and parallel force Q, of the same sense, acting at C together with a couple of moment Q × CH, CH being perpendicular to the line of Q. Q acting at C produces a pressure on the main bearing and hence on the engine frame; the couple Q × CH modifies the turning moment on the crank.

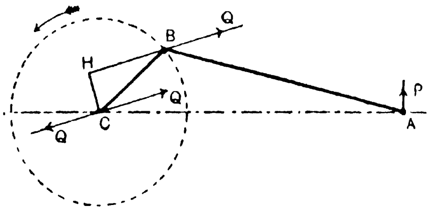

FIG. 519.—Reactions of the engine frame due to the inertia of the connecting rod.

Bending moment on the connecting rod produced by its inertia. Assuming that the inertia effects on the connecting rod will be greatest when the crank and connecting rod are at 90° to each other, the bending-moment diagram may be drawn as follows: In Fig. 520

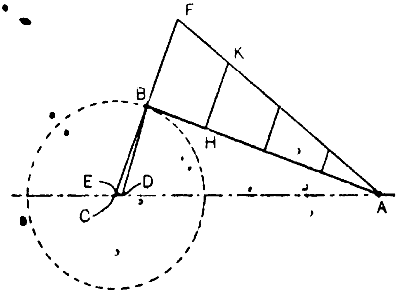

FIG. 520.—Transverse inertia load on the connecting rod.

ABC is 90°, and BD is the acceleration image of the rod. The acceleration of B will be towards C and will be ω^2R feet per second per second, where ω is the angular velocity of the crank in radians per second and R is the radius of the crank in feet. The acceleration of A will be represented by DC, and its component perpendicular to AB will be found by drawing DE perpendicular to CB. CE will be the required component. Neglecting the very small acceleration represented by CE, the acceleration of any point on the connecting rod may be found by making BF perpendicular to AB and equal to ω^2R and by joining FA. The acceleration normal to AB of any point H in the rod will be represented by HK, perpendicular to AB.

Let m be the mass of the rod in pounds per inch length at H; then the transverse inertia load on the rod at H will be

$$\text{Inertia load per inch length} = \frac{m \times \text{HK}}{g} \text{ lb. weight,}$$

HK being measured to the acceleration scale in feet per second per second. A similar calculation should be made for a number of

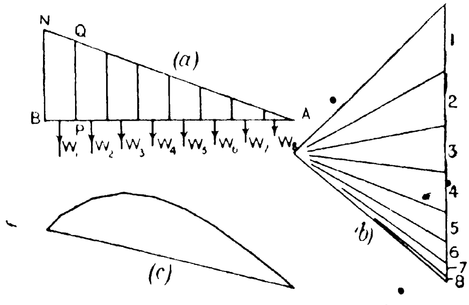

FIG. 521.—Bending moment diagram produced by the inertia load on the connecting rod.

points on the rod; the calculations are somewhat simpler in the case of a rod of uniform cross section, and in other cases may sometimes be simplified by taking m to be the average mass per inch length. A load curve is constructed then by drawing AB (Fig. 521 (a)) to represent the length of the rod and setting off the calculated loads per inch at the various points chosen, as at PQ. Considering the portion BP, the average load per inch will be $\frac{1}{2}(\text{BN}+\text{PQ})$ and the load on BP will be $W_1 = \frac{1}{2}(\text{BN}+\text{PQ})\text{BP}$.

W_1 will act through the centre of area of BNQP. Carrying out the same process for the other portions of the rod, we obtain the equivalent system of concentrated loads W_1, W_2, W_3, etc. The bending-moment diagram may be drawn now by the link polygon method (p. 140), the construction being shown in Fig. 521 (*b*) and (*c*).

Simple slide-valve gear. In ordinary reciprocating engines, the valve employed to distribute the steam to each end of the cylinder consists of an inverted rectangular box V (Fig. 522), which slides on a flat face formed on the cylinder. Two steam passages, or ports S_1 and S_2, lead to each end of the cylinder, and another E leads to the atmosphere or to the condenser. Movement of the valve to the right will permit steam to flow through S_1 into the left-hand side of the cylinder, and at the same time permits the steam in the right hand side to flow out through S_2 and E. Movement of the valve to the left will admit steam to the right-hand side through S_2, and will permit exhaust from the left-hand side through S_1 and E.

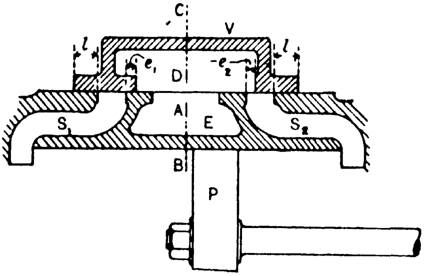

FIG. 522.—Simple slide valve for a steam engine.

CD is the centre line of the valve and AB is the centre line of the cylinder ports; the valve is in its mid-position when these lines are coincident vertically. In this position, the valve generally laps over the edge of the ports; *l* is called the outside lap, and gives an earlier cut-off than if there were no outside lap; e_1 is called positive inside lap; if the inside lap is made as shown at e_2 it is called negative; the inside lap determines the point at which the exhaust steam is stopped from flowing out of the cylinder. Cut-off of the steam supply is effected when some fraction of the piston stroke has been completed; the remainder of the stroke is then completed under the expansive action of the steam. Closing of the exhaust is effected before the end of the return stroke in order to entrap some of the exhaust steam in the cylinder; this is compressed by the returning piston, and acts as a cushion in bringing it to rest. To understand the complete distribution of the steam, it is necessary that the displacement of the valve from its mid-position should be known for any given crank position, and hence for any piston position.

The slide valve is driven generally by means of an eccentric, consisting of a disc A secured to the crank shaft B and revolving with it (Fig. 523). The hole in the disc is bored at a small distance from the centre of the disc. A strap C surrounds the disc and is connected by an eccentric rod DE to the valve rod EF. The eccentric is equivalent to a crank having a radius equal to the distance from the centre of the shaft to the centre of the disc. This radius is generally very small compared with the length of the connecting rod, hence it may be assumed that the motion of the valve is simply harmonic.

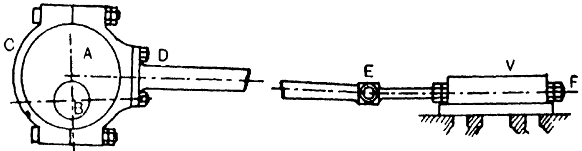

FIG. 523.—Eccentric driving a slide valve

Let the circle ABCD (Fig. 524) represent the path described by the centre of the eccentric, which is rotating in the direction of the arrow. AC may be taken to represent the travel of the valve, which will be in its mid-position O when the eccentric is at OB or at OD. Assuming simple harmonic motion, when the eccentric is in any position, such as OE, if EM and EN are perpendicular to AC and to BD respectively, OM or NE will be the displacement of the valve. If the angle separating the eccentric and the crank be EOK, then OK will be the corresponding crank position. Since the valve displacement towards the right must be equal to l at admission, if OL, equal to l, be measured and LE_1 drawn perpendicular to AC, OE_1 will be the position of the eccentric at admission. By setting off the angle E_1OK_1 equal to EOK, the position of the crank at admission OK_1 may be found. Producing E_1L to E_2 will

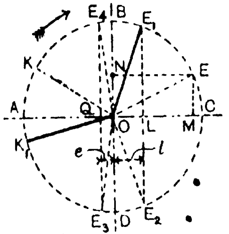

FIG. 524.—Crank positions at admission, cut off, release and cushioning

determine the eccentric position OE_2 at cut-off. Release and cushioning are controlled by the inner edge of the valve. Make OQ equal to *e*, measuring it to the left of O if positive and to the right if negative, and draw E_3QE_4 perpendicular to AC; OE_3 and OE_4 will be the eccentric positions at release and cushioning respectively. In each case, the corresponding crank position may be obtained by setting back angles equal to EOK.

Valve diagrams are based on the idea of obtaining direct the displacement of the valve by merely drawing the crank in any given position In the Reuleaux valve diagram (Fig. 525), OS_1 and OS_2 are the crank positions at admission and cut-off; it is evident that the angle S_1OS_2 in this figure is equal to E_1OE_2 in Fig. 524. Hence D_1D_2, parallel to S_1S_2, will correspond to BD in Fig. 524, and the valve displacement for any crank position OK will be KN, which is drawn perpendicular to D_1D_2, and hence corresponds to EN in Fig. 524. Draw S_1S_2 and E_1E_2 parallel to D_1D_2 and at distances from it equal to *l* and *e* respectively. Then, of the total displacement KN, SN is equal to the outside lap, and hence KS will be the amount by which the valve edge has opened the port to steam. Similarly, at OK′, K′N′ will be the displacement on the left of the mid position and EK′ will be the amount by which the inner edge of the valve has uncovered the port to exhaust. At admission and cut-off, the crank will be at OS_1 and OS_2 respectively, as has been used in the construction, in these positions the port opening is zero. At release and cushioning the crank will be at OE_2 and OE_1 respectively, because in these positions the exhaust opening will be zero.

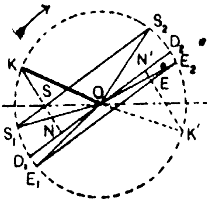

FIG. 525. Reuleaux valve diagram.

There are many other types of valve diagrams; any standard text book may be consulted for the principles on which they are based.*

Component eccentrics for a valve gear. In Fig. 526 (*a*), OC is the crank on the dead centre and OE is the corresponding eccentric position; the angle EOV is called the **angle of advance** and is denoted by *α*. Suppose that, instead of using OE, **component eccentrics** OV and OH are employed, these being found by drawing EV and EH respectively parallel and at right angles to CO. It is to be understood that the actual motion of the valve is to be obtained by adding together the displacement produced by the component eccentric OH and that produced by OV. OV and OH are called the 90° and the 180° components respectively, and are given by

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* A complete discussion is given in *Valves and Valve Gear Mechanisms*, by Prof. W. E. Dalby; (Arnold).

$$OV = OE \cos \alpha.$$
$$OH = OE \sin \alpha.$$

In Fig. 526 (*b*), the crank has moved through an angle θ from the dead point. We have

Displacement produced by $OE = OM = OE \sin (\alpha + \theta)$
$= OE \sin \alpha \cos \theta + OE \cos \alpha \sin \theta.$

Displacement produced by $OV = ON = OV \sin \theta = OE \cos \alpha \sin \theta$

Displacement produced by $OH = ON' = OH \cos \theta = OE \sin \alpha \cos \theta.$

Sum of displacements produced by OH and $OV = OE \sin \alpha \cos \theta + OE \cos \alpha \sin \theta.$

Hence the resultant displacement produced by the component eccentrics is equal to that produced by the actual eccentric.

Let OH and OV be written a and b respectively; then

$$\text{Displacement of the valve} = a \cos \theta + b \sin \theta. \quad \ldots\ldots (1)$$

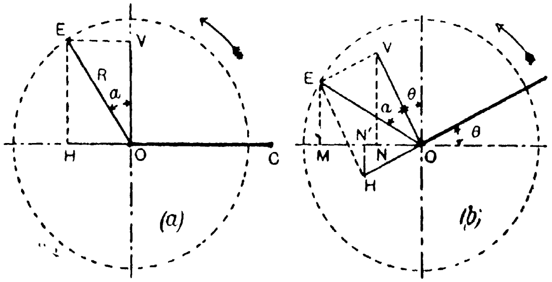


Fig. 526.—Component eccentrics.

a and b are both constant in a simple valve motion having a single eccentric. In many cases of more complicated valve gears the motion of the valve may be represented approximately by an equation of form similar to (1), showing that the valve may be imagined to derive its motion from a single eccentric acting direct on the valve.

Hackworth valve gear. An example is shown in Fig. 527 of a Hackworth valve gear; BC is the crank and AB is the connecting rod. The eccentric CE is at 180° to the crank and is connected to a rod EF, the end F of which may slide on a guide GH. GH is pivoted at K, and the angle β which it makes with CK may be adjusted by hand. Alterations in the cut off and reversal of the direction of motion of the engine are effected by altering β. The valve rod LN is connected to EF at L.

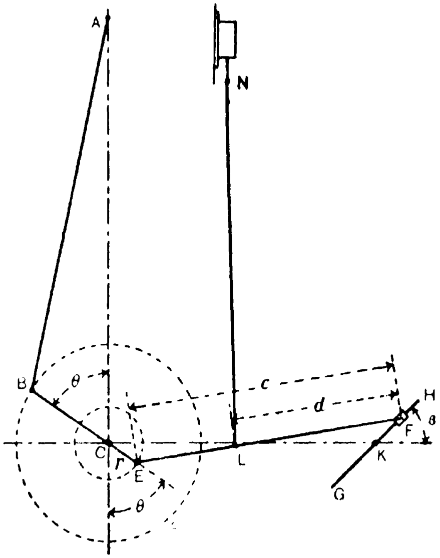


FIG. 527.—Hackworth valve gear

The displacements of the valve will be very nearly equal to the displacement of L vertically.

Let r = the radius of the eccentric.

c = EF.

d = LF. Then

Displacement of E vertically from CK $= r\cos\theta$.

If F were to move in the straight line CK, the displacement of L vertically owing to the above displacement of E would be

$$\text{Vertical displacement of L} = \frac{d}{c}\,r\cos\theta. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Again, Displacement of E horizontally from AC $= r\sin\theta$.

In Fig. 528 CP represents this displacement, and F is supposed to be connected direct to P. KQ will be very nearly equal to CP.

Hence,
$$\frac{FQ}{KQ} = \tan\beta,$$
$$FQ = KQ\tan\beta = CP\tan\beta$$
$$= r\sin\theta\tan\beta.$$

Hence, Vertical displacement of $L = \frac{c-d}{c} \cdot r\sin\theta\tan\beta$. .. . ...(2)

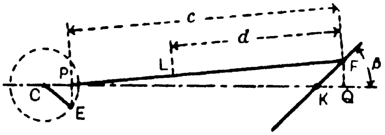


FIG. 528.—Approximations in the motions in the Hackworth valve gear.

To obtain the total displacement of L, take the sum of (1) and (2), noting that the result of (2) may be either positive or negative, depending on the setting of the guide bar HG. Thus

$$\text{Displacement of } L = \left(\frac{d}{c}r\right)\cos\theta \pm \left(\frac{c-d}{c}r\tan\beta\right)\sin\theta. \ . \ .. \ (3)$$

The result shows that the radii of the component eccentrics are

$$\text{Radius of the } 180^\circ \text{ component eccentric} = a = \frac{d}{c}r. \ .. \ .. \ ..(4)$$

$$\text{Radius of the } 90^\circ \text{ component eccentric} - b = \pm\frac{c-d}{c}r\tan\beta. \ ...(5)$$

The first of these, a, is evidently constant. The other, b, depends on the value of β. To obtain the maximum value of b take the maximum values of β, positive and negative, and hence of $\tan\beta$, and obtain the numerical value of (4) and (5). In Fig. 529 CY is made equal to the 180° component eccentric a, and CX and CX′ are respectively equal to the maximum positive and negative values of the 90° component eccentric b. The centre of the resultant eccentric will lie on E′YE, drawn parallel to XX′, for all settings of the guide bar GH. Its limiting positions will be CE and CE′ respectively. The resultant eccentric for any other setting of the guide bar may be obtained by calculating CX″ from (5); the resultant eccentric will then be CE″.

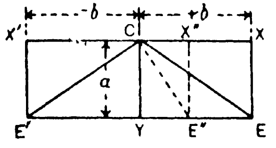


FIG. 529.—Characteristic line for the Hackworth valve gear.

The motion of the valve may be obtained thus for any setting of the guide bar GH by connecting it direct to the resultant eccentric found in this manner. E′YE in Fig. 529 is called the **characteristic line of the gear.**

Oscillating engine mechanism. This mechanism is illustrated in Fig. 530 (*a*), a crank BC revolves about B, and the cylinder is capable of oscillating about an axis or trunnion at A. There is no connecting rod, the piston rod is connected direct to the crank pin as shown. In Fig. 530 (*b*) is shown a slider crank chain, in which the crank BC revolves about C and A moves in the straight line AC. Comparison of (*a*) and (*b*) will show that (*a*) has been produced from (*b*) by an inversion of the mechanism. In (*b*), AB is capable of swinging about AC, while in (*a*) AB is fixed and AC is capable of swinging about AB.

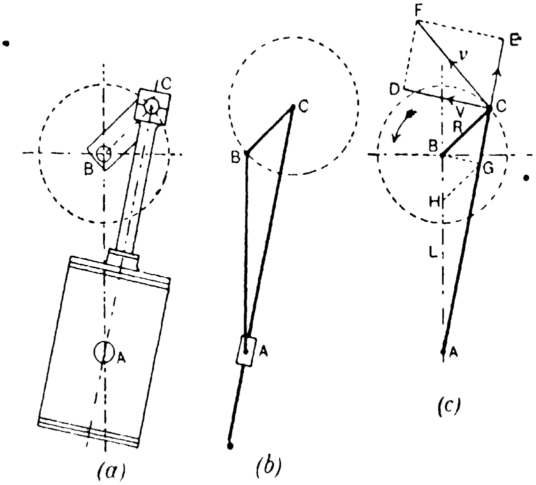


Fig. 530.—Oscillating engine mechanism.

In Fig. 530 (*c*), let the lengths of AB and BC be denoted by L and R respectively in feet, and let the velocity of the crank pin be uniformly equal to v feet per second. The angular velocity of the crank pin will be given by

$$\omega_1 = \frac{v}{R} \text{ radians per second. } \ldots\ldots\ldots\ldots\ldots (1)$$

To obtain the angular velocity of the cylinder, resolve v into velocities respectively along and perpendicular to AC by means of the parallelogram CEFD. CD = V will be the component perpendicular to AC. It is evident that the angular velocities of the cylinder and of the piston rod AC about A will be equal. Hence the angular velocity of the cylinder is

$$\omega_2 = \frac{V}{AC} \text{ radians per second. } \dots\dots\dots\dots(2)$$

Draw BG perpendicular to AC and GH parallel to BC; the triangles BCA and HGA will be similar. Hence,

$$\frac{CA}{CG} = \frac{BA}{BH} = \frac{L}{BH};$$

$$\therefore \ CA = \frac{CG}{BH} L.$$

Substitution in (2) gives

$$\omega_2 = \frac{V.BH}{L.CG}. \ \dots\dots\dots\dots\dots\dots(3)$$

Again, the triangles BGC and FDC are similar. Hence,

$$\frac{CG}{CB} = \frac{CD}{CF} = \frac{V}{v};$$

$$\therefore \ CG = \frac{V}{v} \cdot R = \frac{V}{\omega_1}.$$

Substitute this value in (3), giving

$$\omega_2 = \frac{V.BH}{L} \cdot \frac{\omega_1}{V} = \frac{BH}{L} \cdot \omega_1;$$

$$\therefore \ \frac{\omega_2}{\omega_1} = \frac{BH}{L}. \ \dots\dots\dots\dots\dots\dots\dots\dots(4)$$

This result shows that the angular velocity of the cylinder is represented by BH to a scale in which the angular velocity of the crank pin is represented by the constant length L. The cylinder has zero angular velocity in the two positions in which the crank and piston rod are at right angles. For positions of the crank below these, the cylinder is swinging towards the right, and its angular velocity may be called positive; for crank positions above the zero positions, the angular velocity will be of opposite sense, and may be described as negative. A polar diagram of angular velocity (Fig. 531) may be drawn by carrying out the construction for crank angles differing by 30°. BH (Fig. 530 (*c*)) is measured for each position and set off from C along the radius BC, towards B for negative and away from B for positive values.

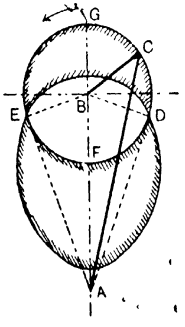


FIG. 531.—Angular velocity diagram for an oscillating cylinder.

The points D and E may be found by describing a circle on AB as diameter, the angles BEA and BDA will then be each 90°. At G and F the angular velocities of the cylinder will be v/AG and v/AF respectively.

Examining Fig. 531, it will be noted that the crank, rotating uniformly, takes a longer time to traverse the arc DGE than it does to traverse the arc EFD. Hence the average angular velocity of the cylinder towards the left will be less than that towards the right. Inspection of the angular-velocity diagram will illustrate the same point.

Quick-return motions. Advantage is taken of these facts in the quick return motion fitted sometimes to shaping machines (Fig. 532).

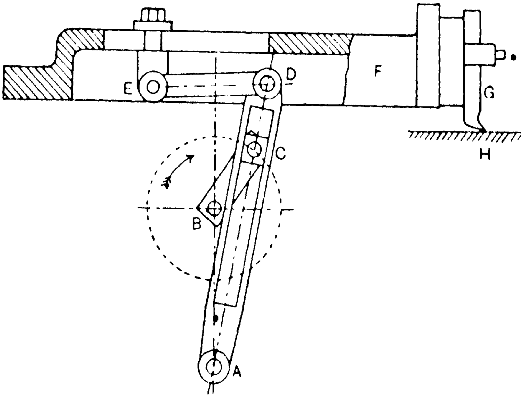


FIG. 532.—Quick return motion for a shaping machine.

The tool G cuts the work H on the stroke towards the right only, and it is advantageous that this cutting stroke should be executed at a slower speed than that of the idle return stroke. The tool is fixed to a sliding ram F, guided so as to move horizontally. F is connected by a short link ED to the top of a slotted bar AD, which may oscillate about A. AD is driven by means of a uniformly-rotating crank BC, the crank pin of which engages a block at C which may slide in the slot of AD. The tool will be at the ends of its travel when BC is in the positions BK and BL, both of which are at 90° to AD (Fig. 533).

Neglecting the effect of the obliquity of DE (Fig. 532), the travel of the tool may be found thus

Half travel of tool = DO (Fig. 533).

Also,
$$\frac{DO}{BK} = \frac{AD}{AB};$$

$$\therefore\ DO = \frac{BK \cdot AD}{AB} = \frac{AD}{AB} \cdot R,$$

and
$$\text{Travel of tool} = d = 2\frac{AD}{AB} \cdot R,$$

where R is the radius of the crank BC.

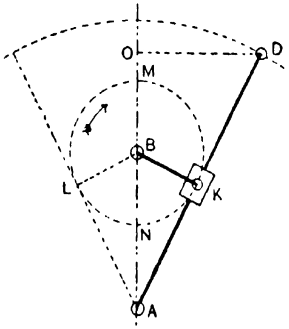


FIG. 533.—Quick return motion; mechanism at the ends of the travel.

The average speed on the cutting and return strokes may be calculated by first finding the times taken by the crank to traverse the arcs LMK and KNL respectively.

Let T = time of 1 revolution of crank in seconds.

t_c = time to traverse arc LMK, in seconds.

t_r = time to traverse arc KNL, in seconds.

Then $t_c : t_r : T$ = arc LMK : arc KNL : $2\pi R$.

$$t_c = \frac{\text{arc LMK}}{2\pi R} \cdot T.$$

$$t_r = \frac{\text{arc KNL}}{2\pi R} \cdot T.$$

Also,
$$\text{Average cutting speed} = \frac{d}{t_c}.$$

$$\text{Average return speed} = \frac{d}{t_r}.$$

The maximum cutting and return speeds may be obtained easily from Figs. 534 (*a*) and (*b*) respectively.

$$\frac{V_C}{v} = \frac{AD}{AM}; \quad \therefore V_C = v\frac{AD}{AM}.$$

And $$\frac{V_R}{v} = \frac{AD}{AN}; \quad \therefore V_R = v\frac{AD}{AN}.$$

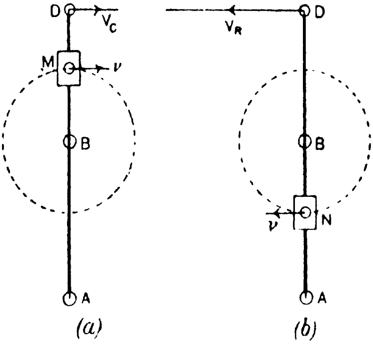


FIG. 534.—Quick return motion; maximum cutting and return speeds.

The **Whitworth quick-return motion** (Fig. 535) is produced by another inversion of the slider-crank chain. A slotted link CD revolves on an axis at C, and is connected to the ram of the shaping machine by the rod DK. Motion is given to CD by means of a crank AB revolving round an axis at B, its crank pin A has a block bearing which slides in the slot of CD. Inspection of the outline diagram (Fig. 536) will show that BC is the crank in the slider crank-

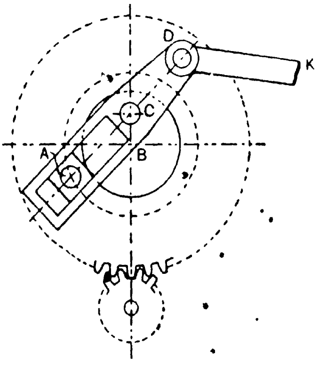


FIG. 535.—Whitworth quick-return motion.

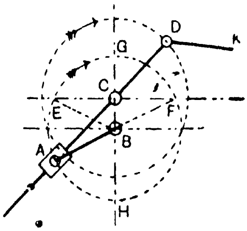


FIG. 536.—Outline of Whitworth quick-return motion.

chain, and now is fixed; AB was formerly the connecting rod, and ACD was the line of stroke.

The travel of the tool will be twice CD, and the tool will be at the ends of the stroke when CD is passing through its horizontal positions CE and CF. The arc through which BA turns during the cutting stroke is FHE, and the arc during the return stroke is EGF. The average speeds may be calculated in the same manner as for the quick-return motion discussed above. The maximum speed during the cutting stroke will occur when A is at H, and that during the return stroke will occur when A is at G.

Let v = velocity of A, assumed uniform.

V_C = maximum cutting velocity.

V_R = maximum return velocity.

Then $$\frac{V_C}{v} = \frac{CD}{CH}, \text{ or, } V_C = \frac{CD}{CH} \cdot v. \quad \ldots\ldots\ldots \quad (1)$$

Also, $$\frac{V_R}{v} = \frac{CD}{CG}, \text{ or, } V_R = \frac{CD}{CG} \cdot v. \quad \ldots\ldots\ldots \quad (2)$$

Comparison of (1) and (2) shows that

$$V_C : V_R = CG : CH.$$

Cams. Cams are employed when the reciprocating motion to be given to the end of a rod or lever is of an irregular character. In Fig. 537 (*a*) the rod OA reciprocates vertically in the line OA, and is driven by a disc B fixed to a revolving shaft C. The rim of B is shaped so as to give the required motion to the rod. In Fig. 537 (*b*) is shown another example, in which the direction of motion of OA does not pass through the axis of the shaft C.

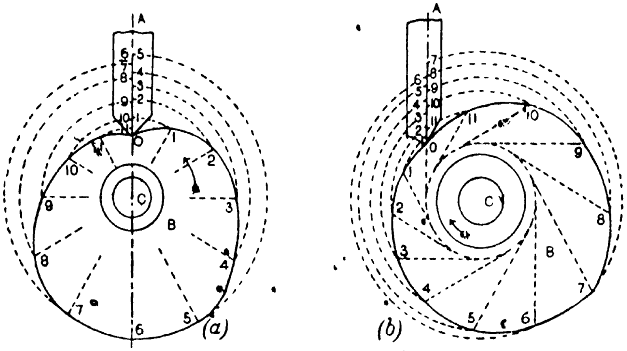


FIG. 537.—Two examples of cams.

The outline of the cam in Fig. 537 (*a*) is drawn by setting off equal angles oC1, 1C2, 2C3, etc., and marking off the distances along OA, through which the rod is to travel while the cam is describing these equal angles. The points of division on OA are then brought to the corresponding radii by means of arcs described with C as centre. The outline in Fig. 537 (*b*) is obtained by first drawing a circle with centre C and touching the line AO produced. Tangents are then drawn to this circle at equal angular intervals, and the distances along OA to be described during each of these intervals are brought to the corresponding tangent by means of arcs drawn with centre C.

EXERCISES ON CHAPTER XIX.

1. A link AB, 2 feet long, is horizontal at a given instant and is moving in a vertical plane. The velocity of the left-hand end A is 10 feet per second upwards to the right at 45 degrees to AB. The velocity of B relative to A is 4 feet per second downwards. Find the actual velocity of B.

2. In Question 1, find the instantaneous centre of the rod and the velocity of the centre of the link.

3. In a crank and connecting-rod mechanism, the crank CB is 1 foot long and the connecting rod AB is 4 feet long. The velocity of the crank pin B is uniform and equal to 10 feet per second Divide the crank circle into intervals of 30 degrees, and find the velocity of the crosshead A for each of the crank positions so determined. Draw diagrams of velocity for a complete revolution (*a*) on a time base, (*b*) on a space base, of length to represent twice the stroke of the crosshead.

4. In a double crank and connecting rod (four-bar chain) the cranks are AB, 1·5 feet long, and CD, 2·5 feet long; the connecting link BC is 2 feet long; the bar AD is fixed and is 2·25 feet long. B has a linear velocity of 2 feet per second. For the position in which AB makes 60 degrees with AD, find (*a*) the angular velocity of AB, (*b*) the velocity of C, (*c*) the angular velocity of DC Use the instantaneous centre method, and check the result for (*c*) by the ratio given on p 460.

5. A four-bar chain has cranks AB and DC 2 and 1·75 feet long respectively; the connecting link BC is 1 5 feet long and the bar AD is fixed horizontally and is 1·5 feet long. The cranks are crossed. Draw the mechanism when AB makes 45 degrees with AD, and find a point E in the link BC (produced if necessary) which is moving in a vertical direction in this position of the mechanism.

6. In the parallel motion shown in Fig. 498, AC is $1\frac{1}{8}$ inch and BC is $\frac{1}{2}$ inch long. The line joining AB is horizontal and $1\frac{5}{8}$ inch long. AD is 1·5 inch long and is vertical when AC is horizontal. Find the length of CP, and confirm the result by drawing for the positions when AC makes 15 degrees with AB.

7. In the mechanism shown in Fig. 502, the crank is 3 inches long and has a uniform speed of 60 revolutions per minute. The mass of the reciprocating parts is 80 pounds. Find the forces in lb. weight required to overcome inertia for crank positions differing by 30 degrees throughout the revolution. Plot these forces on a base line representing twice the stroke.

8. In Question 7 the steam cylinder driving the mechanism is 5 inches in diameter; the net pressure of the steam urging the piston is 50 lb. per square inch up to half stroke, and is 35, 24 and 20 lb. per square inch at 0·6, 0·8 and the end of the stroke respectively. Draw the pressure diagram, and find the turning moment on the crank for each position given in Question 7, making allowance for inertia. Draw the turning-moment diagram.

9. Two parallel shafts, 30 inches axis to axis, have each a crank 6 inches long connected by a uniform steel rod 30 inches long, 2 inches deep and 1 inch wide. The shafts are driven at equal speeds. Take the density of steel to be 0·28 pound per cubic inch, and find the maximum upward and downward forces due to the inertia and weight of the rod when the speed is 150 revolutions per minute.

10. In Question 9, the stress due to bending of the rod is limited to 10,000 lb. per square inch. What is the maximum permissible speed of the shafts in revolutions per minute?

11. Take the data of Question 3 and find the acceleration of the crosshead in the line of the stroke when the crank has travelled 60 degrees from the inner dead point. Do this by use of Klein's construction. Calculate also the accelerations when the crosshead is at each end of the stroke. The mass of the reciprocating parts is 500 pounds; calculate the forces required to overcome their inertia in these positions.

12. In Question 11, the total effective steam pressure on the piston when the crank is 60 degrees from the inner dead point is 9000 lb. What will be the turning moment on the crank, (*a*) neglecting inertia, (*b*) taking account of inertia?

13. With the data of Question 3, find the acceleration image of the connecting rod when the crank is at 90 degrees to the connecting rod, and hence find the acceleration of the centre of the rod.

14. In Question 13, take the connecting rod to be of uniform section and of mass 3 pounds per inch length. Find the resultant force which must act on the rod in order to overcome its inertia.

15. Draw the bending-moment diagram for the connecting rod as given in Question 14, and state the value of the maximum bending moment.

16. In an oscillating-engine mechanism, the crank is 2 feet long and makes 50 revolutions per minute. The distance between the centre of the crank shaft and the cylinder trunnion is 5 feet. Find the angular velocity of the cylinder when the crank is passing each dead point. Answer the same when the crank is at 45 degrees from the outer dead point.

17. In Question 16, if the crank rotates clockwise, find the time of swing of the cylinder (*a*) from left to right, (*b*) from right to left. Calculate the angle through which the cylinder oscillates.

18. The axis of a vertical rod passes through the axis of a horizontal shaft when produced downwards. The rod has a roller 1 inch diameter at its lower end and is driven vertically by a cam fixed to the shaft. The minimum radius of the cam is 2 inches, and it gives simple harmonic motion to the rod during the upward travel of 2 inches, followed by a period of rest while the shaft rotates through 45 degrees. The downward travel is simple harmonic. Draw the outline of the cam.

19. Answer Question 18, supposing that the axis of the rod passes 0·5 inch from the shaft axis when produced.

20. Describe the character of the straining actions to which the coupling-rod of a locomotive engine is subject, and sketch an appropriate form of transverse section. In a locomotive having driving wheels of 6 feet 6 inches diameter, the coupling rod is 8 feet long between its centres, and is attached to cranks of 1 foot radius. Suppose the locomotive to travel at 60 miles an hour, and the weight W of the coupling-rod to be uniformly distributed along the 8 feet of its length, estimate the maximum bending moment to which it will be subjected. (I.C.E.)

21. Describe, without proof, a construction for determining the acceleration of the slider in the slider-crank mechanism. Apply the construction to find the acceleration of the piston of an ordinary direct-acting engine when the crank is 30° from the inner dead centre. Length of crank, 8 inches. Length of connecting rod, 36 inches. Speed of crank shaft, 200 revolutions per minute. State the answer in feet per second per second. (L.U.)

22. In a slider-crank chain AB is the connecting rod, 30 inches long, BC the crank and AC the horizontal line of stroke. In AB produced beyond B a point P is taken, BP being 18 inches. If the locus of P is an approximately vertical straight line, while AB travels through angles from 0° to 30° with the line of stroke, find a suitable length for BC. A load of 2000 pounds at P acts at right angles to the line of stroke; find the pressure on the crosshead required to equilibrate, and find also the thrusts on the guides and crank when BAC = 30°. (L.U.)

23. In a four-bar chain ABCD, AB is the driving, CD the driven crank, and BC the coupler, DA being fixed. BC, produced if necessary, cuts AD in P. Show that the ratio of the angular velocity of CD to that of AB is PA/PD. Draw the velocity diagram for this chain when AB, BC, CD and DA are 1, 6, 3 and 7 feet respectively, the angle BAD being 90° and AB and CD being on the same side of AD. If the velocity of B is 1 foot per second, find the velocity of C, and check by using the ratio given above. (L.U.)

24. A simple slide valve driven by an eccentric has a travel of 5 inches. The cut-off is at $\frac{5}{8}$ of the stroke of the piston, and the release takes place at $\frac{7}{8}$ of the stroke. The lead is $\frac{1}{4}$ inch. Assuming that the valve and piston have simple harmonic motions, find the outside and inside laps of the valve and the position of the piston when compression begins. (L.U.)

25. A connecting rod is 5 feet long and 5 inches in diameter, assumed uniform throughout its length. Stroke of piston, 2 feet 6 inches. Revolutions per minute, 180. Weight of material, 480 lb. per cubic foot. When the crank angle is 60° measured from the inner dead centre, draw the load and bending-moment curves on the connecting rod due to its inertia, and state the value of the maximum bending moment. (L.U.)

CHAPTER XX.

FLYWHEELS. GOVERNORS.

Fluctuations in angular velocity. It frequently is the case tha it is important to secure uniformity of angular velocity in some shaf belonging to a machine. This condition is usually very desirable i engines supplying motive power. In such cases there may be tw kinds of disturbance owing to lack of equality in the rates of suppl of energy to the engine and of abstraction of energy for drivin purposes. In any machine we have the following equation (p. 328 for the balance of energies during a stated time:

Energy supplied = energy abstracted + energy wasted in the machine.

Suppose the energy supplied exceeds that required in order t satisfy the above equation, then the machine must be increasing its speed, as the excess energy must be disposed of, and the only method available is by the storage of additional kinetic energy in the parts of the machine. The converse will be the case if the energy supplied falls below that required in order to satisfy the equation.

For simplicity, suppose the moment of resistance to rotation of the engine shaft, supplied by the machinery to be driven, to be uniform. There will be a demand then for a constant amount of energy during each revolution of the shaft. But the rate of supply of energy to the shaft is never uniform, depending as it does on the turning-moment diagram (p. 470), which may show great lack of uniformity. The result would be evidenced in rapid alterations in the angular velocity of the shaft, a jerky action which it is the function of the flywheel to remedy. This the flywheel does by storing the excess energy in the kinetic form, and its large moment of inertia enables it to do so with comparatively small changes in its speed. It is evident that, if the energy supplied during the revolution is exactly sufficient to satisfy the equation, the angular velocity of the flywheel at the end of the revolution will be equal to that at the beginning, *i.e.* there is no net gain or loss of speed despite intermediate small fluctuations.

A second kind of variation in the angular velocity may occur during a period extending over several revolutions of the shaft. This would be owing to the supply of working fluid to the engine being too large or not enough, and would be evidenced by a steady rise or fall in the speed of rotation. The flywheel alone is incompetent to deal with this matter, which must be remedied by a contrivance called a **governor.** The governor is driven by the engine, and is constructed so that the relative positions of its parts alter with change of speed. These movements may be made to operate valves which control the supply of working fluid, and thus the shaft is kept rotating at a speed which may vary only slightly above and below the mean speed. Absolute steadiness of speed cannot be attained, for change of speed must occur before the governor will move into another position, and so operate the control valve.

Kinetic energy of a flywheel. Some calculations regarding the capacity for energy of a given wheel and of its change of angular velocity in giving up a stated amount of energy will be found on p. 420.

Let I = the moment of inertia of the wheel, pound and foot units.
ω = its angular velocity, in radians per second.

Then Kinetic energy of wheel $= \frac{\omega^2}{2g} I$ foot-lb......... ..... (1)

Let the wheel change its speed from N_1 to N_2 revolutions per minute. Then

$$\text{Change in kinetic energy} = \frac{\omega_1^2}{2g} I - \frac{\omega_2^2}{2g} I.$$

Also,
$$\omega_1 = \frac{N_1}{60} . 2\pi = \frac{\pi N_1}{30},$$

$$\omega_2 = \frac{\pi N_2}{30}. \text{ Hence,}$$

$$\text{Change in kinetic energy} = \frac{I}{2g}\left(\frac{\pi^2 N_1^2}{900} - \frac{\pi^2 N_2^2}{900}\right)$$

$$= \frac{\pi^2 I}{1800g}(N_1^2 - N_2^2)$$

$$= 0{\cdot}00548 \frac{I}{g}(N_1^2 - N_2^2) \text{ foot-lb....(2)}$$

Again, the kinetic energy at any speed N revolutions per minute varies as N^2; hence, if **M** be the kinetic energy at one revolution per minute, the kinetic energy at N revolutions per minute will be $\mathbf{M}N^2$. Hence, the change in kinetic energy in passing from N_1 to N_2 revolutions per minute may be written:

$$\text{Change in kinetic energy} = \mathbf{M}(N_1^2 - N_2^2). \quad \ldots\ldots\ldots\ldots (3)$$

The **fluctuation in speed** of the wheel may be defined as $(N_1 - N_2)$, and the **coefficient of fluctuation of speed** is taken as the ratio which the fluctuation in speed bears to the mean speed. It is sufficiently accurate to write

$$\text{Mean speed} = \tfrac{1}{2}(N_1 + N_2).$$

Hence, $\text{Coefficient of fluctuation of speed} = \dfrac{N_1 - N_2}{\frac{1}{2}(N_1 + N_2)}.$

In practice, values of the coefficient of fluctuation of speed are found varying from 0·05 to 0·008 depending on the type of machinery.

Dimensions of an engine flywheel. In estimating the dimensions of a flywheel for an engine, sufficient information must be given or assumed to enable the fluctuation of energy during a complete cycle to be ascertained. The process consists in reducing the driving effort on the piston to a tangential force acting on the crank pin, making proper allowance for the inertia of the reciprocating parts of the engine. A **crank-effort diagram** showing the values of this force throughout a cycle is drawn. Another diagram is drawn on the same base, showing the driving resistances to be overcome reduced to another tangential force acting at the crank pin. Comparison of these diagrams will enable the fluctuation of energy to be obtained.

The turning-moment diagram may be used in order to calculate R, the tangential force acting on the crank pin. Thus,

$$T = R \times BC,$$

$$\therefore\ R = \frac{T}{BC}\ \text{lb.},$$

where BC is the length of the crank in feet and T is the turning moment in lb.-feet.

Values of R for a steam engine having a single cylinder are set off in Fig. 538 on a base having a length equal to the circumference of the crank-pin circle. The resulting crank-effort diagram OCBDA shows the work done on the crank pin per revolution, neglecting friction.

The whole of the work done on the crank pin is utilised in overcoming (*a*) frictional resistances in the engine, (*b*) the external resistances which are opposed by the machinery being driven.

Assuming both of these to be uniform when reduced to a tangential force at the crank pin, it is evident that both will be taken account of by constructing a rectangular diagram OEFA of height equal to the average height of the crank-effort diagram. This rectangle will express graphically the equality of energy supplied and energy abstracted during the revolution. The driving effort and the resistance to be overcome are equal at G, H, K and L.

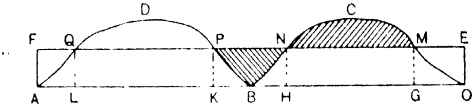


FIG. 518.—Fluctuation in energy in a steam engine.

Consider the portions of the diagram showing the energies while the crank pin travels through the arc represented by GH. Work is done on the crank to an amount represented by the area GMCNH, and the work abstracted is represented by the area GMNH. Hence, surplus work, represented by the area MCN, has been done, with the result that the flywheel will have its angular velocity increased while the crank is passing from G to H.

In the same way, while the crank is passing from H to K, energy represented by the sum of the areas HNB and BPK has been given to the crank and energy represented by the rectangle HNPK has been abstracted. Insufficient energy, represented by the area NBP, has been supplied during this interval and the flywheel will be decreasing its angular velocity. Hence, maximum speed will occur when the crank is at H and minimum speed when the crank is at K. Assuming that the speeds of the flywheel at G and K are equal, it follows that the excess energy represented by MCN will be equal to the deficient energy NBP, and it may be said that the energy represented by the area MCN has been given to the flywheel and taken away again while the crank is passing from G to K. The fluctuation of energy is therefore given by either of the equal areas MCN or NBP. In the same way, the area FDQ is equal to the sum of the areas QAF and EMO, and represents the fluctuation of energy during the remainder of the revolution.

The **coefficient of fluctuation of energy** may be defined as the ratio of the maximum fluctuation in energy to the total work done in one cycle, the cycle occupying one revolution in a steam engine and two revolutions in an internal combustion engine working on the four-stroke cycle.

Let E = the area MCN, expressed in foot-lb., representing the maximum fluctuation.

ω_1 and ω_2 = the angular velocities of the flywheel at H and K respectively, in radians per sec.

I = the moment of inertia of the wheel, in pound and foot units.

Then
$$\frac{\omega_1^2}{2g}I - \frac{\omega_2^2}{2g}I = E,$$

$$\frac{I}{2g}(\omega_1^2 - \omega_2^2) = E. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let
$$\omega_2 = n\omega_1,$$
where n is a fraction expressing the minimum permissible ratio of ω_2 to ω_1. Then
$$\frac{I}{2g}(\omega_1^2 - n^2\omega_1^2) = E,$$

$$\frac{\omega_1^2 I}{2g}(1 - n^2) = E,$$

or
$$I = \frac{2gE}{\omega_1^2(1 - n^2)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The moment of inertia which the flywheel must possess may be calculated from this equation.

Centrifugal tension in flywheels. In Fig. 539 is shown the rim of a revolving flywheel, the other parts of the wheel being disregarded in what follows. The centrifugal forces produce radial loads on the rim of a kind similar to those produced by internal pressure on a cylindrical shell (p. 94).

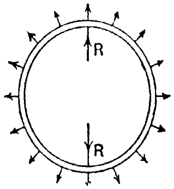


FIG. 539.—Rim of a revolving flywheel

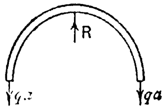


FIG. 540

Let v = the velocity of the rim, in feet per second.

r = the mean radius of the rim, in feet.

m = the mass of the rim, in pounds per foot circumference.

Then Centrifugal force per foot circumference $= \frac{mv^2}{gr}$ lb.

The resultant centrifugal force for half the wheel, corresponding to $(p \times d)$ in the cylindrical shell, will be

$$R = \frac{mv^2}{gr} \times 2r = \frac{2mv^2}{g} \text{ lb.}$$

Let a = sectional area of the rim, in sq. inches.
q = tensile stress on a, lb. per sq. inch.

Then, assuming q to be distributed uniformly,

$$R = 2qa \quad \text{(Fig. 540)},$$

or
$$\frac{2mv^2}{g} = 2qa,$$

$$q = \frac{mv^2}{ga} \text{ lb. per sq. inch.}$$

Let ρ = the density of the material, in pounds per cubic foot.

Then $m = \rho \times 1 \times \frac{a}{144}$ pounds per foot circumference,

and
$$q = \frac{\rho a v^2}{144 g a}$$

or
$$= \frac{\rho v^2}{144 g} \text{ lb. per sq. inch.} \quad \ldots\ldots \quad (1)$$

This result shows that the stress due to centrifugal force is independent of the sectional area of the rim and of the radius of the wheel. Equation (1) may be written

$$v^2 = 144 g \frac{q}{\rho},$$

$$v = 12\sqrt{g}\sqrt{\frac{q}{\rho}}. \quad \ldots\ldots \quad (2)$$

We may deduce from this result that, for a given material having a density ρ, there is a maximum speed of rim corresponding to a safe stress q for the material in question, and that this speed is independent of the dimensions of the wheel.

Governors. In Fig 541 is shown a simple type of governor such as was used by James Watt for controlling the speed of steam engines. Two heavy revolving masses A_1 and A_2 are suspended by links A_1B and A_2B to the upper end of a shaft BF; the joints at B permit of A_1 and A_2 moving outwards or inwards in circular arcs. Another pair of links A_1C_1 and A_2C_2 connect A_1 and A_2 to a sleeve D, which will move upwards if A_1 and A_2 move to a larger radius. The sleeve is connected by means of a bent lever, pivoted at G, and a rod H to a throttle valve K, which is situated in the pipe supplying steam to the engine and controls the supply of steam. The shaft BF is driven by the engine by means of bevel wheels at F, and hence the masses A_1 and A_2 will revolve about the axis of BF. The action of the centrifugal force, the weight, and the pull of the links on each revolving mass, will cause the mass to take up a definite radius depending on the engine speed. The working positions of the revolving masses are settled by the considerations that when they are at the extreme outer or inner working radius, the throttle valve should be closed completely or opened fully respectively. Each of these radii will correspond to a definite speed of rotation, and the engine controlled by the governor will be capable of a range of speed between these limits. In order that the range of speed should not be too great, the difference between the extreme working radii of the revolving masses should not be too large.

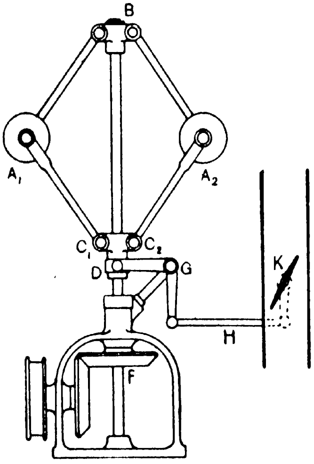


FIG. 541.—Simple unloaded governor

In Fig. 542, three forms of simple governor are shown in outline; these differ merely in the position of the point of suspension of the

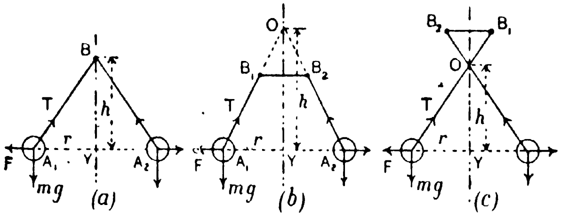


FIG. 542.—Forms of simple governors.

upper links. In (*a*), the joint B is on the axis of rotation; in (*b*), the joints B_1 and B_2 are outside the axis, and are situated at the ends of a short cross piece B_1B_2 which is fixed to the shaft; the same arrangement is used in (*c*), but the links are open in (*b*) and are crossed in (*c*). The following argument applies equally to each of these cases:

Let ω = the angular velocity of the governor shaft, in radians per sec.

m = the mass of either A_1 or A_2, in pounds.

r = the radius in feet of the revolving masses, corresponding to ω.

h = the height in feet of the cone of revolution described by the links and shown by YB in (*a*) and by YO in (*b*) and (*c*).

T = the pull in each link.

Considering one revolving mass, it will be in equilibrium under the action of three forces, viz. its weight mg poundals, the centrifugal force $\omega^2 mr$ poundals, and the pull T. It is evident that A_1YB in (*a*) and A_1YO in (*b*) and (*c*) will be the triangle of forces for these forces in equilibrium. Hence,

$$\frac{F}{mg}=\frac{r}{h}, \text{ or, } \frac{\omega^2 mr}{mg}=\frac{r}{h},$$

$$\omega^2=\frac{g}{h}, \qquad \text{.......................} (1)$$

and

$$h=\frac{g}{\omega^2}. \qquad \text{..........} (1')$$

This result neglects the effects of the mass of the link and also friction, and shows that the height is independent of the weight of the revolving masses.

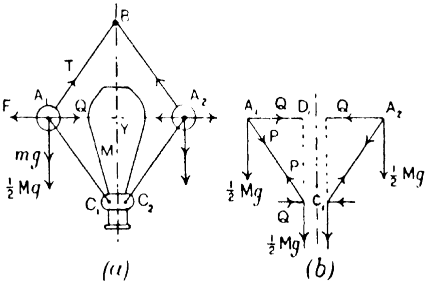
Such governors can be used for low speeds only. For example, if $\omega = 4\pi$ radians per second, corresponding to 120 revolutions per minute, h would be 0·2 foot nearly, a height which is not practicable. Running at low speeds, comparatively small forces will be available for operating the throttle valve unless the revolving masses are made heavy. Accordingly, simple governors usually have revolving masses of large dimensions, and are run at speeds rarely exceeding 60 revolutions per minute.

Loaded governor. The speed may be increased and the revolving masses kept small by the addition of a load on the sleeve (Fig. 543 (*a*)). If M be the mass of this load, half its weight Mg will be carried by each pin C_1 and C_2. C_1 is in equilibrium under the action of the three forces $\frac{1}{2}Mg$, the pull P in C_1A_1 and a horizontal force Q supplied by the sleeve (Fig. 543 (*b*)). The pull P is transmitted to A_1 by the link, and applies a force P to the revolving mass which may be resolved into a vertical force $\frac{1}{2}Mg$ and a horizontal force Q. If all four links be equal, the triangle of forces $A_1D_1C_1$ will give

$$\tfrac{1}{2}Mg : Q = h : r,$$

where h and r are equal respectively to BY and A_1Y in Fig. 543 (*a*).

Hence,
$$Q = \frac{Mgr}{2h}.$$



(*a*) (*b*)

FIG. 543.—Loaded Porter governor.

The revolving mass A_1 is now subjected to three forces, viz. T, the resultant $(F - Q)$ of the centrifugal force F and Q, and the resultant $(mg + \frac{1}{2}Mg)$ of the weights (Fig. 543 (*a*)). A_1BY will be the triangle of forces. Hence,

$$\frac{F - Q}{mg + \frac{1}{2}Mg} = \frac{r}{h},$$

or
$$\frac{\omega^2 mr - \dfrac{Mgr}{2h}}{mg + \frac{1}{2}Mg} = \frac{r}{h};$$

$$\therefore \ \omega^2 mh - \tfrac{1}{2}Mg = mg + \tfrac{1}{2}Mg,$$

$$\omega^2 = \frac{(M + m)}{m} \cdot \frac{g}{h} \qquad \ldots(2)$$

$$= \left(\frac{M}{m} + 1\right) \cdot \frac{g}{h}, \qquad \ldots(2')$$

$$h = \left(\frac{M}{m} + 1\right) \cdot \frac{g}{\omega^2}. \qquad \ldots(2'')$$

Comparison of these results with equations (1) and (1') for the simple governor will show that, for the same value of ω, h in the loaded governor will be greater than that for the simple governor in the ratio of $\left(\frac{M + m}{m}\right)$.

Equation (2″) for the loaded governor shows that h will be made larger or smaller by an increase or diminution of M without alteration in the values of m and ω. The arrangement therefore admits of adjustment of the working radii of the revolving masses by means of varying the load.

Frictional effects in the governor mechanism may be taken into account by the artifice of eliminating the frictional forces and applying instead a force to the sleeve, which will have the same effect. This force must be applied always of sense opposite to that of the direction of motion of the sleeve. The effect might be produced by imagining a mass M_1 pounds to be added to the load M if the sleeve is rising, and to be abstracted if the sleeve is falling. Equation (2) then becomes:

$$\omega^2 = \left(\frac{M \pm M_1 + m}{m}\right)\frac{g}{h} = \left(\frac{M \pm M_1}{m} + 1\right)\frac{g}{h},$$

or,

$$h = \left(\frac{M \pm M_1}{m} + 1\right)\frac{g}{\omega^2}, \quad \ldots \quad \ldots \qquad (3)$$

the positive sign being used if the sleeve is rising or attempting to rise, and the negative sign if the sleeve is falling or attempting to fall.

Two extreme values of h may thus be calculated from (3), indicating that, owing to friction, the governor may remain at any height intermediate between these extreme values while running at a given steady speed ω. The effect on the engine is to permit some variation in speed to occur before the governor will begin to respond by altering its height.

Effect of the governor arms. In Fig. 544, AB is a rod hinged at A and rotating about the vertical axis AK. Centrifugal force and gravity will compel the rod to assume an angle α to the vertical, the value of α depending on the speed of rotation. Steady conditions will be obtained when the total moment of gravity about A is equal to the total moment of the centrifugal force. Consider a small portion of the mass of the rod at P, and let the rod be uniform.

Let m = the mass of the small portion, in pounds.

r = the radius of the small portion, in feet.

y = its distance from A, in feet.

M = the total mass of the rod, in pounds.

Y = the distance of the centre of mass G from A in feet.

L = the length of the rod, in feet.

R = the radius of B, in feet.

H = the vertical height of A over B, in feet.

Then, taking moments about A of the forces acting on the small portion, we have

$$mg \times NP = m\omega^2 r \times AN,$$

or

$$mg \times y \sin \alpha = m\omega^2 y \sin \alpha \times y \cos \alpha, \quad \ldots (1)$$

$$g \,.\, my = \omega^2 \cos \alpha \,.\, my^2.$$

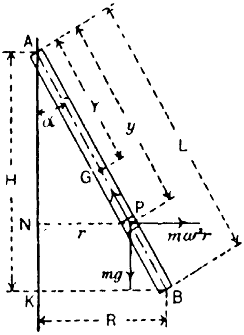


Fig. 544.—A revolving uniform load

The total moments for the whole rod will be obtained by integrating both sides of this equation, giving

$$g \int_0^I my = \omega^2 \cos \alpha \int_0^I my^2,$$

$$gM \frac{L}{2} = \omega^2 \cos \alpha I_A,$$

where I_A is the moment of inertia of the rod with respect to A, viz. $\frac{1}{3}ML^2$ (p. 415). Hence,

$$\cos \alpha = \frac{3}{2} \frac{g}{\omega^2 L}. \quad (2)$$

This result determines the required relation of α and ω for a given uniform rod.

In the actual governor the arm is constrained by the action of the revolving mass to rotate at an angle to the vertical, differing from that given for a free arm in (2) above. In this case, the moment of the weight of the arm may be calculated still as above by imagining that the whole arm is concentrated at the centre of gravity. The moment of the centrifugal force may be calculated by first imagining that the whole arm is concentrated at the centre of mass and calculating the centrifugal force f produced thereby. Then find the position of f (Fig. 545) in order that its moment may agree with the integrated result of the right-hand side of (1). Thus,

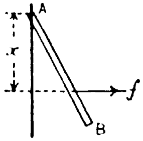


Fig. 515

$$f = M\omega^2 \frac{R}{2}.$$

$$\text{Moment of } f = fx = M\omega^2 \frac{R}{2} x = \omega^2 \sin \alpha \cos \alpha \int_0^I my^2$$

$$= \omega^2 \sin \alpha \cos \alpha \frac{ML^2}{3}$$

$$= \omega^2 \frac{R}{L} \cdot \frac{H}{L} \cdot \frac{ML^2}{3}; \quad \text{(see Fig. 544)}$$

$$\therefore \quad \omega^2 \frac{MR}{2} x = \omega^2 \frac{RHM}{3},$$

$$x = \tfrac{2}{3} H. \quad \ldots (3)$$

Effect of the arms in a loaded governor. This result may be applied to a **loaded Porter governor** having equal arms (Fig. 546 (*a*)). mg is the weight of each arm, and in the case of AB is equivalent to a force $\frac{1}{2}mg$ at A and an equal force at B (Fig. 546 (*b*)). f for each arm is equivalent to a force $\frac{2}{3}f$ acting at B, together with a force $\frac{1}{3}f$ acting at the vertical spindle AC. The mass of the revolving mass being M, its weight will be Mg and the centrifugal force will be $M\omega^2R$.

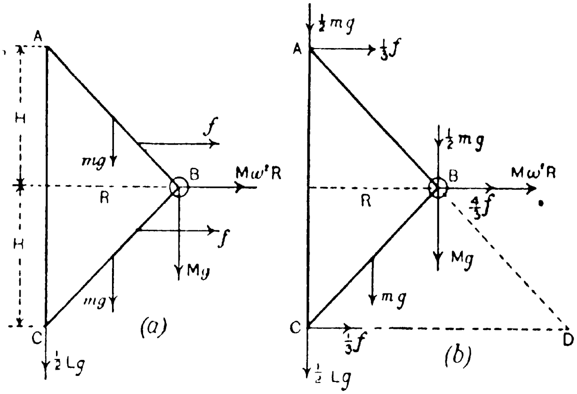


FIG. 546.—Forces in a loaded Porter governor, including effects due to the arms.

The mass of the load is L, and half its weight, viz. $\frac{1}{2}Lg$, will be borne by the right-hand arms as shown. The forces at A are balanced direct by the reactions of the pin securing AB to the spindle at A. The force $\frac{1}{3}f$ at C is balanced by the reaction on the sleeve produced by the spindle. Draw CD perpendicular to AC, and produce AB to cut CD at D (Fig. 546 (*b*)).

Take moments about D of the remaining forces, remembering that $f = m\omega^2\frac{R}{2}$.

$$(M\omega^2R + \tfrac{4}{3}f)H = (Mg + \tfrac{1}{2}mg)R + mg\tfrac{3}{2}R + \frac{Lg}{2}\cdot 2R,$$

$$\left(M\omega^2R + \tfrac{4}{3}m\omega^2\frac{R}{2}\right)H = MgR + \tfrac{1}{2}mgR + \tfrac{3}{2}mgR + LgR,$$

$$\omega^2(M + \tfrac{2}{3}m)H = Mg + 2mg + Lg,$$

or

$$H = \left(\frac{M + 2m + L}{M + \frac{2}{3}m}\right)\frac{g}{\omega^2}. \qquad \ldots\ldots(4)$$

Stability of a governor. Considering one revolving mass of a governor, the centrifugal force is given by

$$F = \omega^2mr. \qquad (1)$$

Suppose ω to be increased by a small amount $\delta\omega$, and that, in consequence, r increases by δr and F by δF. For the new position we have

$$F + \delta F = m(\omega + \delta\omega)^2(r + \delta r)$$
$$= m(\omega^2 r + 2\omega r.\delta\omega + \omega^2.\delta r), \quad \ldots (2)$$

by neglecting the square of $\delta\omega$, and also the term involving the product of the small quantities $\delta\omega$ and δr. Subtraction of (2) and (1) gives

$$\delta F = m(2\omega r.\delta\omega + \omega^2.\delta r). \quad \ldots\ldots(3)$$

Dividing (3) by (1), we have

$$\frac{\delta F}{F} = \frac{2\omega r.\delta\omega + \omega^2.\delta r}{\omega^2 r} = 2\frac{\delta\omega}{\omega} + \frac{\delta r}{r},$$

or

$$2\frac{\delta\omega}{\omega} = \frac{\delta F}{F} - \frac{\delta r}{r}. \quad \ldots\ldots (4)$$

If $\delta\omega$ is an increase in angular velocity, the left-hand side is positive; hence $\frac{\delta F}{F}$ must be greater than $\frac{\delta r}{r}$, *i.e.* the rate of increase of the centrifugal force must be greater than the rate of increase of the radius. The result expresses the condition of stability in a governor, *i.e.* the moving to a definite new radius and remaining there when the revolving masses suffer a change in speed.

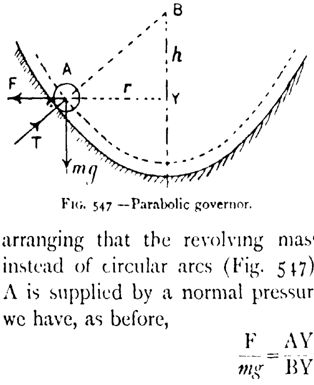


FIG. 547 --Parabolic governor.

An interesting example of a governor which exhibits neutral equilibrium is produced by arranging that the revolving masses move about B in parabolic instead of circular arcs (Fig. 547). Here the pull of the link on A is supplied by a normal pressure T given by the guide Hence we have, as before,

$$\frac{F}{mg} = \frac{AY}{BY} = \frac{r}{h},$$

or,

$$\frac{\omega^2 mr}{mg} = \frac{r}{h},$$

$$\omega^2 = \frac{g}{h}.$$

Now $h = $ BY is the subnormal to the dotted parabola, and it is known from geometry that the subnormal to a given parabola is constant; hence h is constant, and therefore ω must also be constant.

A governor of this type is isochronous, *i.e.* it will run at one speed only if friction be absent, and any change from this speed will send the revolving masses immediately to one or other extreme end of the range.

The question of sensitiveness of a governor is allied closely to its stability. The change of radius for a given fractional change in speed is large in a sensitive governor, but if too large, as in the parabolic governor, the stability may disappear.

Hartnell governor. In the **spring loaded Hartnell governor** (Fig. 548), the revolving masses A_1 and A_2 are supported by bent levers, which are pivoted at B_1 and B_2 on pins supported by a bracket (not shown in the illustration) which is fixed to and driven by the shaft. A spring E bearing on a sleeve D presses downwards the ends C_1C_2 of the bent levers. The revolving masses travel a small distance only from the verticals passing through B_1 and B_2, hence the effect of their weights in exercising control may be neglected. Supposing, for simplicity, that A_1B_1 and B_1C_1 are equal, then, by taking moments about B_1, we see that the centrifugal force F acting on A_1 will be equal to one half of the total force 2Q exerted by the spring. Provided that the adjustment of the spring is correct, this governor will possess great sensitiveness, but easily may be made unstable.

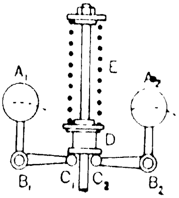


FIG. 548 Hartnell governor

Suppose that the revolving masses are displaced from r_1 to a slightly greater radius r_2 without changing the angular velocity ω, the centrifugal force will be increased to $F + \delta F$, and Q will be increased to $Q + \delta Q$, owing to the additional compression of the spring. Assuming that these forces are equal, we have

$$F + \delta F = Q + \delta Q,$$

or

$$m\omega^2 r_2 = Q + \delta Q.$$

Also, initially,

$$m\omega^2 r_1 = Q;$$

$$\therefore \; m\omega^2 (r_2 - r_1) = \delta Q.$$

Hence,

$$\frac{m\omega^2 (r_2 - r_1)}{m\omega^2 r_1} = \frac{\delta Q}{Q},$$

or

$$\frac{\delta r}{r} = \frac{\delta Q}{Q}.$$

The condition, therefore, that the governor may remain in the new position with the speed unaltered is that the rate of increase of the force in the spring is equal to the rate of increase of the radius of the revolving masses. This condition may be secured by adjusting the spring; but as the stability of the governor then would be neutral, the practical adjustment is made so as to disagree with the condition above expressed.

Effort of a governor. Suppose that the speed of a governor is increased from ω_1 to ω_2, and that the sleeve is held so as to prevent outward movement of the revolving masses. There will be additional centrifugal force, and consequently an effort will be exerted on the sleeve which may be utilised in overcoming the resistance offered by the control valve mechanism. It is evident that the effort will diminish if outward movement of the revolving masses be permitted, and will attain zero value when they reach the position corresponding with the new speed of rotation. The effort of the governor may be defined as the average effort exerted on the sleeve during a given change of speed executed in the manner described above, and may be taken as 0·5 of the maximum effort.

Taking a simple governor (p. 501) for which

$$\omega_1^2 = \frac{g}{h}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

if P is the maximum effort in lb., it may be imagined that P is produced by the weight of a load M, arranged as in the loaded governor (p. 502). Hence,

$$\omega_2^2 = \left(\frac{m + M}{m}\right)\frac{g}{h} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Hence, from (1) and (2),

$$\frac{\omega_2^2}{\omega_1^2} = \frac{(m + M)g}{mg},$$

or

$$\frac{\omega_2^2 - \omega_1^2}{\omega_1^2} = \frac{Mg}{mg},$$

$$Mg = \left(\frac{\omega_2^2 - \omega_1^2}{\omega_1^2}\right) mg. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let P = weight of M, in lb.

w = weight of m, in lb.

Then

$$P = \left(\frac{\omega_2^2 - \omega_1^2}{\omega_1}\right) w = \left(\frac{\omega_2^2}{\omega_1^2} - 1\right) w.$$

Let

$$\omega_2 = n\omega_1,$$

so that n expresses the fractional change in speed. Then

$$P = \left(\frac{n^2\omega_1^2}{\omega_1^2} - 1\right) w$$

$$= (n^2 - 1) w;$$

∴ effort of a simple governor = $\frac{1}{2}(n^2 - 1)w$ lb. . . . . . (4)

In the case of a loaded governor (p. 502), P may be taken as being equivalent to the weight of an additional mass M_1 applied to M. Similar reasoning to that employed for the simple governor will give

$$\text{Effort of a loaded governor} = (n^2 - 1)\left(\frac{w + W}{2}\right)\text{lb}, \ldots \quad (5)$$

where w and W are the weights of one revolving mass and of the load respectively in lb.

Balancing. The complete treatment of the principles of balancing the moving parts of an engine or other machine is beyond the scope of this book.* Reference will be made to some of the easier principles.

Two rotating bodies may be made to balance each other if both have their centres of mass in the same plane which is perpendicular to the axis of rotation, and if the centre of mass of the combined bodies is in the axis of rotation (p. 426).

Thus, in Fig. 549, m_1 and m_2 will balance, provided the forces F, F are equal and are in the same straight line. This will be the case if $m_1r_1 = m_2r_2$ and if the line joining G_1 and G_2 passes through the axis at right angles.

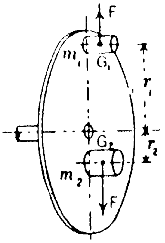


FIG. 549.—Balance of two revolving bodies

Three revolving bodies may balance, provided the resultant centrifugal force of two of them, F_1 and F_3 in Fig. 550, is equal and opposite to the centrifugal force, F_2, of the other. It is thus evident that all three centres of mass must be contained by the same plane

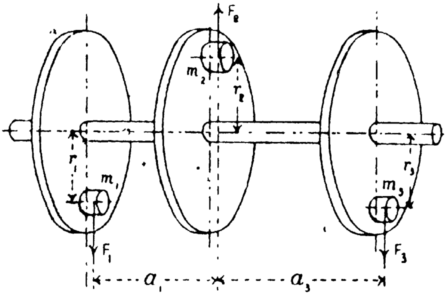


FIG. 550.—Balance of three revolving bodies

which also contains the axis of rotation, and that the bodies must be disposed as shown in Fig. 550.

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* For a complete discussion on this subject, the student is referred to *The Balancing of Engines*, by Prof. W. E. Dalby; (Arnold).

Taking dimensions as indicated in Fig. 550,

$$F_1 + F_3 = F_2 \qquad \ldots(1)$$

and

$$F_1a_1 = F_3a_3. \qquad (2)$$

These equations indicate the conditions of equilibrium to be fulfilled, and may be reduced thus:

From (1),

$$\omega^2 m_1r_1 + \omega^2 m_3r_3 = \omega^2 m_2r_2,$$

or

$$m_1r_1 + m_3r_3 = m_2r_2. \qquad \ldots(3)$$

This result shows that the centre of mass of the combined bodies falls on the axis of rotation.

From (2),

$$\omega^2 m_1r_1a_1 = \omega^2 m_3r_3a_3,$$

or

$$m_1r_1a_1 = m_3r_3a_3. \qquad (4)$$

This equation secures that there shall be no rocking couple set up.

In this case, there are two equations, (3) and (4), and eight quantities involved, hence six of these must be given or assumed

The balancing of four or more revolving masses is capable of many solutions, and graphical or semi graphical methods are best.

Locomotive balancing. As an illustration of the method by means of which the balancing of four revolving masses may be carried out, the following example of a locomotive should be studied.

Equal masses m_1 and m_2 are given (Fig. 551), rotating at equal radii r, r, and symmetrically disposed in relation to the wheels A and B in the planes of which **balance weights** are to be placed. The term "balance weights" is used to denote bodies which must be attached to the mechanism for the purpose of obtaining balance. a is the distance of m_1 from A and of m_2 from B; b is the distance of m_1 from B and of m_2 from A.

Balance the centrifugal force of m_1 separately by attaching balance weights to A and B at the same radius r, if these balance weights be represented by A_1 and B_1 respectively, then masses will be given by

$$A_1 + B_1 = m_1 \qquad (1)$$

and

$$A_1a = B_1b. \qquad (2)$$

From which,

$$A_1 = \frac{m_1b}{a+b} \qquad \ldots(3)$$

and

$$B_1 = \frac{m_1a}{a+b}. \qquad \ldots(4)$$

In the same way, balance F_2 by attaching balance weights to A and B at the same radius r. Let these be A_2 and B_2 respectively. Then it is evident from symmetry that A_1 and B_2 are equal; also A_2

is equal to B_1. These balance weights are shown in the elevations of the wheels in Fig. 551; the views are taken in the directions of the arrows c and d shown in the plan. Find the centres of gravity of A_1A_2 and also of B_1B_2 by joining their centres and dividing the distances in G_A and G_B so that

$$A_1 : A_2 = A_2G_A : A_1G_A$$

and

$$B_1 : B_2 = B_2G_B : B_1G_B.$$

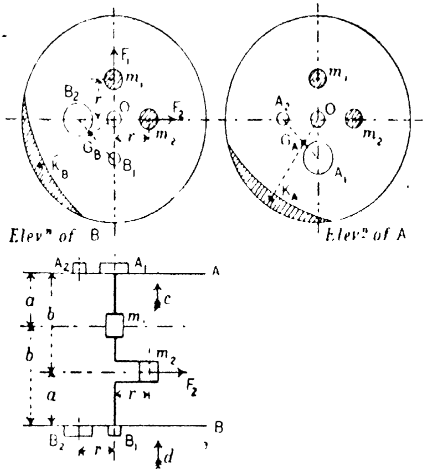

FIG. 551.—Balancing in an inside cylinder locomotive.

Now, since $A_1 + B_1 = m_1$ and $B_1 = A_2$, it follows that $A_1 + A_2 = m_1$, and for a similar reason $B_1 + B_2 = m_2$. Hence, if instead of A_1, A_2, B_1, B_2, a mass equal to m_1 be placed at G_A, and if another equal to m_2 be placed at G_B, the four masses will be in balance. Or the ordinary practical solution may be obtained by applying balance weights having their centres of mass in OG_A and OG_B produced. Let M_A and M_B be their masses respectively. Then balance will be secured if

$$M_A \times K_AO = m_1 \times G_AO$$

and

$$M_B \times K_BO = m_2 \times G_BO.$$

M_A and M_B will be equal, provided their radii K_AO and K_BO are equal.

Graphical solution of balancing problems. This solution depends on the principles that the centrifugal forces must not produce (*a*) a resultant force; (*b*) a resultant couple. Reference is made to Fig. 552.

Since the angular velocities of all the bodies are equal, the centrifugal forces F_1, F_2, etc., may be represented by the products m_1r_1, m_2r_2, etc. Each force, such as F_1, lies in a plane which also contains the axis of rotation, and may be moved along this plane until it comes into a reference plane OZ which is perpendicular to the axis of rotation. To leave matters unaltered, with F_1 acting in OZ, a couple must be applied in the plane containing F_1 and the axis of rotation; the moment of this couple will be $L_1 = F_1a_1$, where a_1 is the distance of the plane of rotation of F_1 from the reference plane.

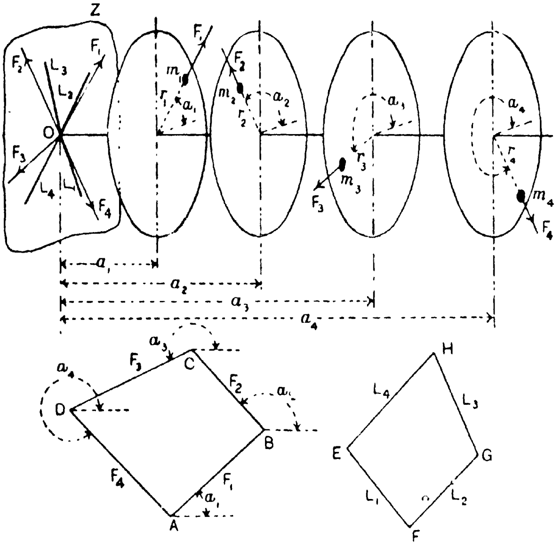

FIG. 552.—Graphical solution of balancing four revolving bodies.

The couple can be represented by an axis, or vector similar to that used in representing angular velocities (p. 400), and this may be drawn as L_1 from O in the reference plane. Treating similarly the other forces, we have four forces F_1, F_2, F_3 and F_4 acting in the reference plane at O, together with four couples represented by the axes L_1, L_2, L_3 and L_4 also in the reference plane.

The first condition of equilibrium will be satisfied if the polygon of forces ABCD closes. The second condition of equilibrium will be satisfied if the polygon of axes of couples EFGH closes.

It will be clear that, in order to satisfy these conditions, some attention must be paid to the data. The polygon of forces will be impossible or insoluble if there be less or more than two unknown quantities. These may be the magnitudes of two of the forces, or the direction of two forces, or one magnitude and one direction.

Similarly, the polygon of couples requires two unknowns, viz. the magnitudes of two couples, or the directions of two axes, or one magnitude and one direction.

Apparatus for testing balance. The above solution may be applied to any number of revolving masses exceeding three in number. A convenient apparatus for testing its truth is illustrated in Fig. 553. A wooden frame is slung from a support by three chains and carries a shaft having four discs. Various weights may be attached to the discs, which may be placed at any angle relative to one another and may be fixed at any place on the length of the shaft. The shaft is driven by means of a small electric motor also carried by the frame. If the revolving masses are in balance, no vibration of the frame will occur when the machine is running. A problem worked out on paper therefore can be tested easily. Another interesting point illustrated by this apparatus may be noticed as the speed rises; if there be want of balance, violent vibrations will occur at a certain speed, viz. that speed at which the natural period of oscillation of the whole apparatus is equal to the speed of rotation of the shaft.

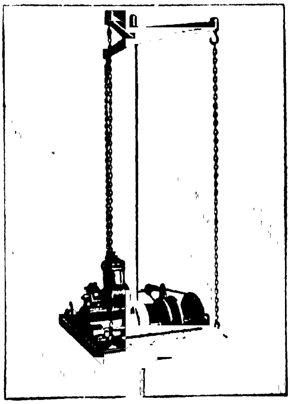

FIG. 553.—Apparatus for experiments on the balancing of revolving bodies.

EXPT. 49. The following data will serve to illustrate one of the many problems which may arise. Let there be four revolving masses, m_1, m_2, m_3, m_4, all known and of values selected from the weights supplied with the apparatus. Let the radii be equal. Then m_1, m_2, m_3 and m_4 may be taken to represent F_1, F_2, F_3 and F_4 respectively. Assume the directions of F_1 and F_2, and find, by the polygon of forces, the remaining two directions. This will also settle the directions of all the axes of couples, and, as there must be still two unknowns, assume values for a_1 and a_2 in Fig. 552, and find the remaining axes by use of the polygon of couples.

The values taken to represent the couples may be m_1a_1, m_2a_2, etc., as the radii are equal. The polygon gives the values of m_3a_3 and m_4a_4, and a_3 and a_4 may be found by dividing these by m_3 and m_4 respectively. Having worked out the solution on paper, arrange the apparatus in accordance with your solution, and then test by actually running it.

Balance of reciprocating masses. Referring to Fig. 554, in which is shown a set of four balanced rotating masses, m_1, m_2, m_3 and m_4, the balance will not be affected if we imagine F_1, F_2, F_3 and F_4 to be resolved horizontally and vertically. It will be evident now that the horizontal components must balance independently, and so also must the vertical components.

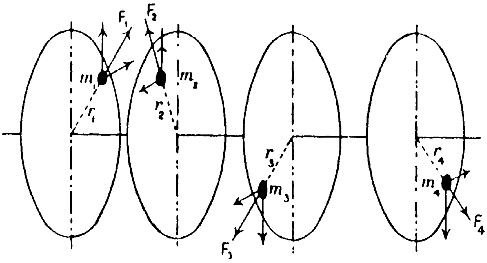

FIG. 554.—Components of the centrifugal forces in a set of four balanced revolving bodies.

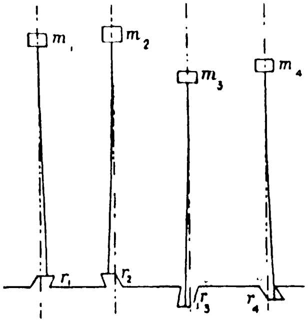

FIG. 555.—The bodies in Fig. 554 arranged as reciprocating masses.

Let the masses be removed and arranged so that they may be driven in vertical lines by means of cranks taking the places of the discs, and connected by rods of length sufficient to give the masses practically simple harmonic motion (Fig. 555). It is evident that we have got rid simply of the horizontal components of the forces F_1, F_2, etc., and retained the vertical components. Hence, if the masses were in balance in their original positions on the discs, the forces due to their inertia will also be in balance when the same masses vibrate in the manner illustrated in Fig. 555 with simple harmonic motion. This leads to the rule that **primary balance** (*i.e.* balance neglecting the oblique action of the connecting rods) may be secured by imagining the reciprocating masses to be attached to discs, and treating the problem as one in revolving masses.

Approximate equations for the velocity and acceleration of the reciprocating parts. In Fig. 556, let the crank and connecting rod be R and L feet long respectively, and let the crank make an angle α with the centre line, the angle which the connecting rod makes with the same line being β. BD is perpendicular to AC and x is the distance between A and C.

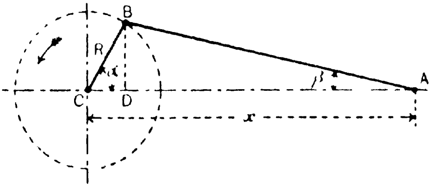

FIG. 556.—Motion of the reciprocating masses.

Let the angular velocity ω of CB be uniform. Then

$$x = CD + DA$$
$$= R\cos\alpha + L\cos\beta. \qquad (1)$$

Also,
$$BD = R\sin\alpha = L\sin\beta,$$
$$\therefore \sin\beta = \frac{R}{L}\sin\alpha;$$

and
$$\cos\beta = \sqrt{1-\sin^2\beta}$$
$$= \left(1-\frac{R^2}{L^2}\sin^2\alpha\right)^{\frac{1}{2}}. \qquad (2)$$

Substituting this value in (1) gives

$$x = R\cos\alpha + L\left(1-\frac{R^2}{L^2}\sin^2\alpha\right)^{\frac{1}{2}}.$$

On expanding the factor in brackets by the binomial theorem, two terms only need be taken, as, for ordinary ratios of R to L, the remaining terms are negligible. Hence,

$$x = R\cos\alpha + L\left(1-\frac{1}{2}\frac{R^2}{L^2}\sin^2\alpha\right)$$
$$= R\cos\alpha + L - \frac{R^2}{2L}\sin^2\alpha. \qquad (3)$$

To obtain the velocity of A in the direction of AC, differentiate x with respect to the time, giving

$$V_A = \frac{dx}{dt} = -R\sin\alpha\,.\,\frac{d\alpha}{dt} - \frac{R^2}{2L}\,.\,2\sin\alpha\,.\cos\alpha\,.\,\frac{d\alpha}{dt}.$$

Also,

$$\frac{d\alpha}{dt} = \omega$$

and

$$2\sin\alpha\cos\alpha = \sin 2\alpha.$$

Hence,

$$V_A = \frac{dx}{dt} = -\omega R\sin\alpha - \frac{\omega R^2}{2L}\sin 2\alpha. \quad \ldots\ldots \quad (4)$$

From this equation, the velocity of A may be calculated for any crank position. To obtain the acceleration of A in the line of AC, differentiate (4) with respect to the time, giving

$$a_A = \frac{dV_A}{dt} = -\omega R\cos\alpha\,\frac{d\alpha}{dt} - \frac{\omega R^2}{2L}(\cos 2\alpha)\,2\,\frac{d\alpha}{dt}$$

$$= -\omega^2 R\cos\alpha - \frac{\omega^2 R^2}{L}\cos 2\alpha. \quad \ldots\ldots\ldots\ldots \quad (5)$$

The acceleration of A may be calculated from this equation for any crank angle. Let M be the mass of the reciprocating parts. Then the force required in order to overcome their inertia when the crank is at an angle α is

$$P = -M\omega^2 R\cos\alpha - \frac{M\omega^2 R^2}{L}\cos 2\alpha. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Suppose that the mass M is concentrated at the crank-pin centre B (Fig. 557). Then the central force required will be $M\omega^2 R$, and the component of this force parallel to the centre line AC will be $M\omega^2 R\cos\alpha$. Evidently this is equal to the first term of (6). The factor $\cos 2\alpha$ in the second term, having reference to an angle 2α, which will be double of the crank angle in all crank positions, may be interpreted by reference to an imaginary crank rotating at twice the angular velocity of the real crank, *i.e.* ω_0 for the imaginary crank would be equal to 2ω. Hence,

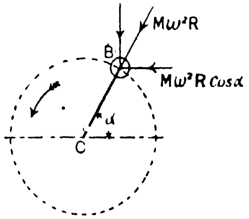

FIG. 557.—Equivalent imaginary primary mass.

$$\omega = \frac{\omega_0}{2},$$

$$\omega^2 = \frac{\omega_0^2}{4}.$$

Therefore the second term in (6) becomes

$$\frac{M\omega^2 R^2}{L}\cos 2\alpha = \frac{M\omega_0^2 R^2}{4L}\cos 2\alpha. \quad \ldots\ldots\ldots\ldots \quad (7)$$

Let a mass equal to M be concentrated at a crank radius r (Fig. 558), set at an angle 2α and rotating with angular velocity ω_0. Then

Central force $= M\omega_0^2 r$.

Component of this force parallel to the line of stroke $= M\omega_0^2 r \cos 2\alpha$.

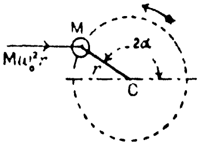

FIG. 558 Equivalent imaginary secondary mass

If this be made equal to (7), we have

$$M\omega_0^2 r \cos 2\alpha = \frac{M\omega_0^2 R^2}{4L} \cos 2\alpha,$$

or

$$r = \frac{R^2}{4L} \qquad \qquad (8)$$

Hence the second term in equation (6) would be produced by a mass M equal to that of the reciprocating parts, concentrated at a crank radius $\frac{R^2}{4L}$, its crank rotating at an angular velocity double that of the engine crank and making an angle with CA in Fig 556 double of that made by the engine crank. The complete equivalent system is shown in Fig. 559, where CB is the real crank and CD is the imaginary crank. The balancing of the effects of M at B is called **primary balancing**, and balancing the effects of M at D is called **secondary balancing**. The disturbances produced in the direction of the line of the stroke, if no attempt at balancing is made, may be calculated easily from the first term of equation (6) for primary disturbances and from the second term of the same equation for secondary disturbances. It will be understood, of course, that, if the disturbances on the engine frame are being calculated, the senses of the forces shown in Fig. 559 must be reversed.

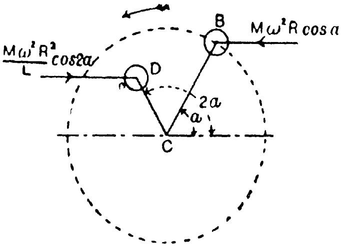

FIG. 559 Effects of the reciprocating masses produced by an imaginary revolving system.

EXAMPLE. A horizontal engine, stroke 2 feet, mass of reciprocating parts 300 pounds, has a speed of 240 revolutions per minute. Find the primary and secondary disturbances on the frame when the crank is at 0°, 45°, 90°, 135° and 180° from the inner dead point. The connecting rod is 4 feet long.

Primary disturbance : $P_1 = \frac{M\omega^2R}{g} \cos \alpha$ lb. weight.

$$\omega = \frac{240}{60} \cdot 2\pi = 8\pi \text{ radians per sec.}$$

$$P_1 = \frac{300 \times 64\pi^2 \times 1}{32 \cdot 2} \cos \alpha$$

$$= 5880 \cos \alpha \text{ lb. weight.}$$

α - - -	0°	45°	90°	135°	180°
$\cos \alpha$ - - -	+1	$+\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1
P_1, lb. weight -	+5880	+4160	0	-4160	-5880

P_1 is denoted positive when the disturbance on the frame is in the sense from B towards A (Fig. 556), and negative when of the opposite sense.

Secondary disturbances :

$$P_2 = \frac{M\omega^2R^2}{gL} \cos 2\alpha$$

$$= \frac{300 \times 64\pi^2 \times 1}{32 \cdot 2 \times 4} \cos 2\alpha$$

$$= 1470 \cos 2\alpha \text{ lb. weight.}$$

The same convention regarding signs being adopted, the disturbances will have values as given below :

α - - -	0°	45°	90°	135°	180°
2α - - -	0°	90°	180°	270°	360°
$\cos 2\alpha$ - - -	+1	0	-1	0	+1
P_2, lb. weight -	+1470	0	-1470	0	+1470

The **combined primary and secondary disturbances** will be obtained by taking the algebraic sum of the corresponding values of P_1 and P_2 :

α - - -	0°	45°	90°	135°	180°
$(P_1 + P_2)$, lb. weight	+7350	+4160	-1470	-4160	-4410

Whirling of shafts. In Fig. 560 is shown a vertical shaft AB running in swivel bearings at A and B; these bearings do not in any way restrain the directions of the shaft axis at A and B; hence, bending of the shaft will correspond to the case of a beam simply supported at the ends. A heavy wheel is mounted on the shaft midway between the bearings, and it is assumed that its centre of mass C does not fall quite in the shaft axis. The effects of centrifugal force may be examined as follows:

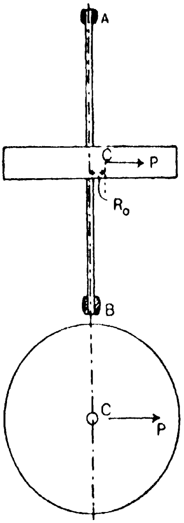

FIG. 560 – Whirling of a loaded shaft

Let M = the mass of the wheel, in pounds.
R_0 = the distance in feet of the centre of mass of the wheel from the shaft axis.
Δ = the deflection produced by centrifugal force, in feet.
ω = the angular velocity, in radians per sec.
L = the length of the shaft, in inches.
I = the moment of inertia, or second moment of area, of the shaft section about a diameter, inch units.
E = Young's modulus, lb. per sq. inch.

$$\text{Centrifugal force} = P = \frac{M\omega^2(R_0+\Delta)}{g} \text{ lb. weight} \quad \ldots\ldots (1)$$

Also,

$$12\Delta = \frac{PL^3}{48EI} \text{ inches (p. 169)},$$

$$\Delta = \frac{PL^3}{576EI} \text{ feet},$$

and

$$P = \frac{576EI\Delta}{L^3}. \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Equating (1) and (2), we have

$$\frac{M\omega^2(R_0+\Delta)}{g} = \frac{576EI\Delta}{L^3};$$

$$\therefore\ M\omega^2L^3(R_0+\Delta) = 576EIg\Delta,$$

or

$$M\omega^2L^3R_0 = 576EIg\Delta - \omega^2ML^3\Delta;$$

$$\therefore\ \Delta = \frac{M\omega^2L^3R_0}{576EIg - \omega^2ML^3}. \quad \ldots\ldots\ldots\ldots (3)$$

It is evident that a critical speed will occur when the denominator of this fraction becomes zero; the deflection will become very large then, and the shaft is said to **whirl.** To obtain this speed, we have

$$576\text{EI}g=\omega^2\text{ML}^3,$$

$$\omega^2=\frac{576\text{EI}g}{\text{ML}^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

If the shaft is of steel, this equation will reduce to the following form by using the usual values of the coefficients:

$$\omega=746000\sqrt{\frac{\text{I}}{\text{ML}^3}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

If the wheel in Fig. 560 be removed, the plain shaft will whirl, but at a much higher speed of revolution. The effect is owing to somewhat similar conditions to those which produce elastic instability in a long strut (p. 228), viz. want of perfect straightness and of perfect uniformity in elastic properties. Any slight deflection will be increased indefinitely when the whirling speed is attained.

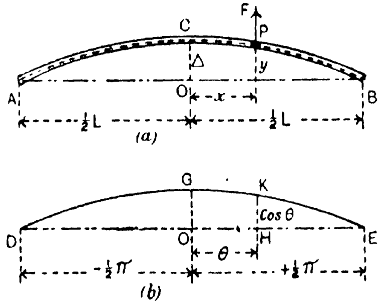

Fig. 561.—Whirling of a uniform shaft having swivel bearings.

Fig. 561 (*a*) shows a uniform shaft in swivel bearings at A and B and deflected to the curve ACB by whirling.

Let m = the mass in pounds per inch length.

L = the length of the shaft, in inches.

y = the radius in inches at P, distance x inches from O.

Δ = the maximum radius OC, in inches.

ω = the angular velocity, in radians per second.

F = the centrifugal force at any point, in lb. weight per inch length.

M = the bending moment at any section, in lb.-inches.

I = the moment of inertia of the shaft section, in inch units.

E = Young's modulus, in lb. weight per square inch.

g = acceleration due to gravitation, **inches** per second per second.

Then, at P, $$F = \frac{m\omega^2 y}{g} \text{ lb. weight per inch length;} \quad \ldots(1)$$

$$\therefore \quad F \propto y.$$

This result indicates that the curve in Fig. 561 (*a*) not only represents the deflection, but also the load per unit length to another scale. Hence, we may write

$$F = cy,$$

where c is a numerical coefficient rectifying the scale.

Now, if the coordinates y and x refer to a given deflection curve, the second differential coefficients, when plotted, will represent a curve of bending moments, and the fourth differential coefficients will represent a curve of loads which would produce the given deflection curve. Hence, in the present case,

$$\frac{d^4y}{dx^4} \propto F = cy. \quad \ldots(2)$$

From this expression, the shape of the deflection curve has to be obtained, and may be inferred to be a curve of cosines. Thus, take the equation $y = \cos\theta$ and obtain the fourth differential coefficient:

$$y = \cos\theta; \; \frac{dy}{dx} = -\sin\theta; \; \frac{d^2y}{dx^2} = -\cos\theta; \; \frac{d^3y}{dx^3} = \sin\theta, \; \frac{d^4y}{dx^4} = \cos\theta.$$

Therefore, in the curve representing the equation $y = \cos\theta$,

$$\frac{d^4y}{dx^4} = \cos\theta = y.$$

In Fig. 561 (*a*), there is zero deflection at A and B and maximum deflection at C. Hence the corresponding cosine curve (Fig. 561 (*b*)) will have the origin at O, OE and OD will represent $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$ respectively (for which the cosines are zero), and OG will represent $\cos 0 = 1$. HK, corresponding to y in Fig. 561 (*a*), will represent $\cos\theta$, where θ is the angle represented by OH, corresponding to x in Fig. 561 (*a*). From the diagrams, we have

$$\theta = \frac{x}{\frac{1}{2}L} \cdot \frac{\pi}{2} = \frac{\pi}{L}x.$$

Also, $$\frac{y}{\Delta} = \frac{\cos\theta}{\cos 0} = \cos\frac{\pi}{L}x;$$

$$\therefore \; y = \Delta\cos\frac{\pi}{L}x. \quad \ldots(3)$$

Obtaining the fourth differential coefficient of this,

$$\frac{d^4y}{dx^4} = \Delta\frac{\pi^4}{L^4}\cos\frac{\pi}{L}x. \quad \ldots(4)$$

Now,
$$\frac{d^2y}{dx^2}=\frac{M}{EI};$$
$$\therefore \frac{d^4y}{dx^4}=\frac{1}{EI}\cdot\frac{d^2M}{dx^2}.$$

Also,
$$\frac{d^2M}{dx^2}=F;$$
$$\therefore \frac{d^4y}{dx^4}=\frac{F}{EI}. \qquad \text{(5)}$$

Hence, from (1) and (5),
$$\frac{d^4y}{dx^4}=\frac{m\omega^2y}{EIg}. \qquad \text{(6)}$$

Equating (4) and (6),
$$\frac{m\omega^2y}{EIg}=\Delta\frac{\pi^4}{L^4}\cos\frac{\pi}{L}x. \qquad \text{(7)}$$

This equation is true for any corresponding values of y and x. Take the value $x=0$, when
$$\cos\frac{\pi}{L}x=\cos 0=1$$

and
$$y=\Delta.$$

Hence,
$$\frac{m\omega^2\Delta}{EIg}=\Delta\frac{\pi^4}{L^4}. \qquad \text{(8)}$$

The deflection cancels from both sides of this equation, indicating that a critical speed ω has been attained, and giving the result
$$\omega^2=\frac{\pi^4}{L^4}\frac{EIg}{m},$$

or
$$\omega=\frac{\pi^2}{L^2}\sqrt{\frac{EIg}{m}}. \qquad \text{(9)}$$

This result expresses the whirling speed. If the bearings restrain axially the directions of the shaft at A and B (Fig. 561 (*a*)), then it may be shown that the whirling speed is given by
$$\omega=\frac{4{\cdot}74^2}{L^2}\sqrt{\frac{EIg}{m}}. \qquad \text{(10)}$$

EXERCISES ON CHAPTER XX.

1. Find the **M** of a flywheel which, when running at 200 revolutions per minute, will increase its speed by ·1 per cent. while storing 5000 foot-lb. of energy.

2. A solid disc of cast iron, density 450 pounds per cubic foot, is 8 inches in diameter by 2 inches thick and runs at 2500 revolutions per minute. What percentage increase in speed will occur if it is called upon to store an additional 200 foot-lb. of energy?

3. A cast-iron flywheel is 30 feet in mean diameter. The safe tensile stress is 2000 lb. per square inch. Find the maximum permissible speed of revolution of the wheel. Take the density as 450 pounds per cubic foot.

4. A mild steel hoop is 18 inches in mean diameter and the elastic limit of the material is 18 tons per square inch. At what speed of revolution would permanent damage begin to occur? Take the density as 480 pounds per cubic foot.

5. What is the limit to the velocity of the rim of an ordinary flywheel? Does it depend on the diameter? Prove your statements. (B.E.)

6. In the manufacture of a large drum for a steam turbine a hollow, red-hot steel billet is, at a high speed, rolled between internal and external rollers, which effect a gradual increase of diameter and diminution of thickness. Show that the intensity of the tangential stress of the material of the drum remains constant during the rolling operation, assuming a constant speed of the rollers. Determine the limiting speed of the rollers to keep the tensile stress within 1 ton per square inch. (Weight of steel, 485 lb. per cubic foot.) (I.C.E.)

7. The indicated horse-power of a steam engine is 100; the mean crank shaft speed is 200 revolutions per minute. The energy to be taken up by the flywheel of the engine between its minimum and maximum speeds is 10 per cent. of the work done in the cylinders per revolution of the crank shaft. If the radius of gyration of the flywheel is 2 feet 6 inches, determine its weight in order that the total fluctuation of speed may not exceed 2 per cent. of the mean speed. (L.U.)

8. Show from first principles that two flywheels of the same dimensions but of materials of different densities will have equal kinetic energies when run at the speeds which give equal hoop stresses. Calculate the kinetic energy stored per pound of rim in a cast-iron flywheel, when the hoop stress is 800 pounds per square inch. Cast iron weighs 450 pounds per cubic foot. (L.U.)

9. In a simple Watt governor, the height of the cone of revolution is 4 inches. What is the speed in revolutions per minute?

10. A Porter governor has revolving masses of 2 pounds each. The arms are all equal and 8 inches long. If the height of the cone of revolution is to be 5 inches at 180 revolutions per minute, find the dead load required.

11. In Question 10, the throttle valve is full open when the height is 5·5 inches and closed entirely when the height is 4·5 inches. Find the limits of speed of revolution controlled by the governor. State the total variation in speed as a percentage of the mean speed of 180 revolutions per minute.

12. A uniform rod 8 inches long, mass 4 pounds, is hinged at its upper end to a vertical axis of revolution. Find the speed at which the arm will describe a cone of semi-vertical angle 45 degrees. Supposing this speed to be doubled without alteration in the position of the rod, what controlling couple must be applied to the rod?

13. The mass of each of the balls of a spring loaded governor arranged as in Fig. 548 is 5 pounds. When the radius of the balls is 6 inches the governor makes 250 revolutions per minute. Find the total compressive force in the spring, and, neglecting friction, find the stiffness, *i.e.* the force per inch compression, of the spring that the governor may be isochronous. Show that the effect of friction would be to make the governor stable. (L.U.)

14. A Porter governor has equal links 10 inches long, each ball weighs 5 pounds and the load is 25 pounds. When the ball radius is 6 inches the valve is full open, and when the radius is 7·5 inches the valve is closed. Find the maximum speed and the range of speed. If the maximum speed is to be increased 20 per cent. by an addition to the load, find what addition is required. (L.U.)

15. Three bodies of 2, 3 and 5 pounds mass respectively revolve at equal radii round a horizontal axis. The axial distance between the outer pair of bodies is 18 inches. Arrange the bodies so that they shall be in balance.

16. The four weights w'_1, w'_2, w'_3, w'_4 (Fig. 562) rotate in one plane about an axis, their magnitudes and the radii at which they act being given in the table:

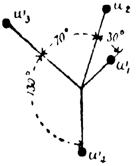

FIG. 562.

Weight	Magnitude in lb	Radius in feet
w'_1	10	0·5
w'_2	8	1·0
w'_3	6	1·25
w'_4	12	0·75

Find graphically the equivalent single mass in magnitude and direction, acting at a radius of 1 foot; and calculate the total displacing force on the shaft when the revolutions are 200 per minute. (I.C.E.)

17. A shaft runs in bearings A, B, 15 feet apart, and carries three pulleys C, D and E, which weigh 360, 400 and 200 pounds respectively, and are placed at 4, 9 and 12 feet from A. Their centres of gravity are distant from the shaft centre line by amounts: C $\frac{3}{16}$ inch, D $\frac{1}{8}$ inch and E $\frac{1}{4}$ inch. Arrange the angular positions of the pulleys on the shaft so that there should be no dynamic force on B, and find for that arrangement the dynamic force on A when the shaft runs at 100 revolutions per minute. (L.U.)

18. Find the positions and magnitudes of the balance weights required to balance all the revolving and $\frac{2}{3}$ of the reciprocating masses in a simple inside cylinder locomotive specified as follows: masses per cylinder at 12 inch radius, revolving 720 pounds, reciprocating 630 pounds; centre to centre of cylinders, 26 inches; planes of balance weights, 58 inches apart; radius of balance weights, 32 inches. (L.U.)

19. The reciprocating masses for the first, second and third cylinders of a four-cylinder engine are 4, 6 and 8 tons, and the centre lines of these cylinders are 13, 9 and 4 feet respectively from that of the fourth cylinder. Find the fourth reciprocating mass, and the angles between the various cranks, in order that these may be balanced. (B.E.)

20. Show that the disturbing effect of a reciprocating mass connected to a crank by the equivalent of an infinite connecting rod is the same as that produced in the line of stroke by an equal mass placed at the crank pin. An engine has three cylinders A, B and C whose axes are parallel. The axis of B is at a distance *a* from the axis of A and a distance *c* from the axis of C. The mass of the reciprocating parts of B is M. Assuming that all the pistons have harmonic motion and the same length of stroke, show how the cranks on the crank shaft must be placed, and find the masses of the reciprocating parts of A and C in order that all the reciprocating parts may be completely balanced. (L.U.)

21. A four-cylinder vertical engine, cranks at right angles, has its cranks equally spaced between the bearings, the pitch being *p*. Taken from the left, the order is A, B, C, D. The revolving mass for each cylinder is M_1 and the reciprocating mass M_2, and the speed is ω radians per second. The crank radius is *r* and the connecting-rod length *l*. Examine the primary and secondary balance, forces and couples when (*a*) the cranks are as shown in Fig. 563, (*b*) the cranks are at 45 degrees to the line of stroke. (L.U.)

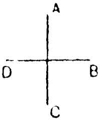

FIG. 563.

22. A vertical steel shaft 1 inch in diameter runs in swivel bearings 36 inches centre to centre. A wheel of mass 20 pounds is mounted at the centre of the shaft, and its centre of mass is at a small distance from the shaft axis. At what speed of revolution will whirling occur? Take E = 30,000,000 lb. per square inch.

23. A steel shaft 2 inches in diameter runs in swivel bearings 9 feet centre to centre. At what speed will whirling occur? Take E = 30,000,000 lb. per square inch and the density 0.28 pound per cubic inch.

24. Answer Question 24 if the bearings constrain the directions of the shaft at its ends.

25 In Question 23, the speed of the shaft is 600 revolutions per minute. Find the limiting distance centre to centre of the bearings.

CHAPTER XXI.

TRANSMISSION OF MOTION BY BELTS, ROPES, CHAINS AND TOOTHED WHEELS

Driving by belt. Motion may be transmitted from one shaft to another by means of a belt running on the rims of pulleys which are fixed to the shafts. The driving effort is transmitted from the belt to the pulley by the agency of the frictional resistance to slipping of the belt on the pulley. A will drive B in the same direction of rotation if the belt is open (Fig. 564), and in the opposite direction if the belt is crossed (Fig. 565). In the latter case, each portion of the belt is given a half turn in order that the same side of the material may bear against the rims of both A and B. In these diagrams the shafts are parallel, and both pulleys are arranged so that their planes of revolution coincide, if this condition be not attended to, the belt will not remain on the pulleys. It is customary also to round slightly the rims of the pulleys (Fig. 566), with a view to enable the belt to ride on the centre of the rim; the action will be understood by reference to Fig. 567, which shows the exaggerated case of two frusta of cones placed base to base. The belt, in bedding down on the conical surface, bends as shown; consequently the points a and a' will be higher up the cone than b

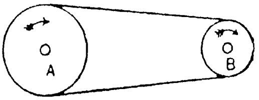

FIG. 564.—Open belt.

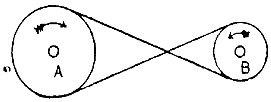

FIG. 565.—Crossed belt.

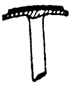

FIG. 566

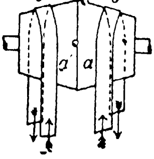

FIG. 567

and *b'*, which came into contact a little before *a* and *a'*. Hence the belt will climb to the highest part and remain there.

It will be evident that the part of the belt which is advancing towards the pulley must be moving in the same plane as that in which the pulley is rotating. The part receding from the pulley may do so in a plane which does not coincide with the plane of rotation. Advantage is taken of these conditions in the case of two shafts having directions at right angles (Fig. 568). A is so arranged on the lower shaft that the part C of the belt leaving it is moving in the plane in which B rotates; similarly B is so arranged that the portion D of the belt leaves B in the same plane as that in which A is rotating. The belt will ride safely on both pulleys, provided that the directions of rotation are not reversed at any time. Reversal of direction must be preceded by a rearrangement of the pulleys. The distance between the shafts should not be small enough to render excessive the angle at which the belt leaves the pulleys.

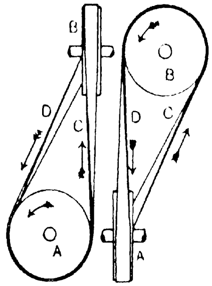

FIG. 568.—Two shafts at 90° connected by a belt

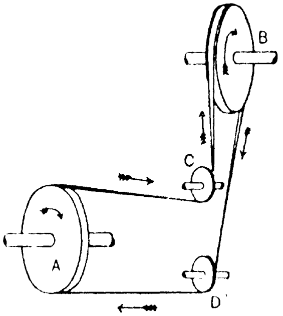

FIG. 569.—Use of jockey pulleys.

In Fig. 569 is shown an arrangement in which A drives B by means of a belt which is guided into the proper planes by **jockey pulleys** running freely at C and D.

Velocity ratio of belt pulleys. A certain amount of slipping is always present in belt driving; in the best cases there may be 1 to 2 per cent. of the motion of the driven pulley lost in slipping. The belt usually comes off the pulleys if the slip exceeds 10 per cent. Neglecting slipping, it will be evident that the speed of the belt will be equal to the speeds of the rims of both pulleys. Referring to Fig. 570,

Let D_A = the diameter of A, in feet.
D_B = the diameter of B, in feet.
V = the velocity of the belt, in feet per minute.
N_A = revolutions per minute of A.
N_B = revolutions per minute of B.

Then, Distance travelled by rim of each pulley = V feet per minute.

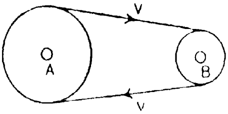

FIG. 570.—Velocity ratio of belt pulleys.

$$N_A = \frac{V}{\pi D_A},$$

$$N_B = \frac{V}{\pi D_B}$$

and $$\frac{N_A}{N_B} = \frac{D_B}{D_A}.$$

Hence the **speeds of revolution are inversely proportional to the diameters of the pulleys.**

Strictly speaking, the diameters should be measured to the mean thickness of the belt, *i.e.* the thickness of the belt should be added to D_A and D_B. The presence of slip usually renders this correction an unnecessary refinement.

In Fig. 571, A is an engine pulley driving a line shaft pulley B; a countershaft has a pulley D driven from a pulley C on the line shaft; a machine pulley F is driven from the countershaft pulley E. Assuming that there is no slip,

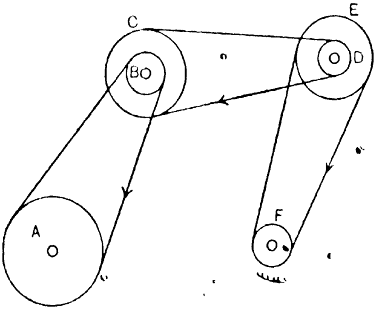

FIG 571.—A belt pulley arrangement.

$$\frac{N_B}{N_A} = \frac{D_A}{D_B}; \quad \therefore N_B = \frac{D_A}{D_B} . N_A.$$

Also,
$$\frac{N_D}{N_C} = \frac{D_C}{D_D}; \quad \therefore N_D = \frac{D_C}{D_D} . N_C.$$

But $N_C = N_B$, $\therefore N_D = \dfrac{D_C}{D_D} \cdot \dfrac{D_A}{D_B} \cdot N_A$.

Again, $\dfrac{N_F}{N_E} = \dfrac{D_E}{D_F}$; $\therefore N_F = \dfrac{D_E}{D_F} \cdot N_E$.

And $N_E = N_D$,

$$\therefore N_F = \frac{D_A \times D_C \times D_E}{D_B \times D_D \times D_F} \cdot N_A.$$

Now A, C and E are drivers and B, D and F are driven pulleys; hence we have the rule: **To obtain the speed of revolution of the last wheel, multiply the speed of the first wheel by the product of the diameters of all the drivers and divide by the product of the diameters of all the driven pulleys.**

Supposing that each pair of pulleys connected by a belt experiences a percentage slip p, *i.e.* the driven pulley loses by slip p revolutions in every 100, then

$$N_B = \frac{D_A}{D_B} N_A \left(\frac{100 - p}{100}\right).$$

$$N_D = \frac{D_C}{D_D} N_C \left(\frac{100 - p}{100}\right).$$

$$N_F = \frac{D_E}{D_F} N_E \left(\frac{100 - p}{100}\right).$$

Since $N_B = N_C$ and $N_D = N_E$ these reduce to

$$N_D = \frac{D_A}{D_B} \cdot \frac{D_C}{D_D} N_A \left(\frac{100 - p}{100}\right)^2$$

and

$$N_F = \frac{D_A \times D_C \times D_E}{D_B \times D_D \times D_F} N_A \left(\frac{100 - p}{100}\right)^3.$$

Friction of a belt on a pulley. The greatest possible difference which can exist between the pulls on the tight and slack sides of a belt will depend on the maximum frictional resistance to slipping of the belt on the pulley. In Fig. 572 (*a*) is shown a pulley having a belt embracing it over an arc of contact AB. Let T_1 and T_2 be the pulls at the ends when the belt is on the point of slipping, and let T_1 be the larger pull. Let the angle subtended by AB at the centre of the pulley be θ radians, and consider a small arc CD subtending a small angle $\delta\alpha$ radian. The portion CD of the belt will be in equilibrium under the action of forces T and $T + \delta T$, these being the pulls at D and C respectively (Fig. 572 (*b*)), together with a normal reaction p from the pulley rim and also the frictional resistance to slipping.

Resolve T into components along and at right angles to p; these will be $T \sin \frac{1}{2}\delta a$ and $T \cos \frac{1}{2}\delta a$ respectively. In the same manner, $T+\delta T$ will have components $(T+\delta T)\sin \frac{1}{2}\delta a$ and $(T+\delta T)\cos \frac{1}{2}\delta a$ respectively along the same lines. The sum of the components along p must be equal to p, hence

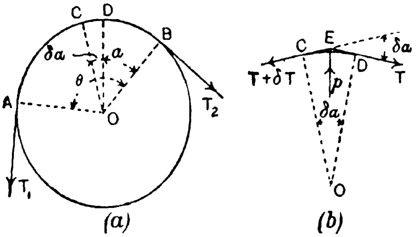

FIG. 572.—Friction of a belt on a pulley.

$$p = T \sin \tfrac{1}{2}\delta a + (T+\delta T)\sin \tfrac{1}{2}\delta a$$
$$= (2T+\delta T)\sin \tfrac{1}{2}\delta a.$$

Neglecting the products of small quantities, this reduces to

$$p = 2T \sin \tfrac{1}{2}\delta a.$$

Again, the difference between the sine of a very small angle and its radian measure is infinitesimal. Hence,

$$p = 2T \cdot \tfrac{1}{2}\delta a$$
$$= T \cdot \delta a. \quad \ldots\ldots\ldots\ldots (1)$$

Let the coefficient of friction be μ. Then

$$\text{Frictional resistance of arc CD} = \mu p$$
$$= \mu T \cdot \delta a. \quad \ldots\ldots (2)$$

This frictional resistance must be equal to the difference in the components of T and $T+\delta T$ taken at right angles to p, hence

$$\mu T \cdot \delta a = (T+\delta T)\cos \tfrac{1}{2}\delta a - T \cos \tfrac{1}{2}\delta a$$
$$= \delta T \cos \tfrac{1}{2}\delta a.$$

The angle $\frac{1}{2}\delta a$ being very small, its cosine may be taken as unity and the equation reduces to

$$\mu T \, \delta a = \delta T,$$
$$\frac{\delta T}{T} = \mu \cdot \delta a. \quad \ldots\ldots\ldots\ldots (3)$$

In the limit, writing $d\alpha$ and dT, and integrating both sides, we have

$$\int_{T_2}^{T_1}\frac{dT}{T}=\mu\int_0^\theta d\alpha,$$

or

$$\log_\epsilon\frac{T_1}{T_2}=\mu\theta. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3')$$

This equation may be written

$$\frac{T_1}{T_2}=\epsilon^{\mu\theta}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where ϵ is the base of the hyperbolic logarithms (p. 11).

The physical meaning of this equation may be understood by dividing the total arc of contact into a number of equal arcs AB, BC, CD, etc. (Fig. 573). Let each arc subtend an angle α at the centre, and let the tensions in the belt at B, C, D, etc., be denoted by T_B, T_C, T_D, etc. Equation (4) above applies to each arc. Hence,

$$\frac{T_1}{T_B}=\epsilon^{\mu\alpha}, \quad \frac{T_B}{T_C}=\epsilon^{\mu\alpha}, \quad \frac{T_C}{T_D}=\epsilon^{\mu\alpha}, \text{ etc.}$$

As the right-hand side is constant in each of these expressions, the ratios of the tensions will be constant, *i.e.*

$$\frac{T_1}{T_B}=\frac{T_B}{T_C}=\frac{T_C}{T_D}=\text{a constant.}$$

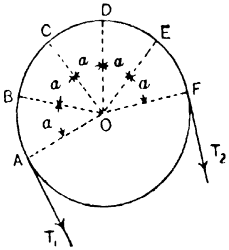

FIG. 573.—Tensions in the belt at different parts of the arc of contact.

Hence, if the value of the constant for a given angle is known, the ratio of the tensions for any angle when slipping is about to occur may be calculated easily.

EXAMPLE 1. A rope is coiled round a fixed drum over an arc of contact of 90°. It is found that slipping occurs when the ratio of the pulls is ¾. Find the ratio of the pulls for an arc of contact of 270°.

$$\frac{T_0}{T_{90}}=\frac{T_{90}}{T_{180}}=\frac{T_{180}}{T_{270}}=\frac{3}{4};$$

$$\therefore \frac{T_0}{T_{90}}\times\frac{T_{90}}{T_{180}}\times\frac{T_{180}}{T_{270}}=\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4},$$

or,

$$\frac{T_0}{T_{270}}=\underline{\frac{27}{64}}.$$

EXAMPLE 2. A leather belt laps 180° round a cast-iron pulley. Taking $\mu=0\cdot5$, calculate the pull on the slack side when slipping is about to occur, if the pull on the tight side is 200 lb.

Here $\theta = 180° = \pi$ radians. Hence,

$$\log_e \frac{T_1}{T_2} = \mu\theta = 0{\cdot}5\pi = 1{\cdot}5708\,;$$

$$\therefore \frac{T_1}{T_2} = 4{\cdot}81,$$

or

$$T_2 = \frac{300}{4{\cdot}81} = \underline{62{\cdot}4 \text{ lb.}}$$

Horse-power transmitted by a belt. It will be observed that the diameter of the pulley does not enter into the expression for the ratios of the pulls of a belt or rope. For example, in the last result, the pulls would be 300 lb. and 62·4 lb. when the belt is embracing a pulley 3 feet in diameter or 6 feet in diameter, provided the arc of contact is 180° in each case. Some other cause must be looked for to explain the known fact that a belt which constantly slips on a certain drive may be remedied by substituting pulleys of larger diameter on both shafts, keeping the ratio of the diameters as at first, so as not to alter the speeds of the shafts. The explanation lies in the fact that the belt is now running at a higher speed, and will therefore do the same work per minute, or will transmit the same horse-power, with a smaller difference in pulls. Thus,

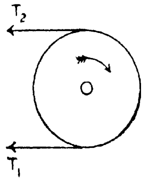

FIG. 574.

Let T_1 = pull on tight side, lb.

T_2 = pull on slack side, lb.

V = velocity of belt in feet per minute.

Considering the driven pulley (Fig. 574), T_1 is urging it to turn and T_2 is tending to prevent rotation; hence the net driving force is $(T_1 - T_2)$.

Work done per minute $= (T_1 - T_2)V$ foot-lb.

$$\text{Horse-power transmitted} = \frac{(T_1 - T_2)V}{33{,}000} \quad \dots\dots\dots\dots (1)$$

Now let V be increased to V_2 feet per minute by the substitution of larger pulleys running at the same revolutions per minute. The horse-power transmitted being the same as at first, we have

$$\frac{(T_1 - T_2)V}{33{,}000} = \frac{(T_1' - T_2')V_2}{33{,}000},$$

where T_1' and T_2' denoted the altered pulls in the belt. This gives

$$(T_1 - T_2)V = (T_1' - T_2')V_2. \quad \dots\dots\dots\dots (2)$$

As V_2 is greater than V, it follows that $(T_1' - T_2')$ must be less than $(T_1 - T_2)$. Hence there is now less frictional resistance to slipping called for, and consequently the risk of slipping is reduced.

Equation (1) above for the horse-power may be written in terms of the maximum pull T_1 in the belt. Thus,

$$\frac{T_1}{T_2} = \epsilon^{\mu\theta};$$

$$\therefore\ T_2 = \frac{T_1}{\epsilon^{\mu\theta}}. \qquad \ldots\ldots\ldots\ldots (3)$$

Substituting in (1) gives

$$\text{Horse-power transmitted} = \frac{\left(T_1 - \frac{T_1}{\epsilon^{\mu\theta}}\right)V}{33{,}000}$$

$$= \left(1 - \frac{1}{\epsilon^{\mu\theta}}\right)\frac{T_1 V}{33{,}000}. \qquad \ldots\ldots\ldots\ldots (4)$$

From this equation the dimensions of a belt suitable for transmitting a given horse-power may be obtained. The strength of a belt is stated in pounds per inch of width generally.

Let b = width of belt in inches.
p = safe pull per inch width of belt.

Then $T_1 = pb$

and $$\text{Horse-power} = \left(1 - \frac{1}{\epsilon^{\mu\theta}}\right)\frac{pbV}{33{,}000}. \qquad \ldots\ldots\ldots\ldots (5)$$

The width b may be calculated from this result when the other quantities involved are given.

Driving by rope. Ropes of cotton, hemp, manila or steel wire may be used for transmitting motion. In such cases the rims of the pulleys are grooved to receive the ropes. The section of a pulley rim suitable for ropes of cotton or similar material is given in Fig. 575. The ropes bear on the sides of the wedge shaped grooves, thus increasing the frictional resistance to slipping. In Fig. 576,

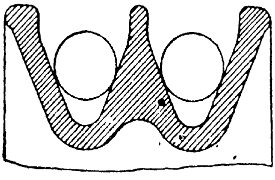

FIG. 575.—Section of the rim of a rope pulley.

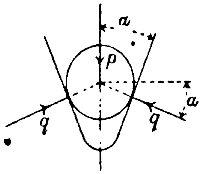

FIG. 576.—Pressures on the groove of a rope pulley.

Let α = half the angle of the wedge.
p = the normal force on a small arc of the rim, in lb.
μ = the coefficient of friction.

Then p will be equal to the sum of the vertical components of the normal pressures q, q on the sides of the groove. Hence,

$$p = 2q \sin \alpha,$$

$$q = \tfrac{1}{2} p \operatorname{cosec} \alpha. \qquad \text{(1)}$$

Now the frictional resistance to sliding on the small arc considered is

$$f = 2\mu q = 2\mu \tfrac{1}{2} p \operatorname{cosec} \alpha$$

$$= p \, . \, \mu \operatorname{cosec} \alpha. \qquad \text{(2)}$$

Had the case been that of a flat belt on an ordinary pulley, the frictional resistance would be μp. Hence, the results already obtained for flat belts may be used for ropes which bear on the sides of the groove by writing $\mu \operatorname{cosec} \alpha$ instead of μ. Thus, from equation (4), p. 531, and equation (4), p. 533, we have

$$\frac{T_1}{T_2} = \epsilon^{\mu\theta \operatorname{cosec} \alpha}. \qquad \text{(3)}$$

$$\text{H.P.} = \left(1 - \frac{1}{\epsilon^{\mu\theta \operatorname{cosec} \alpha}}\right) \frac{T_1 V}{33{,}000}. \qquad \text{(4)}$$

In the case of wire ropes, the rope should not bear on the sides of the groove, as it would suffer injury thereby. Fig. 577 shows a suitable form of rim, in which the rope beds on the bottom of the groove; it is found advantageous to line the bottom of the groove with leather, with a view of increasing the frictional resistance. Where the ropes are very long, idle bearer pulleys may be used at intervals to support the ropes. These run loose on their bearings, and may have rims of a section shown in Fig. 578.

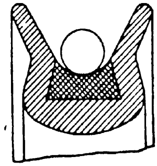

FIG. 577.—Section of a wire rope pulley.

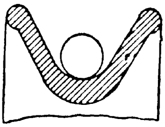

FIG. 578.—Section of an idle pulley for a wire rope.

The equations for a flat belt apply without alteration to the case of a wire rope bedding on the bottom of the groove.

Centrifugal tension in belts and ropes. The portion of a belt or rope which laps on the pulley is subject to centrifugal forces when the belt is running (Fig. 579).

Let m = the mass of the belt per foot run, in pounds.

v = the velocity of the belt, in feet per sec.

r = radius, in feet, to the centre of the belt.

Then the centrifugal force per foot length of arc will be given by

$$f = \frac{mv^2}{gr} \text{ lb. weight.}$$

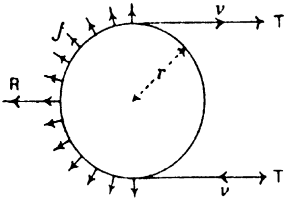

Fig. 579.—Centrifugal tension in a belt.

These radial forces will have a resultant R directed towards the left in Fig. 579, and will be balanced by tensions T, T in the belt, which are in addition to those required for driving purposes. The case is analogous to a boiler shell subjected to internal pressure, and may be solved by the method given on pp. 95 and 96.

$$R = f \times 2r$$

$$= \frac{mv^2}{gr} \cdot 2r = \frac{2mv^2}{g} \text{ lb. weight.}$$

Also, $2T = R$;

$$\therefore \ T = \frac{mv^2}{g} \text{ lb. weight.}$$

For leather belts, m may be taken as

$$m = 0{\cdot}4A \text{ pounds per foot run,}$$

where A is the cross-sectional area of the belt in square inches.

The general effects of centrifugal force are to increase the pulls in the belt, and also to reduce partially the radial pressures on the rim of the pulley. As the latter are relied on for the production of the frictional driving effort, it follows that excessive slipping will occur at speeds which are too high, and the power transmitted will be reduced thereby.

Belt striking gears. The intermittent motion required for driving many classes of machines may be obtained by means of two pulleys on the countershaft driving the machine. In Fig. 580 a pulley A on the main or line shaft drives a countershaft having two pulleys, one B_1 running loosely on the countershaft and the other B_2 fixed to the shaft. The belt may be moved from one pulley to the other by means of forks C, C, which loosely embrace the belt. The forks are operated by a sliding bar D and a handle E, carried to a suitable position for the operator. The pulley A is made specially wide, so as to permit the belt to ride on either B_1 or B_2; in the former case, the countershaft and machine will be at rest.

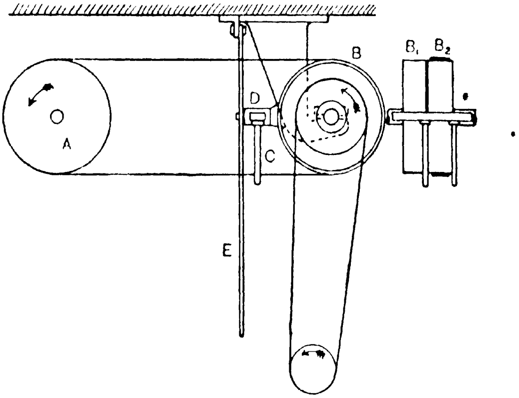

Fig. 580.—Belt striking gear.

Another arrangement is shown in Fig. 581. The countershaft B has two loose pulleys L_1 and L_2, and also a pulley F fixed to the shaft. There are two belts, one D open and one E crossed; these are operated by the belt-striking forks and bar shown at C. No motion will be transmitted to the countershaft if both belts are on the loose pulleys, and motion in either one or the other direction will occur, depending on which belt is made to ride on F. The arrangement forms a convenient reversing gear.

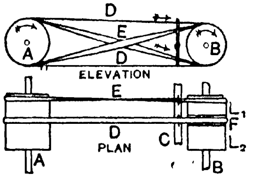

Fig. 581.—Arrangement for reversing a machine.

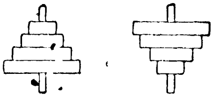

Fig. 582.—Stepped cones.

Variation in the velocity of rotation of the driven shaft may be accomplished by means of stepped cones or speed pulleys (Fig. 582). These consist of a number of pulleys of different diameters mounted on the shafts so as to oppose the smallest and the greatest. The belt may ride on any corresponding pair.

The length of belt required enters into the question of stepped cones, as the belt has to fit any corresponding pair without alteration being made in its length. For a crossed belt it may be shown that the sum of the diameters of any corresponding pulleys should be constant for the whole set. With an open belt there is a small divergence from this rule, which becomes negligible if the distance between the shafts is large compared with the pulley diameters; such is usually the case.

Transmission of motion by chains. In cases where the driving effort is too large to be transmitted by a belt or rope, or where slipping is inadmissible, chains may be used in combination with toothed or sprocket wheels. A few patterns of suitable chains are given in Fig. 583. (*a*) is a block chain in which a number of small blocks are connected by pairs of links and riveted pins. Chains of this pattern are used for conveyors, as the carriers are attached readily to the blocks. (*b*) is a similar pattern, but made entirely of links. (*c*) is a better form, and works more easily. The inner links are connected by a tube riveted over at its ends, and a roller runs on the tube; the outer links are connected by a pin passing through the tube and riveted over at its ends.

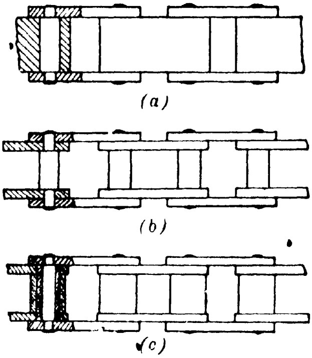

FIG. 583.—Types of driving chains.

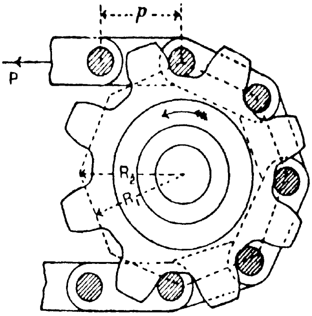

FIG. 584.—Sprocket wheel for chain driving, showing the effect produced by the chain stretching.

A **sprocket wheel** is shown in Fig. 584. The centres of the chain pins lie at the corners of a polygon having sides equal to the pitch p of the chain. The driving force P may act at radii which will vary from R_1 to R_2, and thus cause variations in the turning moment and in the velocity ratio. These variations will be small, provided the number of teeth on the sprocket wheel be sufficiently large.

The form of the teeth may be constructed by first drawing semicircles of radius r equal to that of the chain pin, or roller. Using radii slightly smaller than $(p - r)$ and centres nearly coinciding with the adjacent pin centres, the sides of the teeth may be drawn. These will be such as to enable the chain to leave the wheel at the top without the pin or roller touching the face of the tooth.

In general there is practically no pull on the slack side of a chain; hence, the work done per minute is given by the product of P and the velocity of the chain in feet per minute. The chain is liable to stretching of the links and to wear at the pins, both of which tend to increase the pitch. The effect of this will be ultimately that the top pin, or roller alone, as is shown in Fig. 584, will be bearing against its tooth, and this tooth accordingly will carry the whole load. The effect is manifest in the chain grinding on the teeth, thus introducing additional frictional resistance and also wearing the teeth. These effects may be obviated somewhat by using a roller chain, and by making the spaces between the teeth wider than the diameter of the chain pin or roller. Increase in the pitch is provided for perfectly in the Renold's silent chain. The links are of the form shown in Fig. 585; any increase in the pitch, caused by wear or stretching, has simply the effect of causing the links to ride on the teeth at a larger radius from the centre of the wheel. Speeds of 1250 feet per minute and horse-powers up to 500 have been attained with these chains.

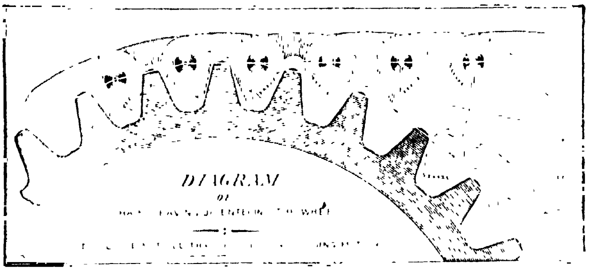

FIG. 585 —Renold's silent chain

Friction gearing. In cases where the shafts are close enough together, motion may be communicated from one to the other by

The length of belt required enters into the question of stepped cones, as the belt has to fit any corresponding pair without alteration being made in its length. For a crossed belt it may be shown that the sum of the diameters of any corresponding pulleys should be constant for the whole set. With an open belt there is a small divergence from this rule, which becomes negligible if the distance between the shafts is large compared with the pulley diameters; such is usually the case.

Transmission of motion by chains. In cases where the driving effort is too large to be transmitted by a belt or rope, or where slipping is inadmissible, chains may be used in combination with toothed or sprocket wheels. A few patterns of suitable chains are given in Fig. 583. (*a*) is a block chain in which a number of small blocks are connected by pairs of links and riveted pins. Chains of this pattern are used for conveyors, as the carriers are attached readily to the blocks. (*b*) is a similar pattern, but made entirely of links. (*c*) is a better form, and works more easily. The inner links are connected by a tube riveted over at its ends, and a roller runs on the tube; the outer links are connected by a pin passing through the tube and riveted over at its ends.

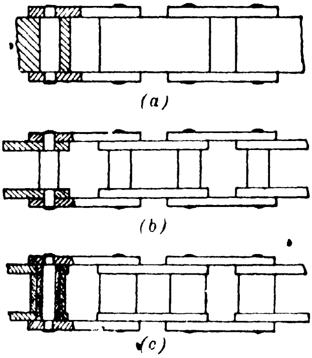

FIG. 583.—Types of driving chains.

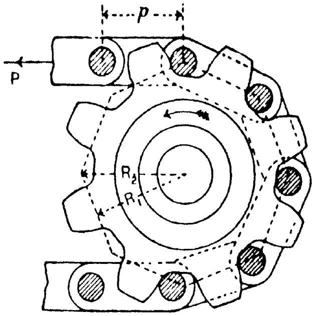

FIG. 584.—Sprocket wheel for chain driving, showing the effect produced by the chain stretching.

A **sprocket wheel** is shown in Fig. 584. The centres of the chain pins lie at the corners of a polygon having sides equal to the pitch p of the chain. The driving force P may act at radii which will vary from R_1 to R_2, and thus cause variations in the turning moment and will be no slip anywhere, *i.e.* the rolling will be perfect. Wheels of this kind are called **bevel wheels.**

Driving by toothed wheels. Motion lost by reason of slipping may be eliminated entirely by the addition of teeth to the rims of friction wheels. Fig. 588 shows two toothed wheels in gear; the original friction wheels are shown dotted, and come into contact at a point on the line joining the centres of the wheels. This point is called the **pitch point,** and the circles are called **pitch circles.** The length of the arc on the pitch circle between the centres of an adjacent pair of teeth is called the **circular pitch** of the teeth. It is evident that the pitch must be the same for both wheels. For practical purposes, the **diametral pitch** is used often, and is the result of dividing the diameter of the wheel by the number of teeth.

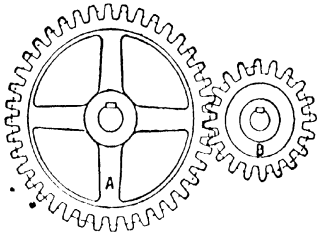

FIG. 588.—Toothed wheels in gear.

Let D = the diameter of the wheel.
n = the number of teeth.
p = the circular pitch.
p_d = the diametral pitch.

Then $$pn = \pi D,$$

or $$p = \frac{\pi D}{n}.$$

Also, $$p_d = \frac{D}{n} = \frac{p}{\pi}.$$

Unless otherwise specified, the term "pitch" will be taken to mean the circular pitch.

Referring to Fig. 589, other definitions are as follows: The portion EFGC of the tooth which lies outside the pitch circle is called the **addendum**; the dotted circle FGL is the **addendum circle**; the working sides of the tooth at EF and CG are called **faces.** The portion EHKC which lies within the pitch circle is called the **root** of the tooth; the dotted circle HKM is the **root circle**; the working sides EH and CK are called **flanks.** EC is the **thickness** of the tooth and CD is the **width of the space** between the teeth.

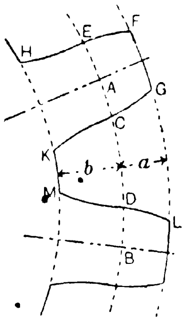

FIG. 589.—Proportions of wheel teeth

Ordinary proportions of teeth may be stated. Reference is made to Fig. 589, and p is the circular pitch.

$$\text{Thickness of tooth} = 0{\cdot}48p.$$
$$\text{Space between teeth} = 0{\cdot}52p.$$
$$\text{Total length of tooth} = (a+b) = 0{\cdot}7p.$$
$$\text{Length of addendum} = a = 0{\cdot}3p.$$
$$\text{Length of root} \quad b = 0{\cdot}4p.$$
$$\text{Width of tooth} = 2p \text{ to } 3\tfrac{1}{2}p.$$

These proportions allow of a clearance equal to $0{\cdot}04p$ between the thickness of the tooth and the space into which it enters on the other wheel; also a clearance of $0{\cdot}1p$ between the point of the tooth and the bottom of the space. With accurate machine cut teeth, these clearances are often made smaller.

Power transmitted by toothed wheels. Let P lb. be the driving effort applied to a toothed wheel tangential to the pitch circle, and let R feet be the radius of the pitch circle. In one revolution, work will be done equal to 2πRP foot lb. If the wheel makes N revolutions per minute, we have

$$\text{Work done per minute} = 2\pi\text{RPN}.$$

$$\text{Horse power transmitted} = \frac{2\pi\text{RPN}}{33000},$$

or

$$\text{P} = \frac{33000 \times \text{horse power}}{2\pi\text{RN}}.$$

If the horse-power be given, P may be calculated, and hence the dimensions of the tooth may be estimated in order that sufficient strength may be secured. It is best to use the rules of proportional strength. Suppose it is known that a certain wheel made of a given material has transmitted a force P_1 successfully, and that the width, length and thickness of its teeth are b_1, l_1 and t_1 respectively. The connection of these dimensions with those of the teeth of another wheel of the same material which has to transmit a force P_2 will be given by (p. 152)

$$\text{P}_1 : \text{P}_2 = \frac{b_1 t_1^2}{l_1} : \frac{b_2 t_2^2}{l_2}.$$

It has been assumed here that P_1 and P_2 are applied at the extreme point of the tooth, as in practice might be the case by accident. Also that the whole of the driving effort may act possibly on one tooth.

Angular velocity ratio of toothed wheels. It is evident from Fig. 590 that two toothed wheels in gear revolve in opposite directions; also that the speeds of the circumferences of the pitch circles will be equal. Hence,

$$\frac{N_A}{N_B}=\frac{D_B}{D_A}.$$

Also,

Number of teeth on $A = n_A = \frac{\pi D_A}{p}$;

Number of teeth on $B = n_B = \frac{\pi D_B}{p}$;

$$\therefore \frac{D_B}{D_A}=\frac{n_B}{n_A},$$

and

$$\frac{N_A}{N_B}=\frac{n_B}{n_A} \quad \ldots\ldots\ldots \quad (1)$$

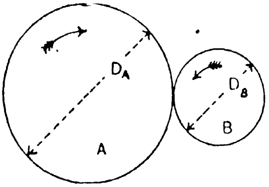

FIG. 590.—Angular velocity ratio of toothed wheels.

Hence, **the revolutions per minute are inversely proportional to the numbers of teeth.** It will be obvious, from what has been said on p. 539 regarding friction bevel wheels, that the same rule applies also to such wheels.

If the wheels A and B are required to revolve in the same direction, an **idle wheel** C may be interposed (Fig. 591). Since the velocities of all three pitch circle circumferences must be equal, it follows that there will be no change in the angular-velocity ratio of A and B. Hence,

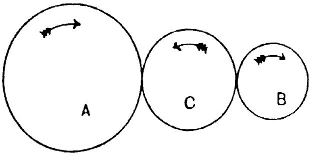

FIG. 591.—Use of an idle wheel

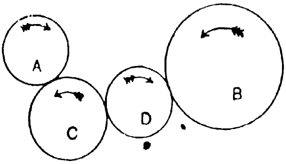

FIG. 592.—Two idle wheels.

$$\frac{N_A}{N_B}=\frac{n_B}{n_A}.$$

Any number of idle wheels (Fig. 592) may be inserted without affecting the angular-velocity ratio of A and B.

Trains of wheels. Fig. 593 shows a train of toothed wheels. In this case we have:

$$\frac{N_E}{N_F}=\frac{n_F}{n_E}; \quad \frac{N_D}{N_C}=\frac{n_C}{n_D}; \quad \frac{N_B}{N_A}=\frac{n_A}{n_B}.$$

Also, $N_E = N_D$; $N_C = N_B$.

Hence,

$$\frac{N_F}{N_E} \times \frac{N_D}{N_C} \times \frac{N_B}{N_A} = \frac{n_E}{n_F} \times \frac{n_C}{n_D} \times \frac{n_A}{n_B},$$

or

$$\frac{N_F}{N_A} = \frac{n_E n_C n_A}{n_F n_D n_B} \quad \ldots\ldots\ldots(2)$$

If F, D and B be called **drivers**, and E, C and A **followers**, the above result gives us the rule that **the angular-velocity ratio of the first and last wheels in the train is equal to the product of the numbers of teeth on the followers divided by the product of the numbers of teeth on the drivers.**

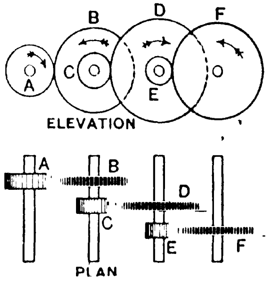

FIG. 593.—Train of wheels.

Fig. 594 shows the gearing wheels used in the Wolseley motor cars for enabling the car to run at different speeds. The shaft AB is driven by the engine, and has a wheel C fixed to it and gearing always with a wheel G on the secondary shaft EF. When the clutch between the engine and AB is "in," the secondary shaft EF will be revolving. H, K and L are wheels of different sizes mounted on, and revolving with, EF. The shaft RS is connected at S to the road wheel axle by means of gearing not shown in Fig. 594; this shaft runs freely in the hollow shaft AB, and is made square between R and S. M, N, P and Q are wheels which may slide on the square shaft RS, and are under the control of the driver by means of an arrangement of interlocking bars (not shown in the figure). The wheel C is hollow, and is furnished with internal teeth at D. M may be slid into C, and, when so situated, AB will drive RS direct, the secondary shaft EF then running idle. Other speeds may be obtained by withdrawing M from C and gearing N with H, or P with K, or Q with L. The lever system for sliding the wheels is so devised

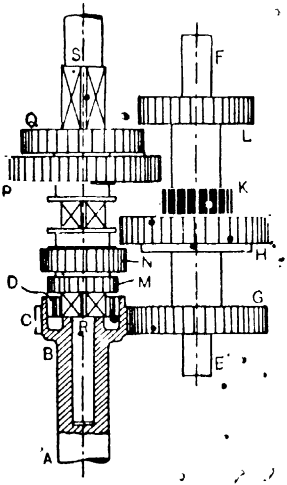

FIG. 594.—Gear wheels for a motor car.

as to prevent two pairs of wheels being in gear simultaneously. Reversal of the car is obtained by sliding idle wheels (not shown in the figure) on another secondary shaft.

Bevel wheels. If the directions of the shaft axes intersect, it has been shown (p. 539) that cones may be used for driving; hence conical pitch surfaces are employed for toothed wheels on intersecting shafts. In Fig. 595, the axes of the shafts intersect at O, and OAB and OBC are the conical pitch surfaces. The dimensions are settled from the relation

$$\frac{N_{OD}}{N_{OE}} = \frac{BC}{AB}$$

To obtain the shape of the teeth, ADB and BEC are other conical surfaces obtained by drawing BD and BE perpendicular to OB. These conical surfaces are developed by describing arcs BF and BG, using D and E respectively as centres. The teeth may then be drawn on these arcs as pitch circles by ordinary methods. The teeth are tapered along the conical surfaces AOB and BOC, and finally vanish at O; hence portions only of the conical surfaces are used, shown in the figure at BKHC and BKLA.

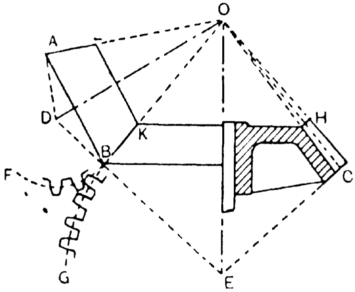

FIG. 595 —Bevel wheels.

Mitre wheels are bevel wheels of equal size on shafts meeting at 90°, and are used in cases where the shafts are to have equal speeds of rotation.

In Fig. 596 is shown an example of the use of mitre wheels. A is a continuously revolving shaft having a mitre wheel B fixed to it, and driving other two mitre wheels C and D which run loose on the shaft EF. Each of the mitre wheels C and D has projecting claws on its inner face, which may engage with the claws of a clutch G. G may slide on the shaft, and has a long feather key which compels it to rotate with the shaft; a pivoted lever H enables the clutch to be operated. In the position shown no motion will be communicated to the shaft EF; motion of either sense of rotation may be obtained by causing G to engage with either C or D; the arrows indicate the directions of rotation. The arrangement thus provides for intermittent motion and for reversal.

Fig. 597 illustrates a common type of **differential gearing** used for the driving axle of a motor car. A toothed wheel A, shown in section, runs loose on the axle EF, and has two bevel wheels B, B mounted on radial spindles. EF is the axle to which the road wheels are attached,

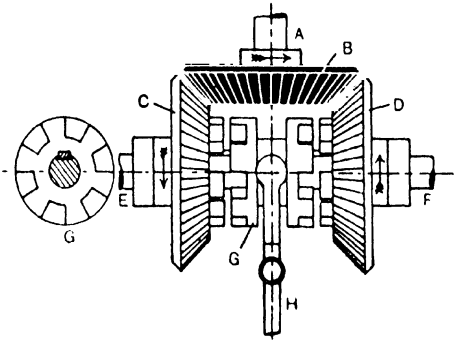

FIG. 596.—Arrangement for intermittent motion and for reversal.

and is made in two pieces. A bevel wheel C is fixed to the portion E, and another bevel wheel D is fixed to F; C and D gear with the bevel wheels B, B. The wheel A is driven by the engine, and, if both road wheels are rotating at the same speed, the wheels B, B will not rotate on their spindles. In rounding a curve, the inner road wheel must rotate at a lower speed than the outer wheel, and this difference in speed is permitted by the bevel wheels B, B

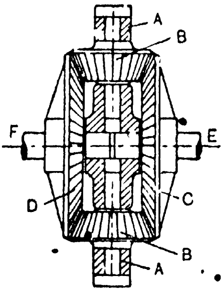

FIG. 597.—Differential gear for a motor car.

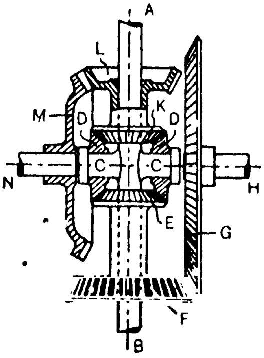

FIG. 598.—Milne's Daimler differential gear.

rotating on their spindles. It will be evident that, if C were held fixed, D would rotate at twice its former speed.

Fig. 598 shows the application of the same arrangement in the Milne's-Daimler differential gear.* AB is a shaft driven by the engine and carries mitre wheels D, D, running loose on cross spindles C, C. These wheels gear into mitre wheels E and K at the inner ends of sleeves which run loose on AB, thus permitting differential motion to the sleeves. F and L are bevel wheels at the outer ends of the sleeves, and gear with wheels G and M fixed respectively to the halves H and N of the road-wheel axle.

Epicyclic trains of wheels. In trains of this kind there is usually one fixed wheel A (Fig. 599)—*i.e.* A does not rotate—together with one or more wheels mounted on an arm D which may rotate about the centre of A. The solution of such trains may be obtained by the following method. Imagine the whole set of wheels to be locked and that the bracket carrying A is free to rotate. Give the whole arrangement one rotation in the clockwise direction, then, keeping the arm fixed in position, apply a correction by giving A one revolution in the anti-clockwise direction. Calling clockwise rotation positive, the process may be tabulated thus:

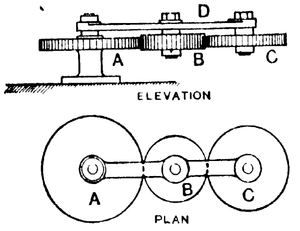

FIG. 599.—An epicyclic train of wheels.

Part - - -	A	B	C	D
Wheels all locked	+1	+1	+1	+1
Correction - -	-1	$+\frac{n_A}{n_B}$	$-\frac{n_A}{n_C}$	0
Final result - -	0	$1+\frac{n_A}{n_B}$	$1-\frac{n_A}{n_C}$	+1

The result shows that, if A and B have the same number of teeth, B will rotate twice clockwise for one clockwise rotation of the arm. If A and C have the same number of teeth, C will not rotate on its spindle; a radial arrow sketched on the upper side of C will point always in the same direction as the arm D is rotated.

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* *Proc. Inst. Mech. Eng.*, 1907.

Epicyclic reducing gears. In Fig. 600, showing an arrangement for reducing the speed of rotation, the wheel D is fixed and has internal teeth; E is an arm fixed to the shaft F, and carries two wheels B and C fixed together so as to revolve as one wheel. C gears with the internal teeth of D, and B is driven by a wheel A. It will be noted that, if D drives C with the arm E fixed, both wheels will have the same sense of rotation. The solution is as follows:

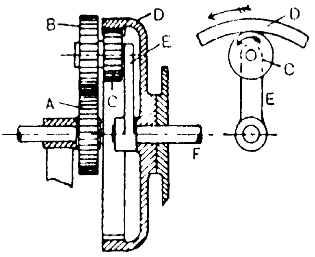

Fig. 600. Speed reduction gear

Part	A	B	C	D	E	F
Wheels all locked	+1	+1	+1	+1	+1	+1
Correction	$+\frac{n_B}{n_A}\frac{n_D}{n_C}$	$\frac{n_D}{n_C}$	$\frac{n_D}{n_C}$	1	0	0
Final result	$+\left(1+\frac{n_B n_D}{n_A n_C}\right)$	$1 \quad \frac{n_D}{n_C}$	$1 \quad \frac{n_D}{n_C}$	0	+1	+1

In Fig. 601, showing another type of speed reduction gear, the shaft AB is driven by a wheel at A, and has an arm C fixed to it

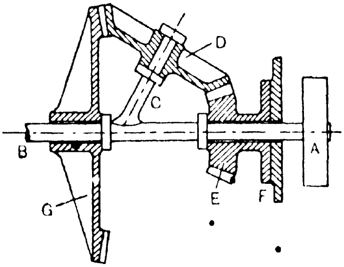

Fig. 601.—Another type of speed reduction gear.

carrying a loose bevel wheel D. D gears with two bevel wheels E and G running loose on the shaft AB. E may be a fixed wheel, or may be rotated in the same or in the opposite sense to that of AB. It is evident that D does not rotate on C during the locked operation, and that E and G will rotate in opposite directions during the correction operation with AB and C fixed, D being an idle wheel during this operation. Supposing E to be a fixed wheel, the solution will be as follows:

Part	AB	C	E	G
Wheels all locked	+ 1	+ 1	+ 1	+ 1
Correction	0	0	− 1	$+\frac{n_1}{n_G}$
Final result	+ 1	+ 1	0	$+\left(1+\frac{n_1}{n_G}\right)$

Suppose now that E is not a fixed wheel, but is rotated ± N_1 times during + 1 revolution of AB. The solution will be:

Part	AB	C	E	G
Wheels all locked	+ 1	+ 1	+ 1	+ 1
Correction	0	0	$-1 + N_1$	$+\frac{n_1}{n_G} + \frac{n_1}{n_G} N_1$
Final result	+ 1	+ 1	$+ N_1$	$1 + \frac{n_1}{n_G}(1 + N_1)$

In the result for G, the − sign is to be taken if A and E are driven in the same direction, and the + sign if they are driven in opposite directions.

The **Humpage gear** is shown diagrammatically in Fig. 602. A is the driving shaft and has a bevel wheel B fixed to it. Two bevel wheels C and D, made in one piece so as to rotate together, run on an arm E fixed to a sleeve F, which runs loose on the shaft A. C gears with a fixed bevel wheel G, and D gears with a bevel wheel H, which is secured to the driven shaft K. For the sake of obtaining balance and of producing practically a driving couple, the arm E and the wheels C and D are duplicated. The solution may be obtained by giving the whole gear + 1 revolution with the wheels locked; apply a correction by keeping the sleeve F and the arm E fixed and giving

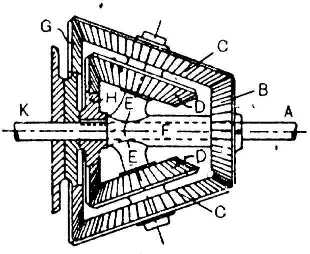

FIG. 602.—Humpage gear.

−1 revolution to G. During this correction, C will act as an idle wheel between G and B; also G will drive H in the anti-clockwise sense through the wheels C and D; the ratio of the revolutions of G and H during this operation will be

$$\frac{N_H}{N_G} = \frac{n_G \times n_D}{n_C \times n_H}.$$

Tabulating the operations, we have

Part - - -	A	F	G	K
Wheels all locked	$+1$	$+1$	$+1$	$+1$
Correction - -	$+\frac{n_G}{n_B}$	0	-1	$\frac{n_G \times n_D}{n_C \times n_H}$
Final result - -	$1+\frac{n_G}{n_B}$	$+1$	0	$1-\left(\frac{n_G \times n_D}{n_C \times n_H}\right)$

Hence,
$$\frac{N_A}{N_K} = \frac{1+\frac{n_G}{n_B}}{1-\left(\frac{n_G \times n_D}{n_C \times n_H}\right)}.$$

Shape of teeth. The shape of the teeth must be such as to fulfil the condition of a uniform ratio of angular velocities in the wheels which gear together. If this condition be neglected, the teeth will work together badly, producing excessive wear and rattling owing to back lash.

Referring to Fig. 603, let P be the point of contact of two teeth, one on the wheel which has its centre at A and the other on the wheel which revolves about B. At P a point on wheel A is moving

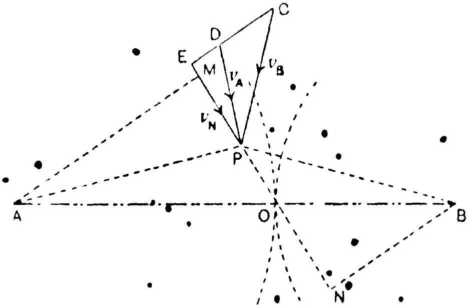

FIG. 603.—Condition for securing a constant angular velocity ratio in toothed wheels.

at right angles to AP, and a point on wheel B is moving at right angles to BP. Let v_A and v_B be these velocities, represented by DP and CP respectively. Let PO be the direction of a common normal to the tooth surfaces at P; it is clear that, if contact is to be maintained, and if there is to be no interpenetration of the teeth on A and B, the components of v_A and v_B along PO must be equal. Resolving v_B along and at right angles to PO by means of the triangle CEP, the equal normal components of v_A and v_B will be represented by $EP = v_N$. Produce the line of v_N, and draw AM and BN perpendicular to v_N. Then, if ω_A and ω_B are the angular velocities of the wheels A and B respectively,

$$\frac{\omega_A}{\omega_B} = \frac{v_N}{AM} \cdot \frac{BN}{v_N} = \frac{BN}{AM}.$$

Again, from the similar triangles AMO and BNO,

$$\frac{BN}{AM} = \frac{BO}{AO} = \frac{\omega_A}{\omega_B}.$$

If O be selected as the pitch point, the ratio BO/AO will be constant, as O is then a fixed point. Hence,

$$\frac{\omega_A}{\omega_B} = \frac{BO}{AO} = \frac{R_B}{R_A} = \text{a constant}.$$

Thus, the condition to be fulfilled in order to maintain a constant angular-velocity ratio is that **the common normal at any point of contact of two teeth must pass through the pitch point.** Theoretically, for a given design of tooth on one wheel, the teeth on the other wheel may be shaped so as to enable the common normal to comply with this condition. In practice, however, cycloidal teeth and involute teeth alone are used, and, in modern machine-cut wheels, the teeth are generally of the involute form.

Cycloidal teeth. The **cycloid** is a curve traced by a point P on

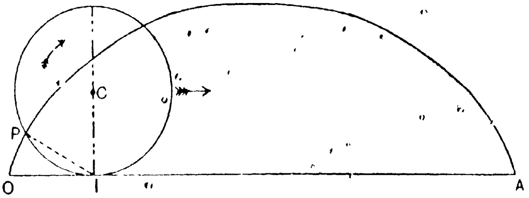

Fig. 604 —A cycloid.

the circumference of a circle which may roll along a straight line (Fig. 604). In any given position, the point of contact I is the instantaneous centre for the rolling wheel; hence the direction of the cycloidal curve at P is perpendicular to IP; therefore the normal at P passes through the point of contact I.

If the rolling circle having a centre C_1 (Fig. 605) rolls on the circumference of another circle having A for centre, an **epicycloid** OD will be traced. If the rolling circle rolls on the inside of the circumference of the circle, a **hypocycloid** OE will be traced (Fig. 605). In the epicycloid, if N_1 is the point of contact of the circles, and P_1 is the corresponding position of the tracing point, it will be clear that the direction of the epicycloidal curve at P_1 is at right angles to N_1P_1, as N_1 will be the instantaneous centre of the rolling circle in the given position. Hence N_1P_1 is the normal to the curve at P_1. For similar reasons, N_2 is the instantaneous centre of the rolling circle when the tracing point is at P_2 on the hypocycloidal curve, and N_2P_2 is the normal to the hypocycloid at P_2.

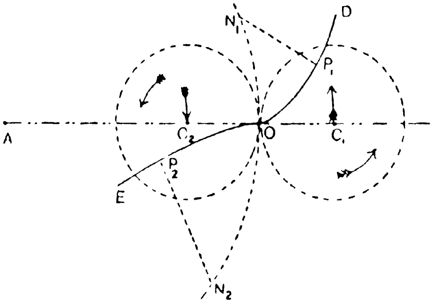

FIG. 605.—Epicycloid and hypocycloid.

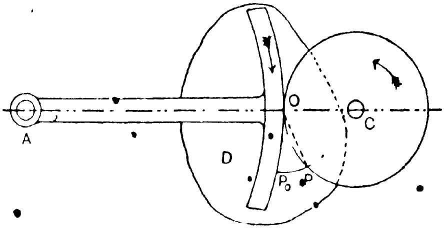

FIG. 606.—Mechanical construction of an epicycloid.

In Fig. 606 is shown a useful way of producing an epicycloid. The wheel A and the rolling circle revolve about fixed centres at A and C, and drive one another in the same manner as friction wheels but without slip. A piece of paper D is fixed to the wheel A and revolves with it, and a pencil P on the rolling circle bears on the paper. The result is the epicycloid P_0P. It is evident that the normal at P passes through the pitch point O. In Fig. 607 is shown a similar method of drawing a hypocycloid by means of another wheel having its centre at B, and the same rolling circle revolving about a fixed centre C. A piece of paper attached to B will have drawn on it a hypocycloid $P_0'P$. If there has been no slip, in each of these figures the arcs OP_0 and OP, on the wheels and on the rolling circles respectively, will be equal. Let the arcs OP_0 in each figure be equal, and imagine that the two figures are superposed, so that the wheels A and B come into contact at the pitch point O (Fig. 608). The arcs OP on the rolling circles in Figs. 606 and 607 will also be equal, and the points P will coincide in Fig. 608. OP will now be simultaneously the normal to the epicycloid and also to the hypocycloid, and these curves will be in contact at P. Therefore the curves comply with the condition that the common normal must pass through the

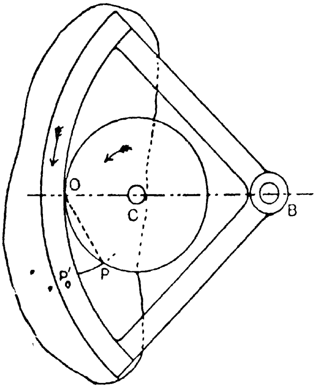

FIG. 607.—Mechanical construction of a hypocycloid.

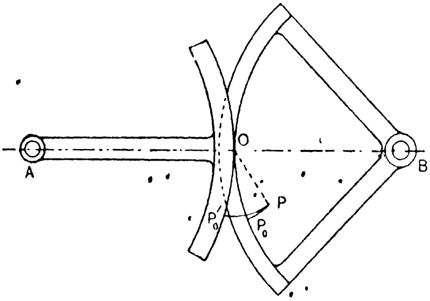

FIG. 608.—The constructions of Figs. 606 and 607 superposed.

pitch point, and thus may be used for the faces of the teeth on A, and for the flanks of the teeth on B. The flanks of the teeth on A and the faces of those on B may be produced in the same manner. It is evidently essential that the same rolling circle must be used both for the faces of A and for the flanks of B; the rolling circle used for the flanks of A and for the faces of B may be of the same or of another diameter. It should be noted that the hypocycloid becomes a straight line, forming a diameter of the wheel if the rolling circle has a diameter equal to the wheel radius; hence the flanks of the teeth would be radial lines. Any larger diameter of rolling circle would produce teeth thin and weak at the roots. In designing a set of wheels, the rolling circle should not have a diameter larger than the radius of the smallest wheel of the set.

Path of contact. From Figs 606, 607 and 608, it will be evident that P_0 and P'_0 on the cycloidal curves were initially in contact at O, and that the point of contact has travelled along the arc OP of the rolling circle. Contact will cease when the circumference of the rolling circle passes outside the addendum circle. In Fig 609, EFG and LMN are parts of the addendum circles of the wheels A and B respectively. These intersect the rolling circles at P and Q respectively; hence the complete **path of contact** is POQ, and is formed of two circular arcs.

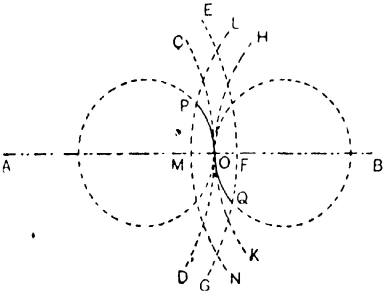

FIG. 609.—Path of contact in cycloidal teeth.

In Fig. 610 two teeth are just starting contact at P. The point C will be in contact when it reaches O, and the arc CO on the pitch circle is called the **arc of approach**. In the same figure, two teeth are just finishing contact at Q; E was in contact when passing through O, and OE is called the **arc of recess**. PO and OQ are called the **paths of approach and of recess** respectively. Let the arc OF be equal to the arc OE. Then COF is the length of arc which passes the pitch point while a tooth on A remains in contact with one on B, and

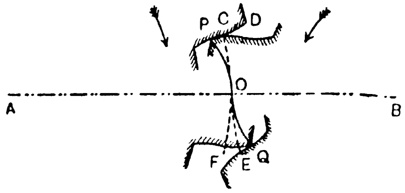

FIG. 610.—Arcs and paths of approach and recess.

may be called the **arc of contact.** If the condition is to be fulfilled that two pairs of teeth are to be in contact always, the arc of contact should be twice the pitch.

Involute teeth. Fig. 611 shows an involute P_0P_4 to a circular curve $P_0O_1O_4$; the curve may be drawn by wrapping a string round the circular curve and having a tracing pencil attached at its end P_0. On the string being unwrapped, the pencil will trace out the involute P_0P_4. It is evident that the string, in any position such as O_2P_2, is perpendicular to the direction of motion of the pencil; O_2 is therefore the instantaneous centre of the string O_2P_2 and O_2P_2 is normal to the involute at P_2.

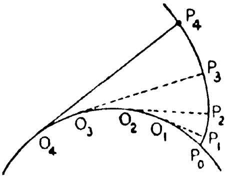

FIG. 611. Involute to a circle

In Fig. 612 is shown a mechanical method of drawing an involute to the circle having A for its centre. Let a crossed belt be passed

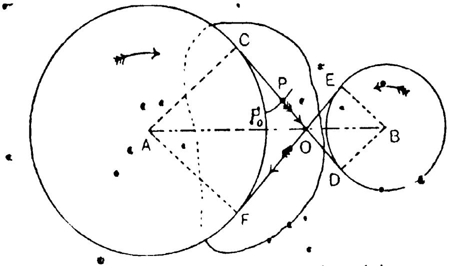

FIG. 612.—Mechanical construction of an involute to the circle having centre at A.

round two wheels revolving about A and B respectively, and let a piece of paper be fastened to wheel A and revolve with it. A tracing pencil secured to the belt at P will draw an involute on the paper. It is evident that CP, the normal to the involute at P, passes always through a point O on AB, and the two parts of the belt intersect in the same point. An involute to the wheel B may be drawn in a similar manner (Fig. 613) by securing the paper to wheel B. The

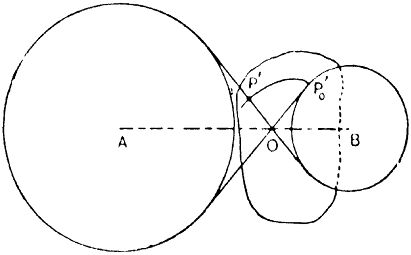

FIG. 613.—Mechanical construction of an involute to the circle having centre at B.

normal at P′ passes through the same point O. If the diagrams (Figs. 612 and 613) be superposed so that P and P′ coincide, it is evident that the two involute curves fulfil the condition that the common normal passes through a fixed point O, which accordingly may be taken for the pitch point of a pair of wheels having teeth shaped to the involute curves.

Let v be the velocity of the belt. Then, in Fig. 612,

$$\frac{\omega_A}{\omega_B} = \frac{v}{AC} \cdot \frac{DB}{v} = \frac{DB}{AC} = \text{a constant.}$$

Also, from the similar triangles AOC and BOD,

$$\frac{DB}{AC} = \frac{BO}{AO};$$

$$\therefore \frac{\omega_A}{\omega_B} = \frac{BO}{AO}.$$

Hence the radii AC and DB of the generating circles should be inversely proportional to the angular velocities of the wheels.

It is clear that part of the straight line CD (Fig. 612) will be the path of contact. Practical considerations rule that this line should make about 45° with the common tangent to the pitch circles at O (Fig. 614). The intersections P and Q of CD with the addendum circles of the wheels will determine the length PQ of the path of contact.

Using the same pair of generating circles connected by a belt as in

Fig. 612, the same involute curves will be produced irrespective of the distance AB separating the wheel centres. As the resulting teeth will be of the same shape as at first, it follows that the distance apart of a pair of involute toothed wheels may be varied to a small extent without interfering with their correct working. This may be advantageous for taking up back lash.

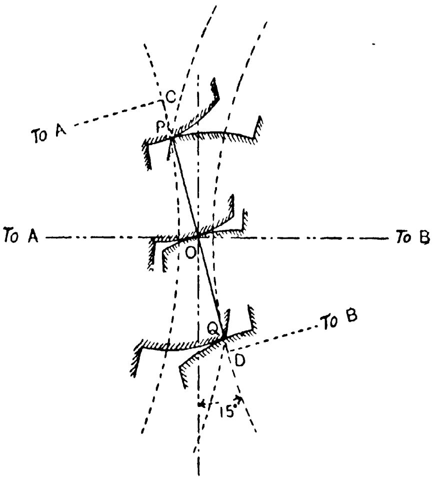

Fig. 614.—Path of contact in involute teeth.

If one of the two wheels in gear becomes of infinitely large radius, the case of a rack is obtained (Fig. 615). The pitch line CD is straight, and the involute is a straight line perpendicular to the line of contact OA. Hence the sides of the teeth in involute racks are straight lines.

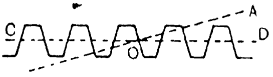

Fig. 615.—An involute rack.

Helical and screw gearing. Greater smoothness of running may be obtained by using wheels possessing several sets of teeth (Fig. 616), each set stepped back a little from the adjacent set on one side. If the steps are made indefinitely narrow (Fig. 617), we obtain a **helical wheel.** Single helical wheels would produce axial thrusts on the shafts, and this objection is obviated, as is indicated in Fig. 617, by employing double helical teeth sloping in opposite ways. Such wheels, with machine-cut teeth, run with remarkable smoothness, and are equally suitable for low and high speeds of running and for heavy loads. When the speed is high, it is best to run the wheels in an oil bath.

A pair of **screw wheels** is shown diagrammatically in Fig. 618. The cylindrical pitch surfaces of two wheels A and B touch at O. CD and EF are the axes of A and B respectively. Imagine a sheet of paper having a straight line GOH drawn on it to be placed between the cylinders. If the paper is wrapped round A, GOH will map out a helix, and, if wrapped round B, a corresponding helix will be described by GOH. These helices define the shape of screw teeth; the other teeth may be produced by having a number of lines parallel to GOH and drawn on the sheet of paper. In Fig. 619 GOH and *ed* show two of these lines. The perpendicular distance O*b* separating these lines is called the **divided normal pitch**, and is evidently the

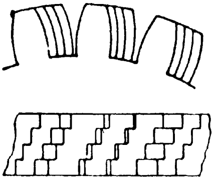
Fig. 616.—Stepped teeth.

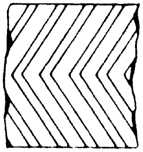
Fig. 617.—Double helical teeth.

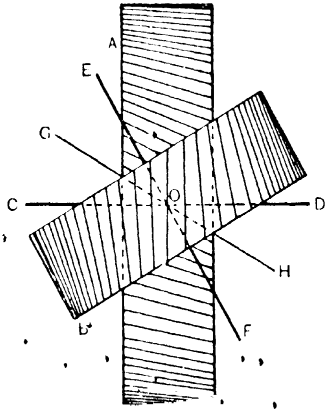

Fig. 618.—Pair of screw wheels.

same for both wheels A and B. *Oa*, measured along the circumference of A, and *Oc*, measured along the circumference of B, are called **divided circumferential pitches**; it will be clear that these pitches must divide evenly into the circumferences of A and B respectively.*

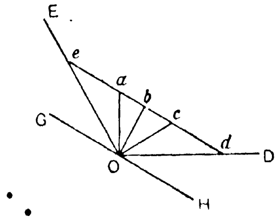

Fig. 619.—Pitches in a screw wheel

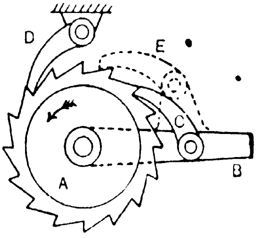

Fig. 620.—Pawl and ratchet wheel.

Ratchet wheels. In Fig. 620, a wheel A is to have intermittent motion to be derived from an arm B which vibrates about the axis of A. A pawl C is pivoted to B, and will engage the teeth of A when B is moving anti-clockwise; the pawl slips over the teeth of A when B is moving clockwise. Clockwise rotation of A may be prevented by a pawl D pivoted to some fixed part of the machine. It will be noted that lost motion to the extent of one tooth may occur between A and B; this may be reduced by means of a second pawl E pivoted to B. The possible lost motion will now be half the former amount. In cycle free wheels, several pawls are often fitted so as to reduce lost motion to a minimum.

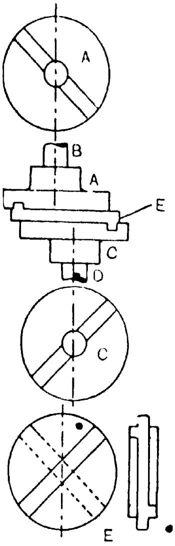

Fig. 621.—Oldham coupling.

Couplings for shafts. The **Oldham coupling** is illustrated in Fig. 621. A flanged coupling A is fixed to a shaft B, and has a groove cut in its face. Another similar coupling C is fixed to the shaft D. The axes of the shafts B and D are parallel. A plate E is interposed between the faces of these couplings, and has a projection on each side which is a sliding fit in the grooves; the projections are at 90° to each other. One shaft can thus drive the other, and as the grooves will always make 90° with each other, the shafts will have equal angular velocities in all positions.

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* For a complete discussion on toothed wheels, see *Machine Design, Part I.*, by Prof. W. C. Unwin. Longmans, 1909.

Hooke's coupling is illustrated in Fig. 622, and is used for connecting shafts in which the axes OA and OB intersect, but are not necessarily in the same straight line. The end of each shaft is formed with a jaw, and the connection is made by means of a cross C, which is free to swivel on the set screws D. The arrangement is shown in outline in Fig. 623, in which the ends of the arms OC and OC_1, attached to the shaft A (Fig. 623 (*b*)), rotate in the circle $YC''XC_1''$ (Fig. 623 (*b*)), the ends of the arms OD and OD_1, attached to the shaft B (Fig. 623 (*b*)), also rotate in a circular path, but this path projects as an ellipse Y_1XY_2 (Fig. 623 (*a*)) owing to the inclination of the shaft axes OA and OB.

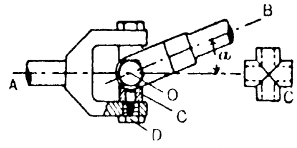
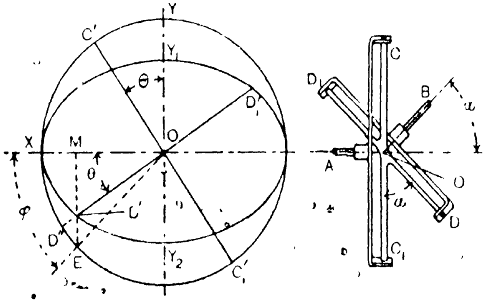


Fig. 622.—Hooke's coupling.

Suppose OC rotates from OY to OC'' through an angle θ (Fig. 623 (*a*)), and that the shafts OA and OB are in the same straight line. Then OD would rotate through an equal angle θ from OX and D would be situated at D'', OD'' being at 90° to OC''. If OB makes an angle α with OA (Fig. 623 (*b*)), then D will occupy a position on



(*a*) (*b*)

Fig. 623.—Diagram of a Hooke's coupling.

the ellipse, obtained by rotating the cross about $C'C_1'$ (Fig. 623 (*a*)); this operation will cause D'' to move at 90° to $C'OC_1'$, and gives the position of D as D' on the ellipse, *i.e.* OC' and OD' are still at 90°. The angle XOD' is not the true magnitude of the angle through which OD′ has rotated from OX; the true angle may be obtained by drawing D′E parallel to OY, cutting the circle at E; XOE = ϕ will then be the angle from OX through which OD rotates while OC rotates through an angle θ from OY. Produce ED′ to cut OX in M, and let α be the angle between the directions of OD and OC$_1$ as seen in Fig. 623 (*b*). Then

$$D'M = ME \cos \alpha.$$

Also,

$$\tan \theta = \frac{D'M}{OM} = \frac{ME}{OM} \cos \alpha$$

$$= \tan \phi \cos \alpha,$$

or

$$\tan \phi = \frac{\tan \theta}{\cos \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This gives the relation of ϕ and θ. The relation of the angular velocities of A and B may be obtained by differentiating both sides of (1) with respect to time. Thus,

$$\sec^2\phi \cdot \frac{d\phi}{dt} = \frac{\cos \alpha \sec^2\theta}{\cos^2\alpha} \cdot \frac{d\theta}{dt} = \frac{\sec^2\theta}{\cos \alpha} \cdot \frac{d\theta}{dt}.$$

Now,

$$\omega_B = \frac{d\phi}{dt}, \quad \text{and} \quad \omega_A = \frac{d\theta}{dt};$$

$$\therefore \quad \omega_B \sec^2\phi = \omega_A \frac{\sec^2\theta}{\cos \alpha},$$

or

$$\frac{\omega_B}{\omega_A} = \frac{\sec^2\theta}{\sec^2\phi \cos \alpha}.$$

Now,

$$\sec^2\phi = 1 + \tan^2\phi$$

$$= 1 + \frac{\tan^2\theta}{\cos^2\alpha} \quad \text{(from (1))};$$

$$\therefore \quad \frac{\omega_B}{\omega_A} = \frac{\sec^2\theta}{\left(1 + \frac{\tan^2\theta}{\cos^2\alpha}\right) \cos \alpha}$$

$$= \frac{\cos \alpha}{\cos^2\theta \cos^2\alpha + \sin^2\theta}$$

$$= \frac{\cos \alpha}{\cos^2\theta (1 - \sin^2\alpha) + \sin^2\theta}$$

$$= \frac{\cos \alpha}{1 - \sin^2\alpha \cos^2\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This ratio will have maxima values when $\cos^2\theta$ is a maximum; this will occur when $\cos \theta$ is $+1$ or -1, *i.e.* when θ is 0° or 180°. The minimum value of the ratio will occur when $\cos^2\theta$ has its minimum value; this occurs when $\cos \theta$ is 0, *i.e.* when θ is 90° or

270°. Equality in the angular velocities occurs when the numerator and denominator in (2) are equal, giving

$$1 - \sin^2\alpha \cos^2\theta = \cos\alpha,$$

$$\sin^2\alpha \cos^2\theta = 1 - \cos\alpha,$$

$$\cos^2\theta = \frac{1 - \cos\alpha}{\sin^2\alpha} = \frac{1 - \cos\alpha}{1 - \cos^2\alpha}$$

$$= \frac{1}{1 + \cos\alpha},$$

$$\cos\theta = \pm\frac{1}{\sqrt{1 + \cos\alpha}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The student will find it a useful exercise to plot values of the ratio of ω_B to ω_A for values of θ from 0° to 360°.

EXERCISES ON CHAPTER XXI.

1. An engine runs at 200 revolutions per minute and drives a line shaft by means of a belt. The engine pulley is 24 inches diameter and the line-shaft pulley is 20 inches diameter. A dynamo is driven from a pulley 36 inches diameter on the line shaft by a belt running on a pulley 8 inches diameter on the dynamo shaft. Find the speeds of the line shaft and of the dynamo (*a*) if there is no slip, (*b*) if there is 5 per cent. slip at each belt.

2. A line shaft runs at 150 revolutions per minute. A machine has to be driven at 1800 revolutions per minute by belts from the line shaft; the pulley on the machine is 6 inches in diameter. In this particular case it is not desirable to use pulleys exceeding 36 inches or less than 6 inches in diameter, and it may be assumed that there will be 4 per cent. slip at each belt. Sketch a suitable arrangement giving the diameters of the pulleys and the speeds of any counter shafts employed.

3. A belt laps 180 degrees round a pulley rim. The larger pull applied is 400 lb. and the coefficient of friction is 0·5. Find the smaller pull, T_2, when slipping just occurs. Find also the pull in the belt at intervals of 30 degrees round the half-circumference of the pulley, and plot these on a base representing angles.

4. A rope is wound three times round a rough post, and one end of the rope is pulled with a force of 20 lb. If the coefficient of friction between the rope and the post is 0·35, what pull at the other end of the rope would cause it to slip round the post? (B.E.)

5. Find, from the following data, what width of leather belt is needed to transmit 25 horse-power to a certain machine: (*a*) Diameter of belt pulley, 30 inches. (*b*) The belt is in contact with $\frac{7}{12}$ of the circumference of the pulley. (*c*) Revolutions of pulley per minute, 150. (*d*) Coefficient of friction between belt and pulley 0·22. (*e*) Safe maximum tension per inch width of belt, 80 lb. (B.E.)

6. A factory engine develops 400 horse-power, which is transmitted to the line shafting in the various mill floors by 20 hemp ropes. Find, from the following data, the maximum tension in any one of the ropes, if they all transmit an equal share of the total power :—(*a*) Diameter of grooved flywheel on which ropes work, 20 feet. (*b*) Angle of groove, 60 degrees. (*c*) Angle of contact of ropes with flywheel rim, 240 degrees. (*d*) Coefficient of friction, 0·18. (*e*) Revolutions of flywheel per minute, 80. (B.E.)

7. A compressor is driven by a gas engine of 18 indicated horse-power, running at 240 revolutions per minute, by means of a belt 0·5 inch thick from the engine pulley, which is 1 foot in diameter. The compressor is double-acting, mean pressure 50 lb. per square inch, cylinder diameter 8 inches, stroke 14 inches. If the mechanical efficiency of the engine is 82 per cent., of the compressor 86 per cent., and if the slip of the belt is 5 per cent., find the maximum speed at which the compressor can be run and the minimum diameter of the pulley fitted to it. (L.U.)

8. A rope drives a grooved pulley, the speed of the rope being 5000 feet per minute. Find the horse-power transmitted by the rope from the following data : $\mu=0{\cdot}25$; angle of groove 45° ; angle of lap 200° ; weight of rope per foot run 0·28 pound ; maximum permissible tension in the rope 200 pounds. (You are expected to make allowance for centrifugal effects on the rope.) (L.U.)

9. A machine demands 6 horse-power, and is driven by means of a spur wheel 18 inches in diameter and running at 150 revolutions per minute. Find the tangential driving effort on the teeth of the spur wheel.

10. In a turning lathe, the slide-rest holding the tool is driven by a leading screw having 3 threads per inch. It is desired to cut a screw of 18 threads per inch. Give suitable numbers of teeth for a wheel train connecting the lathe mandrel to the leading screw.

11. A watch is wound up at the same time each night and the main spring spindle receives 3·5 turns during the winding What is the velocity ratio of the train of wheels connecting the hour hand with the main spring spindle? What is the velocity ratio of the train connecting the minute hand with the hour hand? Give suitable numbers of teeth for the latter train, no wheels to have more than 36 nor less than 8 teeth.

12. The driving wheels of a motor car are 3·5 feet in diameter, and the engine runs at a constant speed of 900 revolutions per minute. Find the velocity ratios of wheel trains suitable for car speeds of 20, 12 and 5 miles per hour respectively.

13. Sketch and discuss the use of a differential gear (*a*) as a suitable means of connecting the driving wheels on a motor car, (*b*) as a speed-reducing gear : show how to calculate the speed ratio (L.U.)

14. In the epicyclic train shown in Fig. 600, the wheels have teeth as follows : D, 48 ; B, 20 ; C, 12 ; A, 30. If F makes one clockwise revolution, find the revolutions of A.

15. In the gear shown in Fig. 601, the numbers of teeth are : D, 40 ; E, 20 ; G, 40. If E is fixed, find the revolutions of G for one clockwise revolution of A. Answer the same if F is driven at the rate of 3 anti-clockwise revolutions for one clockwise revolution of A.

16. In the Humpage gear illustrated in Fig. 602, the wheels have teeth as follows : B, 25 ; C, 30 ; G, 45. Calculate the numbers of teeth on D and H. so that the ratio of the rotational speed of A to that of K is 56 : 5. (L.U.)

17. State and prove the geometrical condition which must be satisfied in order that a pair of spur wheels may gear together with a constant angular-velocity ratio. (I.C.E.)

18. The centres of two spur wheels in gear with one another are 12 inches apart. One wheel has 40 teeth, and the other has 20 teeth. Neglecting friction, the line of pressure between the teeth in gear makes a constant angle of 75° with the line of centres. The teeth are designed so that the path of contact of a pair of teeth in gear is 2 inches long, and is bisected by the line of centres. Draw full size a side elevation of two teeth in gear (L.U.)

19. The axes of two shafts intersect at an angle of 150°. The shafts are connected by a Hooke's coupling. On a straight base 8 inches long, representing 360°, draw a curve whose ordinates represent the angular velocity of the driven shaft for one revolution, the angular velocity of the driving shaft being constant and represented by an ordinate 2 inches long. (L.U.)

CHAPTER XXII.

HYDRAULIC PRESSURE. HYDRAULIC MACHINES.

Some properties of fluids. **A fluid may be defined as a substance which cannot offer permanent resistance to forces which tend to change its shape.** Fluids are either **liquid** or **gaseous**; gases possess the property of indefinite expansion. Liquids alter their bulk but slightly under pressure, and such small changes may be disregarded usually. Gases exist either as **vapours** or as so-called **perfect gases**; the perfect gas was supposed to exist as a gas under all conditions of pressure and temperature; but it is now well known that all gases can be liquefied by great pressure and cold. A vapour may be defined as a gas near its liquefying point, and a perfect gas as the same substance far removed from its liquefying point.

Liquids are said to be **mobile** when they change their shape very easily; chloroform is an example showing great mobility, a property which renders it useful for delicate spirit levels. **Viscous liquids** are those which change their shape with difficulty; examples of such are cylinder oil and treacle.

Change of shape of a body always occurs as a consequence of the application of shearing stresses. A rectangular block under the action of equal push stresses on all its faces will have its volume diminished, but will remain rectangular; shearing stresses applied to the block would alter its shape (p. 107). Hence, if shearing stresses be applied to a fluid, change of shape of the fluid will go on continuously during the application of the stresses, *i.e.* the fluid will be in motion. Conversely, **if the fluid is at rest, there cannot be any shearing stresses acting on it; the stresses must be normal at all parts.** Frictional forces always occur as tangential or shearing forces, and hence must be absent from any fluid at rest.

The principal liquid in use in hydraulics is water, and it will be understood that water is being referred to in the following sections unless some other liquid is specified.

Stress on horizontal immersed surfaces. In Fig. 624 is shown a tank containing a liquid at rest. Consider the equilibrium of a vertical column of the liquid standing on one square foot of the tank bottom. The forces acting will be (*a*) the weight of the column, (*b*) an upward reaction from the tank bottom, (*c*) normal forces which the surrounding liquid applies to the vertical sides of the column. The normal forces, being horizontal, cannot contribute to the support of the weight of the column, which is a vertical force, hence (*a*) and (*b*) equilibrate each other.

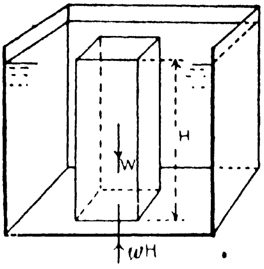


FIG. 624.—Stress on a horizontal immersed surface.

Let H = the height of the column, in feet.

w = the weight of a cubic foot of the liquid, in lb.

A = the total area of the tank bottom, in square feet.

Then Weight of the column $= H \times 1 \times 1 \times w$

$= wH$ lb.,

and this will be the reaction of one square foot of the tank bottom. Hence,

Stress on the tank bottom $= wH$ lb. per square foot.

The tank bottom being horizontal, the stress on any other square foot will be wH; hence the total pressure on the bottom may be calculated by multiplying the area of the bottom by wH. Thus,

Total pressure on the tank bottom $= wHA$ lb.

It will be noted that this result is quite independent of the shape of the tank, provided the bottom is horizontal. All tanks having horizontal bottoms of equal areas and charged with the same liquid to equal depths will have equal total pressures on their bottoms irrespective of the actual weights of liquid in the tanks. The student should guard against the error of supposing that the weight of liquid in the tank gives the pressure on the bottom.

Stress on inclined immersed surfaces. Let *abc* (Fig. 625) be the end elevation of a triangular prism immersed in a liquid and having its axis horizontal. The triangular ends are taken perpendicular to the axis and are vertical. Considering the equilibrium of the prism, the fluid pressures on the ends will evidently equilibrate each other.

If the sides of the prism be taken to be very small, then the weight of the prism may be disregarded, and the fluid stresses p, q and r, acting on ab, bc and ca respectively, may be assumed to be distributed uniformly over the faces of the prism. Let the length of the prism be unity, when the resultant forces on the three faces will be given by

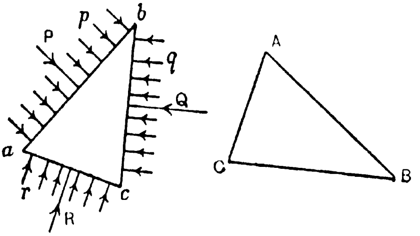


FIG. 625.—Stress on inclined immersed surfaces.

$$P = p \, . \, ab.$$
$$Q = q \, . \, bc.$$
$$R = r \, . \, ca.$$

For equilibrium of the prism, these forces must balance. It will be noted that P, Q and R meet at the centre of the circle passing through a, b and c, and hence comply with the condition that the three forces must pass through the same point. ABC (Fig. 625) is the triangle of forces, in which AB, BC and CA represent P, Q and R respectively. As these sides are drawn perpendicular to ab, bc and ca respectively, the triangles ABC and abc are similar. Hence,

$$P : Q : R = p \, . \, ab : q \, . \, bc : r \, . \, ca \quad \ldots\ldots (1)$$
$$= AB : BC : CA \quad \ldots\ldots (2)$$
$$= ab : bc : ca \quad \ldots\ldots (3)$$

From (1) and (3),

$$p \, . \, ab : q \, . \, bc : r \, . \, ca = ab : bc : ca;$$
$$\therefore \; p = q = r.$$

We may say therefore that the fluid stresses on the faces of the prism are equal. Considering the limiting case of the end elevation of the prism being reduced practically to a point by reason of the sides being taken indefinitely small, the law may be stated thus: **The stress at a point in a fluid is the same on all planes passing through that point, or fluids transmit stresses equally in all directions.** We have already seen that the stress on a horizontal plane is wH lb. per square foot, and it follows that the stress at a point H feet deep on any plane will be given by the same expression.

It will be noted that the stress at any point is proportional to the depth, and hence varies uniformly from zero value at the **free surface** of the liquid, *i.e.* the surface exposed to the atmosphere. **Stress diagrams** may be employed with advantage; such a diagram is given in Fig. 626 for the water stresses on each side of a lock gate having differing depths of water on the two sides. The stress at B will be $p_1 = wH_1$ and that at E will be $p_2 = wH_2$, and these are represented by CB and FE respectively. The complete stress diagrams are ABC and DEF, and their breadths will give the stress at any depth. The term **stress at a point in a fluid** may be defined as **the pressure which would be exerted on unit area embracing that point if the stresses were distributed uniformly.**

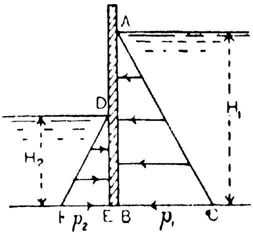


FIG. 626.—Stresses on a lock gate.

Total pressure on an immersed surface. In Fig. 627 (*a*) and (*b*) are shown front and end elevations of immersed surfaces, the former being vertical and the latter inclined. The method of finding the total pressure applies equally to both surfaces. Consider a small area *a* at a depth *y*. The stress on *a* will be

$$p = wy$$

and Force on $a = wya$

The total force on the surface may be found by integrating this expression over the whole area. Thus,

$$\text{Total force} = w\Sigma ay.$$

If the total area is A square feet and the depth of the centre of area is Y feet, then $\Sigma ay = AY$, (pp. 49 and 145). Hence,

$$\text{Total force} = wAY \text{ lb.}$$

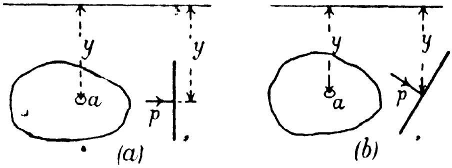


FIG. 627.—Total pressure on immersed surfaces.

The expression $w\mathrm{Y}$ is the stress at a depth Y, and may be defined as the average stress on the immersed surface. Hence the rule: **To find the total pressure on an immersed surface multiply the average stress (which will be found at the centre of area) by the total area.**

It will be noted that the force acting on the small area a, given above as wya, will have the same value in the case of the whole surface being curved. The rule for the total pressure is therefore not confined to flat surfaces, but may be applied to any immersed surface.

Distinction should be made between the terms **total pressure** and **resultant pressure**. The latter term refers to the resultant of all the fluid stresses acting on a surface, and is obtained by resolving these stresses along chosen axes and then reducing by the methods explained in Chapter IV. Usually the operation is simple; for example, the resultant pressure on any vessel containing a liquid is evidently equal to the weight of the contained liquid. A method of dealing with the resultant pressure on floating or immersed bodies will be explained below.

EXAMPLE 1. A cylindrical tank, diameter 7 feet, contains water to a depth of 4 feet. The bottom is horizontal. Calculate the total pressure and the resultant pressure on the wetted surface. Take 62·5 lb. per cubic foot for the weight of water.

$$\begin{aligned}\text{Total pressure on the bottom} &= w\mathrm{A}_1\mathrm{Y}_1\\ &= 62{\cdot}5 \times \frac{\pi d^2}{4} \times 4\\ &= 62{\cdot}5 \times \left(\tfrac{22}{7} \times \tfrac{49}{4}\right) \times 4\\ &= 9625 \text{ lb.}\end{aligned}$$

$$\begin{aligned}\text{Total pressure on the curved surface} &= w\mathrm{A}_2\mathrm{Y}_2\\ &= 62{\cdot}5 \times (\pi d \times 4) \times 2\\ &= 62{\cdot}5 \times \left(\tfrac{22}{7} \times 7 \times 4\right) \times 2\\ &= 11{,}000 \text{ lb.}\end{aligned}$$

$$\begin{aligned}\text{Total pressure on the wetted surface} &= 9625 + 11{,}000\\ &= \underline{20{,}625} \text{ lb.}\end{aligned}$$

The stresses on the curved surface will equilibrate each other; hence the resultant pressure is simply the total pressure on the bottom, or

$$\text{Resultant pressure} = \underline{9625} \text{ lb.}$$

EXAMPLE 2. A spherical vessel $3\frac{1}{2}$ feet in diameter is sunk in sea water, its centre being at a depth of 40 feet. Calculate the total pressure on its surface. Sea water weighs 64 lb. per cubic foot.

$$\begin{aligned}\text{Total pressure} &= w\mathrm{AY}\\ &= 64 \times 4\pi r^2 \times 40\\ &= 64 \times 4 \times \tfrac{22}{7} \times \tfrac{7}{4} \times \tfrac{7}{4} \times 40\\ &= \underline{98{,}560} \text{ lb.}\end{aligned}$$

Resultant pressure on a floating or immersed body. When a body is floating at rest in a fluid which is also at rest, it is subjected to two resultant forces—its weight and the resultant fluid pressure on its surfaces. The weight is a downward vertical force acting through G, the centre of gravity of the body (Fig. 628 (*a*)). The resultant fluid pressure balances W, and therefore must be an upward vertical force R = W, and must act in the same straight line with W. R is due to the buoyant effect of the fluid, and is called the **buoyancy.**

Imagine for a moment that the surrounding fluid becomes solid, and so can preserve its shape, and let the body be removed, leaving a cavity which it fits exactly (Fig. 628 (*b*)). Let this cavity be filled with the fluid, and let the surrounding fluid return again to its original condition. The pressures on the fluid now filling the cavity will be identical with those which acted on the body, and the effect will be the same—the weight of the fluid will be supported. Hence the weights of the fluid filling the cavity and of the body must be equal, as each is equal to R, the resultant pressure of the surrounding fluid. Further, R must act through the centre of gravity of the fluid filling the cavity; this centre is called the **centre of buoyancy,** and from what has been said it will be clear that the centre of buoyancy B (Fig. 628 (*b*)) and G (Fig. 628 (*a*)) must be in the same vertical line. We may state, therefore, that **when a body is floating at rest in still fluid, the weight of the body is equal to the weight of the fluid displaced, and that the centres of gravity of the body and of the displaced fluid are both in the same vertical line.**

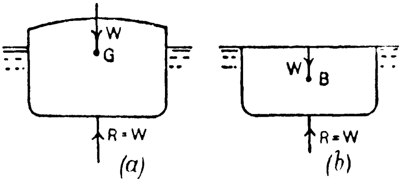


FIG. 628.—Resultant pressure on a floating body.

A ship floating at rest in still water, a submarine boat wholly immersed and at rest, and a balloon preserving constant elevation are examples of this principle. In each case it will be noted that the resultant pressure of the surrounding fluid is equal to the weight of the body, and acts vertically upwards through the centre of gravity of the body.

A body wholly immersed will experience a resultant upward fluid pressure equal to the weight of the fluid displaced; it follows that, to maintain the equilibrium of the body, an upward or a downward force will be required, depending on whether the weight of the body or the weight of the fluid displaced is the greater. Fig. 629 (*a*) illustrates the former case; W is the weight of the body, B is the buoyancy, and, W being greater than B, an upward force P is required given by

$$P + B = W.$$

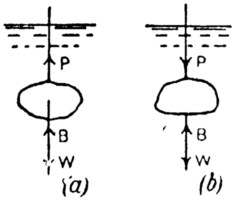


FIG. 629.—Equilibrium of immersed bodies.

Fig. 629 (*b*) shows the case of B being greater than W, when a downward force P is required, given by

$$P + W = B.$$

The **specific gravity** of a substance is defined as its weight in air as compared with the weight of an equal volume of pure water, usually taken at a temperature of 60° F.

Let W_s = weight of a given substance in air.

W_w = weight of an equal volume of water.

Then Specific gravity $= \rho = \frac{W_s}{W_w}$

and $W_w = \frac{W_s}{\rho}$.

It therefore follows that we may calculate the buoyancy of a solid body wholly immersed in pure water by dividing the weight of the body by the specific gravity of its material. This principle may be applied to find the specific gravity of a given substance which is heavier than water. In Fig. 629 (*a*), let P be measured by suspending the body by means of a fine wire or cord from a balance; also weigh the body in air to find W_s. Then

$$P + B = W_s.$$

Also, $$B = \frac{W_s}{\rho};$$

$$\therefore\ P + \frac{W_s}{\rho} = W_s,$$

$$\rho = \frac{W_s}{W_s - P}.$$

Since $(W_s - P)$ is the apparent loss of weight of the body when immersed in water, we may state that **the specific gravity of a body is**

equal to the weight of the body in air divided by its apparent loss of weight when immersed in water.

Centre of pressure. In Fig. 630 (*a*) is shown a flat vertical plate immersed in a liquid. R is the resultant pressure acting on one side of the plate and passes through a point C, which is defined as the **centre of pressure.** The position vertically of C may be found

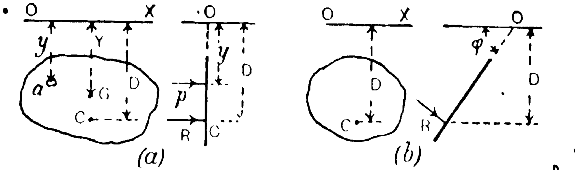


FIG. 630.—Centre of pressure.

by taking moments about OX, the line in which the plate produced cuts the surface of the liquid. Considering a small area a at a depth y, we have

$$\text{Pressure on } a = way,$$

$$\text{Moment of this pressure} = way^2.$$

Integration of this will give the total moment. Thus,

$$\text{Total moment} = w\Sigma ay^2.$$

Now Σay^2 is the second moment of area or moment of inertia (p. 145) of the surface of the plate about OX and may be written I_{OX} or Ak^2, where A is the area of the plate and k is the radius of gyration about OX. Hence,

$$\text{Total moment} = wAk^2. \quad \ldots \quad \ldots \quad (1)$$

Again, if D be the depth of the centre of pressure,

$$R = wAY$$

and

$$\text{Moment of } R = wAYD. \ldots \ldots \ldots \ldots (2)$$

Hence, from (1) and (2),

$$wAYD = wAk^2,$$

or

$$D = \frac{k^2}{Y} \ldots \ldots \ldots \ldots (3)$$

Both k and Y should be taken in foot units, when D will be in the same units.

The case of an inclined surface is shown in Fig. 630 (*b*). If ϕ is the angle of inclination to the horizontal, it may be shown that

$$D = \sin^2\phi \frac{k^2}{Y}, \ldots \ldots \ldots \ldots (4)$$

where k is the radius of gyration about OX, the line in which the plate cuts the surface of the water, and Y is the vertical depth of the centre of area.

In practical examples, usually the position of C horizontally may be easily determined from the symmetry of the plate.

EXAMPLE. A dock gate is 60 feet wide and has water on one side to a depth of 24 feet. Find the centre of pressure.

Let b = the breadth of the wetted surface.
d = the depth " "

Then $I_{OX} = \dfrac{bd^3}{3} = bd \cdot \dfrac{d^2}{3}$, $\therefore k^2 = \dfrac{d^2}{3}$.

Also $Y = \dfrac{d}{2}$;

$$\therefore D = \frac{k^2}{Y} = \frac{d^2}{3} \div \frac{d}{2}$$
$$= \tfrac{2}{3}d$$
$$= \tfrac{2}{3} \times 24 = 16 \text{ feet.}$$

The centre of pressure is therefore at a depth of 16 feet, and lies in the central vertical of the gate.

Stability of a floating body. A body floating at rest in a still liquid will be in **stable equilibrium** when, if rotated through a small vertical angle, it experiences a resultant couple tending to return it to the original position; the equilibrium will be **unstable** if the resultant couple has a moment tending to increase the angle of rotation. In Figs. 631 (*a*) and (*b*) are shown floating bodies which have been disturbed slightly from their positions of equilibrium; the weight, in each case, is a vertical force W, acting through the centre of gravity G; the buoyancy in each case is a vertical force R = W, acting through the centre of buoyancy B. It will be observed that in Fig. 631 (*a*) a couple is formed by R and W tending to restore the body to its original position; the equilibrium in the original position is therefore

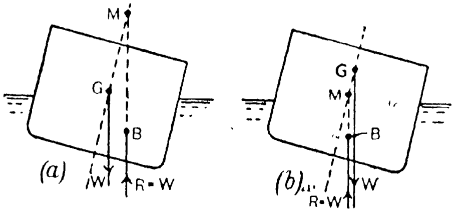


FIG. 631.—Stability of a floating body.

stable. In Fig. 631 (*b*) the couple tends to increase the angle through which the body has been turned, and the equilibrium in the original position is therefore unstable. A floating ball would be in neutral equilibrium.

It will be noticed that the line of R cuts the original vertical through G in a point M, which lies above G in Fig 631 (*a*) and below G in Fig. 631 (*b*). Clearly the sense of rotation of the couple formed by R and W is determined by consideration of the position of M above or below G; the couple will be of **righting moment** if M is above G, and of **upsetting moment** if M is below G. The point M is called the **metacentre.** The metacentre is of importance in calculations regarding the stability of ships; generally the naval architect finds the metacentre for transverse angles of displacement, which affects questions of the ship rolling, and also the metacentre for longitudinal angles of displacement, which affects questions of the ship pitching.

In Fig. 632, G is the centre of gravity and B is the centre of buoyancy of a body floating at rest in still water. G and B must fall in the same vertical, and the conditions of equilibrium are satisfied by the resultant water pressure R being equal to W, the weight of the body, both forces falling in the same straight line BG. To test for stability, the body is rotated through a very small angle θ, which, in order to avoid complication in the figure, has been secured by rotating the water plane from its original position *ab* into the position *a'b'*. G will remain unaltered in position, and B will move to B' in consequence of the body now being immersed deeper on the right-hand side. The weight of the body is now $W' = W$ and acts through G in a direction perpendicular to *a'b'*; the resultant pressure of the water will be $R' = W' = W$, acting through B' and also perpendicular to *a'b'*. R' produced cuts BG produced in the metacentre M.

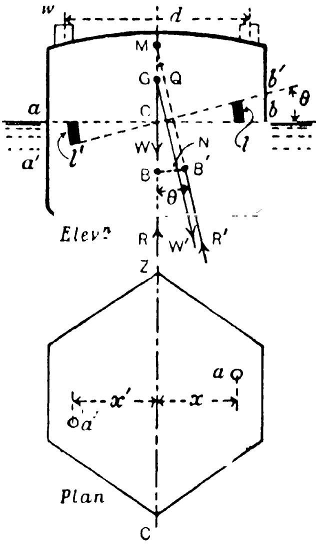


FIG. 632.- Metacentric height for a floating vessel.

As we assume that W′ and W are equal, it follows that the weight of the wedge *bCb′*, which has been added to the volume of water displaced, must be equal to that of the wedge *aCa′*, which has been taken away. In the plan of the plane of flotation *ab* (Fig. 632), small areas *a* and *a′* trace out arcs *l* and *l′* as seen in the elevation.

$$\text{Volume swept by } a = al.$$

Now, $$\frac{l}{x} = \theta;$$

$$\therefore\ l = x\theta.$$

Hence, $$\text{Volume swept by } a = ax\theta.$$

$$\text{Weight of this} = wax\theta, \qquad \ldots\ldots(1)$$

w being the weight of the water per cubic unit.

The total weight of both wedges must be zero from what has been said, and may be obtained by integrating (1) over the whole plane of flotation.

$$\text{Total weight of wedges} = w\theta\Sigma ax = 0.$$

Hence, $$\Sigma ax = 0.$$

This shows that the axis CZ in the plan must pass through the centre of area of the plane of flotation.

The resultant effect of the altered distribution of displacement will be found by calculating the total moment of weight of both wedges about CZ.

From (1), $$\text{Weight of the small volume} = w\theta ax.$$

$$\text{Moment of this about CZ} = w\theta ax^2.$$

$$\text{Total moment of both wedges} = w\theta\Sigma ax^2$$

$$= w\theta I_{CZ}. \qquad \ldots\ldots(2)$$

In this result, I_{CZ} is the second moment of area of the plane of flotation about CZ.

Now, if R′ be brought back to its original position R, we see that the effect of the altered distribution of displacement will be the couple, of moment R × BB′, which must be supplied in consequence of the shift.

$$\text{Moment of couple} = R \times BB' = W \times BB'.$$

Now, $$\frac{BB'}{BM} = \theta;$$

$$\therefore\ BB' = \theta \times BM.$$

$$\therefore\ \text{moment of couple} = W \times \theta \times BM. \qquad \ldots\ldots(3)$$

Hence, from (2) and (3),

$$W \times \theta \times BM = w\theta I_{CZ};$$

$$\therefore\ BM = \frac{wI_{CZ}}{W}. \qquad \ldots\ldots(4)$$

Let V = volume of water displaced by the body.

Then $$W = wV,$$

or $$V = \frac{W}{w}.$$

Substituting in (4), we have

$$BM = \frac{I_{cz}}{V}. \qquad (5)$$

Writing Ak_{cz}^2 for I_{cz}, we obtain a well known equation for BM, viz.:

$$BM = \frac{Ak_{cz}^2}{V}. \qquad (6)$$

From Fig. 632, we have

$$BM = BG + GM.$$

GM will be positive if M falls above G, in which case we have stable equilibrium; the equilibrium will be unstable if G falls below M, leading to a negative value of GM.

It will be noticed that the completion of the calculation depends on a knowledge of the position of the centre of gravity of the body. In the case of a body of simple outline and homogeneous in structure, this point is determined easily, but, in the case of a ship, is obtained only by long and laborious calculation. The calculations for Ak_{cz}^2 and for V required in equation (6), and also for the position of B, are carried out easily for a ship-shape body, and the result may be applied to the finished ship in an experimental determination of the centre of gravity. This is effected by moving weights on board so as to produce a small angle of heel, which is measured carefully by means of long plumb lines suspended in the holds. From a knowledge of the positions of M and B, together with the moment of the weights which have been moved and the angle of heel produced by this movement, the position of G is calculated easily. Thus, referring to Fig. 632, let the line of W′ cut BB′ in N and draw GQ perpendicular to B′M.

Let w = the weight moved, in tons.

d = the distance through which the weight is moved, in feet.

θ = the angle of heel produced by moving w, in radians

Then Capsizing moment due to moving $w = wd$ ton-feet.

$$\begin{aligned} \text{Righting moment} &= R' \times B'N \\ &= W \times GQ \\ &= W \times GM \times \theta. \end{aligned}$$

Hence, $$W \times GM \times \theta = wd,$$

$$GM = \frac{wd}{W\theta}.$$

The author is indebted to Mr. E. L. Attwood, Member of the Royal Corps of Naval Constructors, for the following example of a recent inclining experiment on a large ship.

EXAMPLE. Draught of ship, forward, 24′ 4″.
Draught of ship, aft, 26′ 2½″.

These dimensions correspond to a displacement of 15,357 tons, and a position of the transverse metacentre of 6·76 feet above the load water-line of the ship. 100 tons of ballast was used, arranged on the upper deck, in four lots of 25 tons each. The following measurements were taken by means of pendulums 20 feet in length, one forward, one aft:

Weight moved, tons.	Distance moved, feet.	Direction of movement.	Deflections of pendulums, in inches.	
			Forward.	Aft.
25	62	Port to starboard	7·9	8·1
50	62	,,	15·7	15·8
25	62	Starboard to port	8·0	8·0
50	62	,,	15·6	15·6

Taking the mean of these gives a deflection of 15·84″ for a shift of 50 tons through 62 feet. Hence,

$$GM = \frac{wd}{W\theta}$$

$$= \frac{50 \times 62}{15{,}357 \times \frac{15\cdot84}{240}}$$

$$= \underline{3\cdot06} \text{ feet.}$$

The centre of gravity of the ship is therefore $(6\cdot76 - 3\cdot06) = 3\cdot7$ feet above the load water-line.

Retaining wall for water. Referring to Fig. 633, ABCD is the section of a wall subjected to water pressure on its vertical face AB. In considering the stability of the wall, a portion one foot in length may be taken. For the simple trapezoidal section of wall illustrated, the weight may be calculated easily. Thus,

$$W = w'\left(\frac{AD + BC}{2}\right) H \text{ lb.},$$

where w' is the weight of the material in lb. per cubic foot. H is the height of the wall, and AD and BC are the thicknesses at the top and bottom respectively, all in foot units.

The centre of gravity of the wall section may be found by application of the following graphical method. Bisect AD in *a* and also

BC in *b*; then G lies in *ab*. Make A*c* and C*d* equal to BC and AD respectively, and join *cd* cutting *ab* in G.

If the reservoir is empty, the point *m*, in which the line of W cuts the base BC, will be the centre of pressure of the base of the wall, and the pressure on the base will be owing to W only, and hence will be entirely normal to the base. To find the pressure on the base and the centre of pressure when the reservoir is full, proceed as follows:

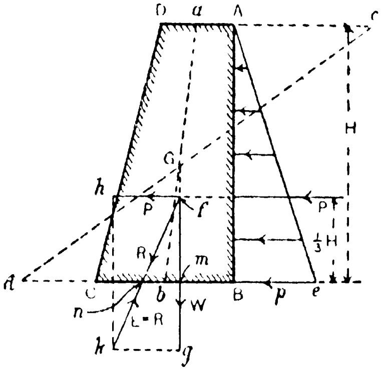


FIG. 633. Stability of a reservoir wall

$$\text{Total water pressure on the wall} = P = wAY \text{ (p. 567)}$$
$$= \frac{wH^2}{2} \text{ lb},$$

where w is the weight of the water in lb. per cubic foot. P will act at $\frac{1}{3}$H feet from B (p. 572), and will meet the line of W at *f*. Construct the parallelogram of forces *fgkh* for P and W acting at *f*, thus finding the resultant pressure R on the wall base. R intersects BC at *n*, thus giving the centre of pressure of the base for the case of the reservoir being full. It is taken usually that the wall will be safe if both *m* and *n* fall within the middle third of the base.

Every horizontal section of the wall will have a centre of pressure for the reservoir empty and another for reservoir full. If these centres be found, curves joining them may be drawn and give the lines of pressure for the wall. Fig. 634 shows how the construction may be carried out for sections 22′, 33′ and 44′. P_1 is the total water pressure on the whole wall, P_2, P_3 and P_4 are the pressures respectively for the portions lying above 22′, 33′ and 44′. W_1 is the total weight, and W_2, W_3 and W_4 are the weights corresponding to P_2, P_3 and P_4. The centres of gravity G_1, G_2, G_3 and G_4 are found

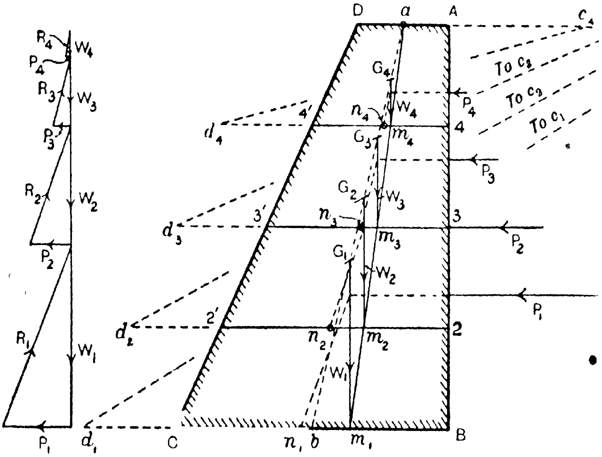


FIG 634.—Lines of pressure for a reservoir wall.

as before, and the lines of weight passing vertically through them give m_1, m_2, m_3 and m_4 on the line of pressure for reservoir empty. R_1 of P_1 and W_1 is found by means of the triangle of forces, and a line drawn from the point of intersection of P_1 and W_1 parallel to R_1, and cutting BC in n_1 gives a point on the line of pressure for reservoir full. The triangles of forces for the remaining forces are shown, and enable points n_2, n_3 and n_4 to be found similarly. The lines of pressure have been drawn separately in Fig. 635 for the sake of clearness. In Fig. 635 uv and st inclose the middle thirds of all sections, and the lines of pressure an_1 and am_1

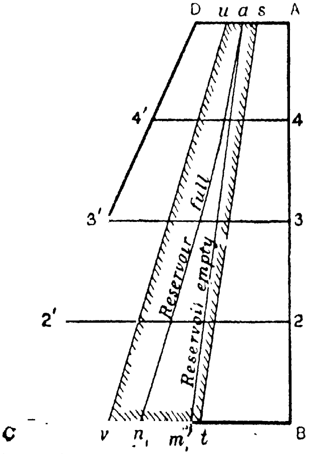


FIG. 635.—Lines of pressure, reservoir full and empty.

fall throughout within the middle thirds. The student will note that the upper ends of the lines of pressure bisect AD in *a*.

Work done by a fluid under pressure. Work may be done by a fluid, either liquid or gaseous, by allowing it to exert pressure on a piston which may move in a cylinder. In Fig. 636,

Let D = the diameter of the cylinder, in feet.
L = the length of the stroke, in feet.
P = the pressure of the fluid, in lb. per square foot.

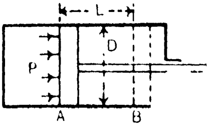


FIG. 636. Work done by a fluid.

Then, if a liquid be employed, owing to the absence of any expansive property, the pressure P must be maintained by continuous admission of liquid to the cylinder. The work done while the piston moves from A to B will be

$$\text{Work done} = P \times \frac{\pi D^2}{4} \times L \text{ foot-lb}$$

Now $\frac{\pi D^2}{4} L$ is the volume swept by the piston, and also represents the volume of liquid admitted in cubic feet; writing this volume V, we have

$$\text{Work done} = PV \text{ foot-lb.}$$

This expression also applies to the case of a gas supplied under constant pressure throughout the stroke.

Fig. 637 shows in outline a hydraulic engine using water as the working fluid. There are three cylinders, A, B and C, arranged at angles of 120°; the water pressure acts on one side of the pistons only, and all the pistons are connected to a single crank DE. The arrangement produces a fairly uniform turning moment. In engines of this type, as the cylinders must be filled completely with water during each stroke, the efficiency will fall very rapidly unless the demand for power is maintained steadily at its maximum amount. Otherwise, devices may be applied by means of which the capacity of the engine may be reduced when a diminished demand for power occurs. These devices usually take

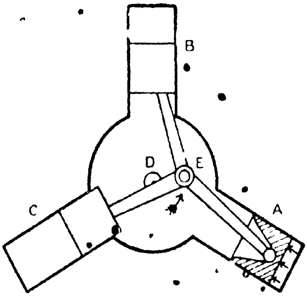


FIG. 637.—Three cylinder hydraulic engine.

the form of having the crank of variable throw; the strokes of the pistons will then vary to correspond. The crank adjustment may be effected either by means of a governor or by means of an automatic spring coupling between the engine and the machine to be driven.

If a gas is used in the cylinder in Fig. 636, advantage may be taken of its expansive property by cutting off the supply after the piston has moved a short distance and allowing the remainder of the stroke to be completed under the continually diminishing pressure of the gas. In Fig. 638 is plotted a curve AB, showing the relation of pressure (vertical) and volume (horizontal) while a gas is expanding and doing work. Usually the law of the curve AB takes the form

$$PV^n = \text{a constant},$$

where P is the pressure of the gas in lb. per square foot measured from zero. On this basis the pressure of the atmosphere is about 14·7 lb. per square inch or 2116 lb. per square foot. V is the volume in cubic feet, n is an index which depends on the conditions under which the expansion is performed. If the temperature is preserved constant, then Boyle's law is being followed, n is unity, and the expansion law will be

$$PV = \text{a constant};$$

n usually lies between 1 and 1·5.

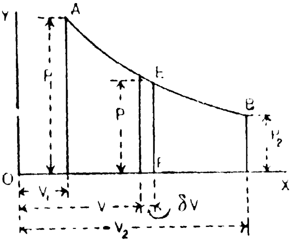


FIG. 638.—Work done by an expanding gas

The work done may be found from the area of the diagram under AB in Fig. 638. Thus, assuming Boyle's law to be followed and taking a narrow strip EF, for which the pressure is P, the volume V and the increase in volume represented by the breadth of the strip is δV, we have

$$\text{Area of the strip} = P\,.\,\delta V.$$

Now, from Boyle's law, $PV = P_1V_1$;

$$\therefore\ P = \frac{P_1V_1}{V}.$$

Hence, $\quad \text{Area of the strip} = P_1V_1\frac{\delta V}{V}.$

$$\text{Total area under AB} = P_1V_1\int_{V_1}^{V_2}\frac{dV}{V};$$

$$\therefore\ \text{work done} = P_1V_1\log_e\frac{V_2}{V_1}\ \text{foot-lb.}$$

If the expansion law is $PV^n = $ a constant, we have, in the same manner:

$$\text{Area of the strip} = P \,.\, \delta V.$$

Also,

$$PV^n = P_1V_1^n,$$

$$P = \frac{P_1V_1^n}{V^n}.$$

Hence,

$$\text{Area of the strip} = P_1V_1^n \frac{\delta V}{V^n}.$$

$$\text{Total area under AB} = P_1V_1^n \int_{V_1}^{V_2} \frac{dV}{V^n}$$

$$= P_1V_1^n \left(\frac{V_2^{1-n} - V_1^{1-n}}{1-n} \right)$$

$$= P_1V_1^n \left(\frac{V_1^{1-n} - V_2^{1-n}}{n-1} \right).$$

Remembering that $P_1V_1^n = P_2V_2^n$, the above becomes, by multiplication,

$$\text{Work done} = \frac{P_1V_1 - P_2V_2}{n-1} \text{ foot-lb.}$$

Hydraulic transmission of energy. In Fig. 639 A and C are two cylinders charged fully with water, and connected by a pipe E. B and D are plungers or rams fitted to cylinders, and carrying loads P and W. Owing to the practical incompressibility of water, any descent of B will produce an ascent of D, and hence work done on P may be transmitted by the medium of the moving water under pressure, and be given out in the form of work done on W. The connecting pipe E may be of any length. In practice, A represents a set of power-driven pumps, which supply water under a pressure of 700 to 1000 lb. per square inch. A pipe system distributes the water over the district to be supplied, and D may be taken to represent one of the machines to be operated. The principal appliances required in a hydraulic power distribution plant are shown diagrammatically in Fig 640. A is one of the power-driven pumps supplying water to the pipe line BC. A safety valve is placed at D. E is an accumulator consisting of a large cylinder fitted with a loaded ram, and connected to the pipe line; its function is to absorb energy by raising the weight if the machines are stopped and the pumps are still working; it also assists in preserving a steady pressure of water. A stop valve F is under the control of the consumer, and another safety valve is placed at G in order to guard against damage to his pipes and machines. H, H represent two of the machines being driven, each is fitted with a control valve K,

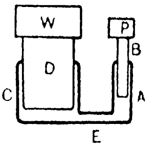


FIG. 639.

K, for the use of the operator. Usually the exhaust water from the machines is collected and passed through a meter, where it is measured for the purposes of charging for power.

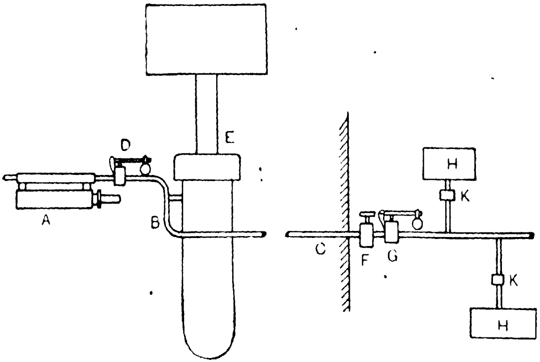


Fig. 640.—Diagram of a hydraulic installation.

Referring again to Fig. 639, let d_1 and d_2 be the diameter of B and D respectively in inches, and let p be the water pressure in lb. per square inch; also let W and P be measured in lb. Then, neglecting friction:

$$P = \frac{\pi d_1^2}{4} p,$$

$$W = \frac{\pi d_2^2}{4} p.$$

Hence,
$$\frac{W}{P} = \frac{d_2^2}{d_1^2}.$$

This gives the mechanical advantage of the arrangement neglecting friction. If P descends one inch, then the volume of water delivered from A into C will be $\frac{\pi d_1^2}{4}$ cubic inches. To accommodate this volume in C, D will rise a height h inches say, and the additional volume in D will be $\frac{\pi d_2^2}{4} h$ cubic inches.

Hence,
$$\frac{\pi d_2^2}{4} h = \frac{\pi d_1^2}{4},$$

$$h = \frac{d_1^2}{d_2^2}.$$

Now $1 \div h$ is the velocity ratio of the arrangement. Hence,

$$\text{Velocity ratio} = \frac{d_2^2}{d_1^2}.$$

Comparison of these results shows that in this hydraulic arrangement, as in other machines, the mechanical advantage when friction is neglected is equal to the velocity ratio (p. 328). The resistance W, which may be overcome by the ram D in this arrangement, may be very large if the ram is made of sufficient diameter. For example, a ram 10 inches in diameter, and supplied with water at 700 lb. per square inch, will exert a total force of about 24½ tons. The principle is made use of in hydraulic presses, forging and other machines.

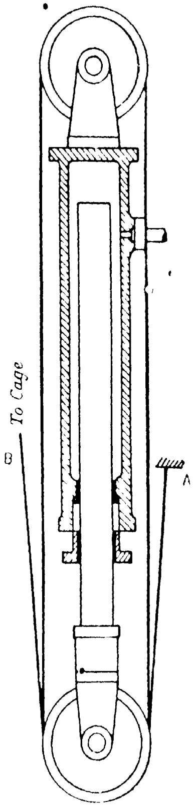


Fig. 641.—Cylinder for a hydraulic lift

Some examples of hydraulic machinery. The cylinder for a **hydraulic lift** is shown in some detail in Fig. 641. The ram passes through a stuffing box in the lower end of the cylinder, and carries two pulleys mounted on its end and both running on the same spindle. Another pulley is placed on the top end of the cylinder. The wire rope used for hoisting the cage is attached to a fixed point at A, and is led round the pulleys, as shown, before being taken away at B to the cage. The object is to multiply the comparatively small movement of the ram into the larger travel required for the cage. The same type of cylinder is made use of in hydraulic cranes.

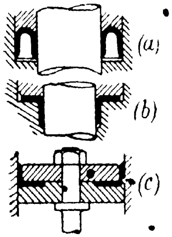


Fig. 642.—Types of leather packing.

Some types of **leather packing** are shown in Fig. 642. (*a*) is a U-leather, used for keeping water-tight rams of fairly large diameter; the water may enter the hollow interior of the U, and presses the leather outwards against the wall of the recess and also against the ram. In (*b*) is shown a hat leather, used for sliding plungers and rods; (*c*) is a cup leather, used for pistons in cases where the water acts on one side of the piston only.

A simple **hydraulic accumulator** is illustrated in Fig. 643. The ram A is fixed to the base plate, and the cylinder B is loaded with a number of cast-iron plates and may move vertically. A tail rod C is fixed to the cylinder, and serves as a guide. Water enters the cylinder by way of an axial hole bored through the ram. When the cylinder is nearing the top of its lift, it raises the end D of the lever DE; the movement of this lever is transmitted to the belt-striking gear on the pump, or to the throttle of the pump engine, and so stops the pump. A spring F pulls the levers back to working position when permitted by the descent of the accumulator, and so starts the pump again. The following simple calculations may be made regarding hydraulic accumulators:

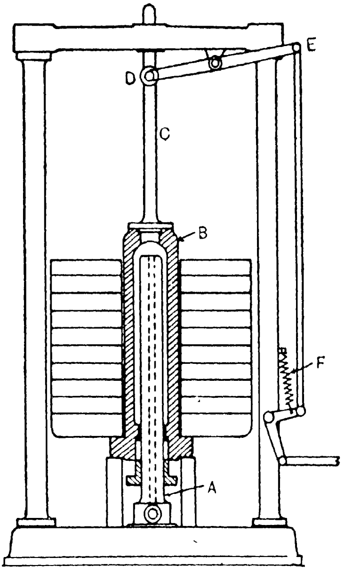


FIG. 643.—Hydraulic accumulator.

Let d = the diameter of the ram, in inches.
p = the water pressure, in lb. per square inch.
W = the total accumulator load, in lb.
H = the height of lift, in inches.

Then $W = p \times \frac{\pi d^2}{4}$ lb., neglecting friction.

When the accumulator is "up," the volume of water stored will be

$$\text{Volume stored} = \frac{\pi d^2}{4} H \text{ cubic inches.}$$

Also, $\qquad$ Energy stored = WH inch-lb.

Occasionally it occurs that a hydraulic machine requires a greater pressure of water than that supplied in the mains, and an **intensifier** is used in order to secure this. In Fig. 644 a cylinder A has a hollow ram B which passes through its right-hand end. A fixed hollow ram C passes into the interior of B as shown. Low-pressure water is supplied at D, and water of a higher pressure is discharged at E.

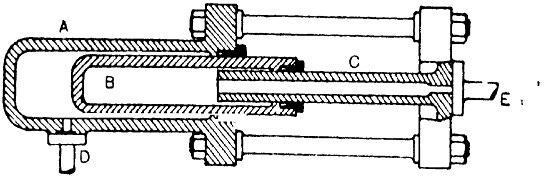


Fig. 644.—Hydraulic intensifier.

Let $\qquad p_1$ = the lower pressure, in lb. per square inch.

p_2 = ,, higher ,, ,, ,,

d_1 = the external diameter of B, in inches.

d_2 = the external diameter of C, in inches.

Then, neglecting friction,

$$p_1 \frac{\pi d_1^2}{4} = p_2 \frac{\pi d_2^2}{4},$$

or

$$p_1 d_1^2 = p_2 d_2^2,$$

$$\frac{p_2}{p_1} = \frac{d_1^2}{d_2^2}.$$

With a ratio of diameters of 2 to 1, the supply pressure of 700 lb. per square inch may be intensified to 2800 lb. per square inch, neglecting friction. Valve arrangements are provided for enabling the lower pressure to be used in the machine, and at the moment when the higher-pressure water is required, low-pressure water is admitted to A in the intensifier, and at the same time the machine is connected to E.

Pumps. Fig. 645 shows a **hydraulic pump** suitable for supplying water for operating hydraulic machines. A cylinder A is fitted with a piston B operated by a plunger rod C. Water enters the cylinder at D, and is prevented from flowing back by the suction valve E. G is a discharge valve opening to the discharge branch H, and F is a passage connecting the right-hand side of the piston to the discharge. The valves E and G are cushioned on lifting against rubber discs, separated by metal washers; the piston packing consists of two cup leathers. The action is as follows: Suppose the piston to be moving

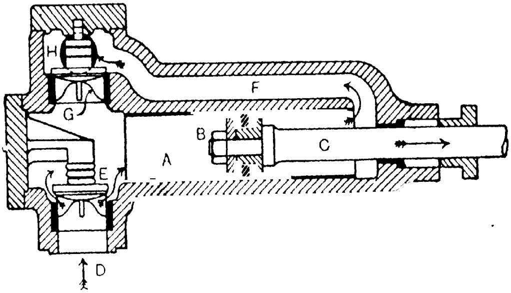


FIG. 615.—Hydraulic pump

towards the right as shown; E is open and G is closed. Water will flow into the pump through E, and will fill the space vacated by the receding piston; at the same time the water on the right-hand side of the piston is being forced into the discharge pipe through F. If the diameters of the piston and of the plunger rod are d_1 and d_2 respectively, and if the stroke is L inches, then the volume discharged from the right-hand side of the piston during this stroke will be $\left(\frac{\pi d_1^2}{4} - \frac{\pi d_2^2}{4}\right)L$ cubic inches.

Now let the piston be moving towards the left; E will be closed and G will open, and the water on the left-hand side of the piston will be discharged through G. The volume so discharged will be $\frac{\pi d_1^2}{4}L$ cubic inches, but a portion of this only will be sent into the discharge pipe, the remainder finding its way through F to the right-hand side of the piston; the amount so passing through F will be $\left(\frac{\pi d_1^2}{4} - \frac{\pi d_2^2}{4}\right)L$ cubic inches; hence the volume discharged from the pump during this stroke will be

$$\frac{\pi d_1^2}{4}L - \left(\frac{\pi d_1^2}{4} - \frac{\pi d_2^2}{4}\right)L = \frac{\pi d_2^2}{4}L \text{ cubic inches.}$$

The pump is thus **double-acting**, *i.e.* water is discharged during both strokes. For equality of **discharge**, we have

$$\left(\frac{\pi d_1^2}{4} - \frac{\pi d_2^2}{4}\right)L = \frac{\pi d_2^2}{4}L,$$

or

$$d_1^2 - d_2^2 = d_2^2,$$

$$d_1^2 = 2d_2^2,$$

$$d_1 = d_2\sqrt{2}.$$

This result may be expressed also by stating that the sectional area of the plunger rod should be half that of the piston.

Fig. 646 illustrates a type of **bucket pump** used in raising water from a lower to a higher level. The piston or bucket is shown ascending, and water is passing into the cylinder A through B and the suction valve C. The water already on the top of the bucket is being discharged through the discharge valve F and the passage G. During this stroke, the bucket valve is closed. On the downward stroke, the suction and discharge valves C and F both close, and the bucket valve opens, permitting water to pass from the lower to the upper side of the bucket. It is not absolutely necessary to have a discharge valve F in this type of pump, but, if fitted, it serves as a check on the suction valve during the downward stroke of the bucket. This pump is single-acting.

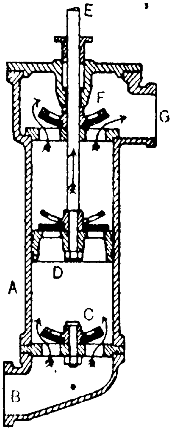


Fig. 646 Bucket pump.

In Fig 647 is shown a single-acting **plunger pump**. On the upward stroke of the plunger B, water enters the pump through the suction valve C, and is delivered, during the downward stroke, through the discharge valve D. E is an air vessel, the function of which is to get rid of shocks. The water coming from the pump flows partly into the air vessel, during the early part of the discharge stroke, and compresses the air contained therein; during the later part of the discharge stroke, and also possibly during part of the suction stroke, the pressure of the compressed air drives some of the water out of the air vessel into the discharge pipe. The acceleration required to be given to the column of water in the discharge pipe in starting it into motion is lowered by the action of the air vessel, and hence the force required is also lowered, and shock is avoided entirely. The cushion of compressed air is also beneficial in quietly closing the discharge valve at the end of the stroke without depending on any backward movement of the mass of water in the discharge pipe; thus hammering of the valve is avoided. The air

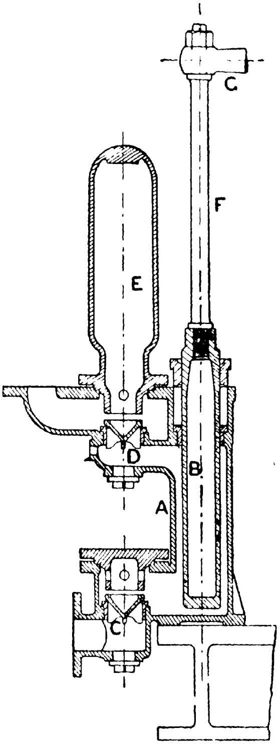


FIG. 647.—Boiler feed pump.

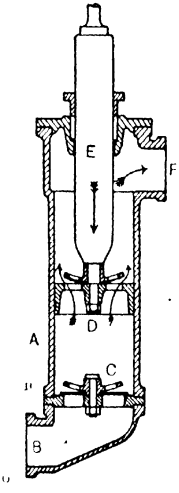


FIG. 648.—Combined plunger and bucket pump.

vessel should be situated always as close as possible to the discharge valve. The type of pump illustrated is much used for forcing the feed water into steam boilers.

Fig. 648 illustrates a **combined plunger and bucket pump.** During the downward stroke, the suction valve C is closed, and the bucket valve D is open, the plunger E is thus operating in discharging water through F. During the upward stroke, the bucket valve D is closed and the suction valve C opens. Fresh water thus flows into the cylinder A from B, and the water already on the top of the bucket is discharged through F. As is the case in the pump shown in Fig. 645, the area of the plunger should be one half that of the bucket for equality of discharge on the two strokes.

Pumps may be placed at some height above the supply water, and in this case depend on the pressure of the atmosphere acting on the supply water and forcing it up the suction pipe into the partial vacuum created by the action of the pump bucket or plunger. The maximum possible height through which the atmospheric pressure will raise water thus is about 34 feet; from 25 to 30 feet is the greatest practical height.

EXERCISES ON CHAPTER XXII.

1. A rectangular tank is 4 feet long, 3 feet wide and 2 feet deep. Find the total pressures on the bottom, on one side and on one end when the tank is full of oil which weighs 50 lb per cubic foot.

2. A tank 10 feet long has a horizontal bottom 4 feet wide. The ends of the tank are vertical, and both the sides are inclined at 45° to the horizontal. Water is contained to a depth of 6 feet. Find the total pressures on the bottom, on one side and on one end. Take $w = 62\cdot5$ lb. per cubic foot.

3. A dock gate is 80 feet wide and has sea water to a depth of 30 feet on one side and 9 feet on the other side. Find the total pressure on each side of the gate, and show the lines of action. Find also the resultant force on the gate, and show its position. Take $w = 64$ lb. per cubic foot.

4. A tank is in the form of an inverted cone, 6 feet diameter at the top and 4 feet vertical depth. When full of oil having a specific gravity 0·8, find the weight of the contained oil and the total pressure on the curved surface of the tank.

5. A rectangular opening in a reservoir wall is 4 feet high and 3 feet wide, and has its top edge 20 feet below the water level. Find the total water pressure on the door or gate closing the opening, and find also the centre of pressure.

6. A rectangular pontoon 100 feet long and 30 feet wide has a draught in fresh water of 8 feet (*i.e.* the bottom of the pontoon is 8 feet below the surface of the water). Find the weight of the pontoon. Supposing the weight to remain unaltered and the pontoon to be floating in sea water, what will be the draught? For fresh water $w = 62\cdot5$, and for sea water $w = 64$ lb. per cubic foot.

7. The weight of a submarine is 200 tons, and it lies damaged and full of water at the bottom of the sea. Supposing the specific gravity of its material to be 7·8, find what total pull must be exerted by the lifting chains in order to raise the vessel from the bottom. Take $w = 64$ lb. per cubic foot.

8. For the pontoon in Question 6, when floating in fresh water, find the heights of the transverse and longitudinal metacentres above the centre of buoyancy.

9. The pontoon in Question 8 carries a crane, and is hoisting a load which produces a transverse capsizing moment of 200 ton-feet. Calculate the angle of heel. It may be assumed that the centre of gravity of the complete pontoon is 0·5 foot below the surface level of the fresh water.

10. A retaining wall for water is triangular in section and has the wetted surface vertical. The height is 30 feet and the breadth of the base is 25 feet. Fresh water has its surface level 3 feet below the top of the wall. The weight of the material is 140 lb per cubic foot. Take one foot length of wall and find the resultant force acting on the base. Answer the same if the reservoir is empty. Do these forces fall within the middle third of the base?

11. Answer Question 10 for sections at 3 feet, 10 feet and 20 feet from the top of the wall, using graphical methods so far as is possible. Plot the lines of pressure for the reservoir full and empty.

12. Water is supplied by a hydraulic company at a pressure of 700 lb. per square inch, and is charged at the rate of 18 pence per thousand gallons. How much water must be used in an hour to obtain one horse-power, and what would be the cost? Neglect waste.

13. A single-acting hydraulic engine has three rams, each $3\frac{1}{2}$ inches diameter by 6 inches stroke. The effective mean water pressure on the rams is 120 lb. per square inch, and the engine runs at 90 revolutions per minute. Neglect all sources of waste and calculate the horse-power. If the efficiency is 65 per cent., what is the useful horse-power?

14. 2 cubic feet of air at an absolute pressure of 80 lb. per square inch are expanded in a cylinder until the volume is 5 cubic feet. Assuming that the law $PV = $ a constant is obeyed, calculate what work is done.

15. Answer Question 14 if the law of expansion is $PV^{1·41} = $ a constant.

16. A hydraulic accumulator has a ram 7 inches in diameter and the lift is 12 feet. If the water pressure is to be 700 lb. per square inch, find the weight required. How much water is stored when the accumulator is up? Find also the energy stored.

17. In the hydraulic lift cylinder shown in Fig. 641, find the velocity ratio if there are three rope pulleys on the ram end and two pulleys on the cylinder top. Suppose the ram to be 4 inches diameter and that the water pressure is 700 lb. per square inch, and calculate the pull on the cage rope, neglecting frictional waste. What is the pull if the total efficiency is 65 per cent.? What stroke of ram is required for a total cage lift of 60 feet?

18. A hydraulic pump, similar to that shown in Fig. 645, has a piston 4·25 inches and a plunger rod of 3 inches in diameter; the stroke is 18 inches. If the pump makes 60 double strokes per minute, how much water will be delivered, neglecting waste? If the water pressure is 750 lb. per square inch, find the force which must be applied to the rod (*a*) when the piston is moving towards the valves, (*b*) when the piston is moving in the contrary direction, assuming the pressure on the suction side to be 15 lb. per square inch. Neglect friction.

19. A bucket pump (Fig. 646) has to raise 400 gallons of water per minute to a height of 30 feet. If the pump makes 30 double strokes (one up and one down) per minute, and if the length of the stroke is 1·5 times the diameter of the bucket, find the stroke and the bucket diameter, neglecting waste. Calculate the useful work done per minute, and, if the efficiency is 60 per cent., find the horse-power required.

20. A boiler feed pump has a plunger 3 inches in diameter. The delivery pipe leading to the boiler is 40 feet in length and 3 inches in diameter. The pressure in the boiler is 100 lb per square inch. There is no air vessel. Supposing that the acceleration of the plunger at the beginning of the stroke (the water in the delivery pipe being then at rest) to be 90 feet per second per second, what total force must be exerted by the plunger in order to start the water into motion? If a perfect-acting air vessel were fitted, what total force would be required?

21. In finding the total force in the axial direction which a fluid exercises upon a piston or ram, we calculate from the cross section of the cylinder or ram; why is the actual shape of the face of the piston or end of the ram of no importance? (B.E.)

22. A vertical flap closes the end of a pipe 2 feet in diameter; the pressure at the centre of the pipe is equal to a head of 10 feet of water. Find the total pressure on the valve in pounds. (You may neglect the atmospheric pressure.) (B.E.)

23. The ram of a vertical accumulator is 4 inches in diameter; the cylinder is 6 inches in internal diameter and 50 feet high. The ram carries a total load of 5 tons. Find the water pressure, in lb. per square inch, at the top and bottom of the cylinder. (B.E.)

24. Determine the depth from the surface of the centre of pressure on a rectangular sluice valve, 6 feet long and 3 feet wide. The centre of the valve is at a depth of 8 feet below the surface of the water, and the valve lies in a plane inclined at an angle of 30 degrees to the horizontal, with one of the long edges of the valve parallel to the surface of the water. (B.E.)

25. A horizontal channel of V section, whose sides are inclined at 45°, is closed at the end by a vertical partition. The water-surface has a width of 4 feet, and consequently a maximum depth of 2 feet. Calculate the total hydrostatic pressure upon the partition and the height of the centre of pressure. (I.C.E.)

26. Define metacentric height. A vessel has a length of 150 feet between perpendiculars and a beam of 28 feet. The mean load draft in sea water is 11 feet, and the coefficient of fineness, or ratio between the product of length, breadth and draft and the displacement volume is 0·47. The second moment of the load water-plane about its fore and aft axis is 63 per cent. of the moment of the circumscribing rectangle about the same axis. The centre of buoyancy is situated 3·95 feet below the water-line. If the transverse metacentric height is to be limited to 3·62 feet, determine the distance from the centre of gravity to the water-line. (L.U.)

27. A masonry dam with vertical water face is 20 feet high and 13 feet wide at the bottom, sloping gradually till it is 6 feet wide at the top. The water reaches 2 feet from the top. Draw the line of thrust throughout the dam. Specific gravity of masonry, 2·25. (L.U.)

CHAPTER XXIII.

HYDRAULICS. FLOW OF FLUIDS.

Fluid friction. We have seen already that there can be no friction in any fluid at rest; considerable frictional resistances exist, however, when the fluid is in motion. For liquids, the **laws of fluid friction**, as deduced from experimental evidence, have been mentioned in Chap. XV., and are stated again for reference as follows:

(*a*) The resistance is proportional to the extent of the surface wetted by the liquid.

(*b*) The resistance is independent of the material of which the boundary is made, but depends on the roughness of its surface.

(*c*) The resistance is independent of the pressure to which the liquid is subjected.

(*d*) Rise of temperature of the liquid diminishes the resistance.

(*e*) At slow speeds the resistance is very small.

(*f*) Below a certain critical speed, the resistance is proportional to the speed; at speeds above this, the resistance is proportional to some power, approximately the square, of the speed.

The critical speed depends on the liquid used and its temperature. Below this speed the motion of the liquid is steady, the particles moving in stream lines; above it, the liquid breaks up into eddies. As the flow of water is the most important case in practice, we will confine attention to this liquid.

Kinds of energy of flowing water. Neglecting effects due to changes of temperature and of volume, we may state that the total energy of a particle of water is made up of (*a*) potential energy, (*b*) pressure energy, (*c*) kinetic energy. The potential energy will be proportional to the elevation of the particle above some datum level; the kinetic energy will be proportional to the square of the velocity of the particle. The pressure energy requires some fuller explanation.

In Fig. 649 is shown a cylinder fitted with a piston and supplied with water from an overhead tank, in which the level is maintained constant. If the piston is allowed to move outwards slowly, work will be done by the water pressure on the piston overcoming the external resistance acting on the other side of the piston or on the piston rod.

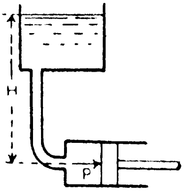


FIG. 649. Pressure energy of water.

Let P = fluid stress on piston, in lb. per square foot.
A = area of piston, in square feet.
L = the distance piston is moved, in feet.

Then Work done = PAL foot lb.

In performing this work, a volume AL cubic feet of water has been admitted to the cylinder, and the work has been done at the expense of the energy of this water. The work done per cubic foot of water may be found by dividing the above result by AL, giving

Work done per cubic foot of water = P foot-lb.

Let w = weight of one cubic foot of water in lb.

Then $\frac{1}{w}$ = volume of one lb. of water.

Hence, Work done per lb. of water $= \frac{P}{w}$ foot-lb.

It has been assumed that there has been no waste of energy; therefore $\frac{P}{w}$ represents the whole energy available in one pound of water due to its pressure. We may say therefore that water at rest and under pressure possesses energy due to its pressure to the amount of $\frac{P}{w}$ foot-lb. per pound of water.

Transformations of energy in flowing water. In Fig. 650 is shown two tanks at different levels, and connected by a pipe so that water may flow from the upper into the lower tank. OX is an arbitrary datum level. Considering a pound of water at A and assuming it to be at rest, there will be no kinetic energy; it will, however, possess H_A foot-lb. of potential energy owing to its elevation H_A feet above OX. The water, being exposed to atmospheric pressure P_a lb. per square foot, will also possess pressure energy to the amount of $\frac{P_a}{w}$ foot-lb. per pound. Hence,

$$\text{Total energy at A} = \left(H_A + \frac{P_a}{w}\right) \text{ft. lb. per lb. of water.}$$

It is reasonable to suppose that the bulk of the water in the upper tank is at rest, only a portion near the pipe entrance, mapped out by the dotted curve *abc* (Fig. 650), will possess any considerable velocity. Hence, at B, a pound of water will have potential energy H_B foot-lb., together with pressure energy $\frac{P_B}{w}$ foot-lb. owing to its absolute pressure P_B lb. per square foot. Therefore,

$$\text{Total energy at B} = \left(H_B + \frac{P_B}{w}\right) \text{ ft.-lb. per lb. of water.}$$

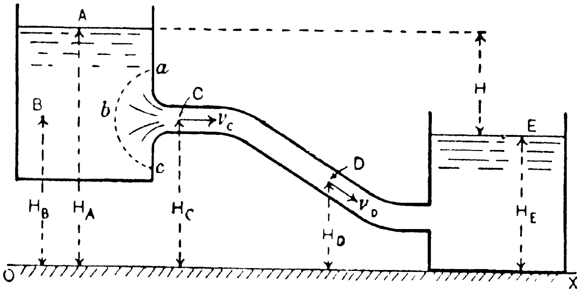


FIG. 650.—Transformation of energy of flowing water.

Consider now a pound of water at C, having acquired a velocity of v_C feet per second, under pressure P_C lb. per square foot and at an elevation H_C above OX. The total energy will be given by

$$\text{Total energy at C} = \left(H_C + \frac{P_C}{w} + \frac{v_C^2}{2g}\right) \text{ ft.-lb. per lb. of water.}$$

In the same way, a pound of water at D will have a total energy given by

$$\text{Total energy at D} = \left(H_D + \frac{P_D}{w} + \frac{v_D^2}{2g}\right) \text{ ft.-lb. per lb. of water.}$$

At the surface level E in the lower tank, the water may be assumed to be at rest again, and also exposed to atmospheric pressure. The total energy here will be

$$\text{Total energy at E} = \left(H_E + \frac{P_a}{w}\right) \text{ ft.-lb. per lb. of water.}$$

We may now trace the transformations of energy which have taken place during the passage of the water from A to E. It may be assumed that a pound of water moves from A to E very slowly, and arrives without any appreciable diminution of energy, since the frictional resistances will be very small. Hence,

Total energy at A = total energy at B,

or
$$H_A + \frac{P_a}{w} = H_B + \frac{P_B}{w}.$$

The water has given up potential energy represented by $(H_A - H_B)$ ft.-lb. per pound, and has acquired an equal amount of pressure energy given by $\left(\frac{P_B}{w} - \frac{P_a}{w}\right)$ ft.-lb. per pound.

During the passage from B to C, considerable velocity has been acquired, and hence the frictional resistance will produce corresponding waste of energy. In passing along the pipe from C to D there will be further frictional waste of energy. If these sources of waste be disregarded we may apply the principle of the conservation of energy in asserting that the total energies at B, C and D are equal. Hence,

Total energy at C = total energy at D,

or
$$H_C + \frac{P_C}{w} + \frac{v_C^2}{2g} = H_D + \frac{P_D}{w} + \frac{v_D^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This equation is the algebraic expression of **Bernoulli's law**, which asserts that **if there be no waste of energy, the total energy of water flowing from one place to another remains constant.** Calculations may be made on this assumption, and then corrections can be applied in order to account for known sources of waste.

Referring again to Fig. 650, the water leaving the pipe and entering the lower tank will produce surging of the water in this tank, accompanied by a considerable waste of energy. The total waste of energy in the complete passage from A to E may be estimated by taking the difference in total energies at these places. Thus,

$$\text{Total waste of energy} = \left(H_A + \frac{P_a}{w}\right) - \left(H_E + \frac{P_a}{w}\right)$$
$$= (H_A - H_E) \text{ ft.-lb. per lb. of water.}$$

It will be noted that $(H_A - H_E)$ is simply the difference in surface levels of the water in the two tanks, H feet say (Fig. 650). Hence, in the case before us, the total waste of energy per pound of water is represented by H foot-lb.

Venturi water meter. In Fig. 651 is shown a straight horizontal pipe, which converges from A to B and then enlarges again between B and C. As the pipe is horizontal, there will be no change in the potential energy of the water flowing through it; there will, however.

be interchanges of pressure and kinetic energies, and if pressure gauges be fitted as shown so that the pressure heads may be measured, it is possible to calculate the velocity of flow, and hence the quantity of water flowing, from a knowledge of the pipe diameters.

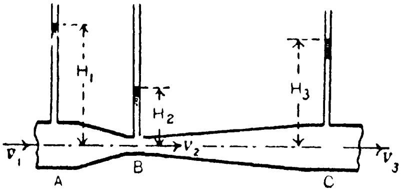


FIG. 651.—Principle of the Venturi meter.

The same quantity of water, Q cubic feet, will pass all sections of the pipe per second. If the sectional areas be A_1, A_2 and A_3 square feet at A, B and C respectively, and if the velocities v_1, v_2 and v_3 be measured in feet per second, we have

$$Q = v_1A_1 = v_2A_2 = v_3A_3. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Applying Bernoulli's law and neglecting any frictional waste, we have

$$H_1 + \frac{v_1^2}{2g} = H_2 + \frac{v_2^2}{2g} = H_3 + \frac{v_3^2}{2g}, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

H_1, H_2 and H_3 being the pressure heads in feet.

If the pipe diameters at A and C are equal, as is usually the case, v_1 and v_3 will be equal, and H_1 and H_3 will also be equal, neglecting friction. Using the first two terms of (2),

$$H_1 - H_2 = \frac{v_2^2 - v_1^2}{2g}.$$

From (1),
$$v_2 = \frac{A_1}{A_2}v_1,$$

Hence,
$$H_1 - H_2 = \frac{\left(\frac{A_1}{A_2}v_1\right)^2 - v_1^2}{2g} = \frac{\left(\frac{A_1^2}{A_2^2} - 1\right)}{2g} \cdot v_1^2$$
$$= \frac{A_1^2 - A_2^2}{2gA_2^2}v_1^2,$$

or,
$$v_1^2 = 2g(H_1 - H_2)\frac{A_2^2}{(A_1^2 - A_2^2)},$$
$$v_1 = \sqrt{2g(H_1 - H_2)}\frac{A_2}{\sqrt{A_1^2 - A_2^2}}.$$

Now
$$Q = v_1A_1$$
$$= \sqrt{2g(H_1 - H_2)}\frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}} \text{ cubic feet per sec.} \quad \ldots(3)$$

Practically, the quantity flowing differs somewhat from the result calculated from equation (3). A coefficient, the value of which is approximately 0·98, may be used for multiplying the right-hand side of (3). Small Venturi meters used in laboratories for testing purposes usually require calibration, especially for low heads and velocities.

Steady motion. Steady motion of a fluid may be defined as that state of motion when all particles passing through any fixed point arrive at the point with the same velocity, both as regards magnitude and direction. Thus, in steady motion, the particles will be travelling in lines or filaments either straight or curved, these filaments being called **stream lines.** For example, if a fine jet of coloured water be injected into a mass of water moving with steady motion, the coloured water will follow the stream line which passes through the point of injection, and will move unbroken through the mass of water, giving a coloured band which will be straight or curved depending on the circumstances of the flow, but will appear to remain fixed in position.

A fluid can only move in straight stream lines provided there is no resultant force acting on the boundary of the filament in a direction perpendicular to that of the motion of the filament. Any such force will produce a change in the direction of the motion, and the path of the filament will be curved, the resultant force being found on the convex side of the filament (Fig. 652).

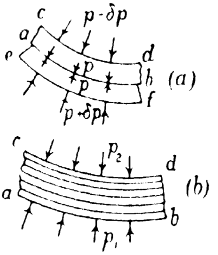


FIG. 652.—Transverse pressures on curved stream lines.

In a mass of fluid moving in curved stream lines, each stream line communicates pressures to the adjacent stream lines and is itself reacted on. As the concave side of any stream line is in contact with the convex side of the adjacent stream line, the pressure on the concave side *ab* of the first will be equal to that on the convex side *ab* of the second; let this pressure be p (Fig. 652 (*a*)). The pressure on the concave side *cd* will be less than p by an amount δp, and that on *ef* will be greater than p by another small amount δp. Applying the same reasoning to all stream lines in a body of fluid moving steadily in a curved path (Fig. 652 (*b*)), we see that the pressure p_1 on the convex boundary *ab* will diminish gradually across the stream, attaining a lower value p_2 at the concave boundary *cd.*

Discharge from an orifice. One of the simplest cases of the flow of water is found in a jet discharged through a small sharp-edged circular hole in a thin plate. In Fig. 653 such a hole is formed in the vertical side of a tank, WL being the free surface level, giving a steady head H feet over the orifice *de*. OX may be taken as a datum level. At A, a pound of water being at rest will have a total energy given by

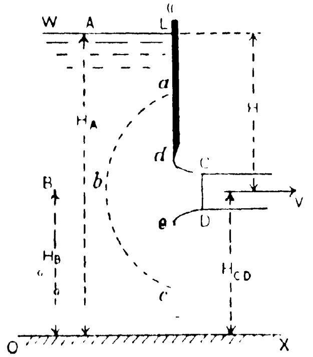


FIG. 653.—Discharge through a sharp-edged orifice.

$$E_A = H_A + \frac{P_a}{w}, \quad \ldots \ldots \quad . (1)$$

P_a being, as before, the atmospheric pressure in lb per square foot, and w the weight of the water in lb. per cubic foot.

Passing to a point B on the same level as the centre of the orifice, some of the potential energy possessed at A will have been converted into pressure energy, giving a total energy at B of

$$E_B = H_B + \frac{P_B}{w}. \quad \ldots \ldots \ldots \quad \ldots . (2)$$

As the motion of a particle passing from A to B will be very slow, it is reasonable to suppose that frictional losses may be disregarded. Hence,

$$E_A = E_B,$$

or

$$H_A + \frac{P_a}{w} = H_B + \frac{P_B}{w}.$$

Again,

$$P_B = P_a + wH$$

$$\therefore \; H_A + \frac{P_a}{w} = H_B + \frac{P_a}{w} + H,$$

or

$$H = H_A - H_B. \quad . \quad . \quad . (3)$$

This equation simply expresses the fact that the superatmospheric pressure head at B is H.

Assuming that the motion inside the region of important velocity, *abc*, is stream line, we may state that a particle situated at *b*, level with the centre of the orifice, will move along a straight horizontal stream line and so pass out; particles crossing the boundary *abc* at other points will approach the orifice in curved stream lines. Clearly the sharp edges of the orifice *de* cannot produce a sudden change in the direction of any stream line; hence the curvature will be maintained for some distance after the plane of the orifice has been passed. This leads to contraction of the issuing jet, and such contraction will not be complete until a section CD has been reached; this section is called the **contracted vein.**

In the body of water between *de* and CD, the stream lines are convex towards the axis of the jet, hence there must be resultant fluid pressures acting transversely to each stream line and directed outwards towards the boundary of the jet. As the boundary is exposed to atmospheric pressure p_a, it follows that superatmospheric pressure, of values gradually increasing towards the axis of the jet, will be found in the interior of the jet, the maximum pressure occurring at the axis. From CD onwards the stream lines will be parallel, hence the water in the jet beyond CD will be under uniform pressure equal to P_a, and will possess pressure energy given by $\frac{P_a}{w}$ ft-lb per pound.

The velocity of any particle has been increased gradually in passing from the boundary *abc* to the section CD, and hence the particle has been acquiring kinetic energy gradually, this being obtained at the expense of its other kinds of energy. For example, a pound of water at *b* has had its superatmospheric pressure energy, H foot lb, changed into an equal quantity of kinetic energy (neglecting frictional losses) while passing from *b* to CD. The conversion is completed on arriving at CD, and hence we find the maximum velocity at this section. Supposing V feet per second to be the velocity of the jet at CD, then the total energy per pound of water at CD will be

$$E_{CD} = H_{CD} + \frac{P_a}{w} + \frac{V^2}{2g} \quad \ldots\ldots(4)$$

Applying Bernoulli's law and neglecting frictional effects, the total energies at A and CD will be equal. Hence,

$$H_A + \frac{P_a}{w} = H_{CD} + \frac{P_a}{w} + \frac{V^2}{2g};$$

$$H_A - H_{CD} = \frac{V^2}{2g},$$

or

$$H = \frac{V^2}{2g} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

This equation may be written

$$V^2 = 2gH,$$

or

$$V = \sqrt{2gH}. \quad \ldots\ldots\ldots\ldots\ldots(6)$$

The actual velocity at CD will be somewhat less than V, due to waste of energy in overcoming frictional resistances in the flow between *abc* and CD.

Experimentally it is found that the actual velocity V_a is about 0·97V, this number being called the **coefficient of velocity**, written c_v. Hence,

$$V_a = c_v\sqrt{2gH}. \qquad (7)$$

The quantity of water discharged can be obtained, provided we know the area of the section CD. For a small round orifice this will be about 0·64 of the area of the orifice; this number is called the **coefficient of contraction**, written c_c.

Let Q = the quantity discharged per second, in cubic feet.

A = the area of the orifice, in square feet.

H = the head over the centre of the orifice, in feet.

Then

$$\begin{aligned} Q &= c_c A V_a \\ &= c_c c_v A V \\ &= c_d A\sqrt{2gH}. \qquad (8) \end{aligned}$$

In this result, $c_d = c_c c_v$ is called the **coefficient of discharge**. For a round orifice its value will be

$$\begin{aligned} c_d &= 0{\cdot}64 \times 0{\cdot}97 \\ &= 0{\cdot}62. \end{aligned}$$

The discharge from a small round sharp edged orifice therefore will be given by $Q = 0{\cdot}62A\sqrt{2gH}$ cubic feet per second.

If the orifice is situated in the tank bottom so that the jet discharges vertically downwards (Fig. 654), contraction does not cease at CD. This is owing to the potential energy of the water in the falling jet continually diminishing; hence the kinetic energy, and therefore the velocity, must be increasing continually. In a steady jet (prior to its breaking up into drops) the same quantity of water passes each section per second, and therefore the area of the jet must be diminishing as the jet recedes from the orifice. The approximate velocity at any section may be estimated from

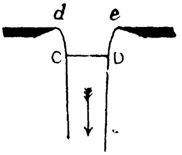


FIG. 654.—Sharp-edged orifice in a tank bottom.

$$v = c_v\sqrt{2gH},$$

where H is the head measured from the free surface level in the tank to the section considered.

Contraction of the jet after passing the orifice may be got rid of by means of a trumpet orifice (Fig. 655). In this case the discharge is estimated by applying a coefficient of velocity only.

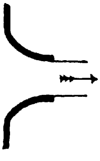


Fig. 655.—Trumpet orifice.

Thus,
$$Q = c_v A\sqrt{2gH}.$$

Flow of a gas through an orifice. Assuming that the pressure in the reservoir containing the gas is only slightly greater than the pressure in the space into which the jet of gas is discharged, and that there is no change in temperature, there will be very little change in the weight of the gas per cubic foot, and the flow through the orifice may be estimated in the same manner as for a liquid.

Let $v_1 = 0$ = the velocity in the reservoir.

v_2 = the maximum velocity of the jet, in feet per second.

p_1 = the pressure in the reservoir, in lb. per square foot.

p_2 = the pressure in the space which the jet enters, in lb. per square foot.

w = the weight of a cubic foot of the gas, in lb., under the conditions existing in the reservoir.

A = the area of the orifice, in square feet.

Then
$$\frac{p_1}{w} + 0 = \frac{p_2}{w} + \frac{v_2^2}{2g},$$
$$\frac{v_2^2}{2g} = \frac{p_1 - p_2}{w},$$
$$v_2 = \sqrt{2g\frac{(p_1 - p_2)}{w}}.$$

Hence, applying a coefficient of discharge C_d, we have
$$Q = c_d A v_2$$
$$= c_d A\sqrt{2g\frac{(p_1 - p_2)}{w}} \text{ cubic feet per sec.}$$

Experiments show that for circular sharp-edged orifices discharging air, the value of c_d is in the neighbourhood of 0 6.

Reaction of a jet. In Fig. 656 is shown a tank mounted on wheels and discharging water through a trumpet orifice in one side. The issuing water has acquired momentum in passing out of the orifice, and a resultant force acting towards the right on the water in the mouthpiece is required in order to produce this change of momentum. There must also be an equal

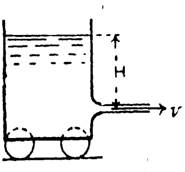


Fig. 656.—Reaction of a jet.

opposite reaction, and hence there will be a tendency to move the tank towards the left when the jet is flowing. The magnitude of this force may be found by estimating the change of momentum per second.

Let H = the head over the centre of the orifice, in feet.
v = velocity of jet, in feet per second.
A = area of jet, in square feet.
w = mass in pounds of a cubic foot of water.

Then Quantity flowing per second = Avw pounds;
$\therefore$ momentum acquired per second = $Avw \cdot v$.

$$\text{Force required} = \frac{Av^2w}{g} \text{ lb.}$$

Neglecting the coefficient of velocity, we have

$$v^2 = 2gH.$$

Hence,

$$\text{Force required} = \frac{Aw \cdot 2gH}{g}$$
$$= 2AwH \text{ lb.}$$

If the orifice be closed by a plate, the pressure on the plate would be AwH lb; hence the reaction of the jet is double the pressure on a plate closing the orifice.

In the Borda mouthpiece, a short tube projects into the interior of the tank, and has its inner edge sharpened (Fig. 657). This orifice produces an effect differing considerably from a trumpet orifice or from a simple hole in the tank side. In the latter cases, owing to the curvature of the stream lines in the vicinity of the orifice, the walls of the tank there are somewhat relieved of pressure, the pressure diminishing from a maximum at the axis of the orifice to a minimum at the boundary. In the Borda mouthpiece, the curved portions of the stream lines are removed sufficiently from the tank side as not to modify the pressures on the sides. Hence, the force producing change of momentum in the issuing water will be simply that which would exist on a plate closing the orifice.

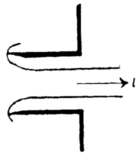


FIG. 657.—Borda mouthpiece.

Let A = area of orifice, in square feet.
H = head of water, in feet.
a = area of jet, in square feet.
v = velocity of jet, in feet per sec.

Then Force producing change of momentum = wHA lb.

Quantity flowing per second = wav pounds;

Change of momentum per second = wav^2;

$$\therefore \text{ reaction of jet} = \frac{wav^2}{g} \text{ lb.}$$

Hence,

$$wHA = \frac{wav^2}{g}.$$

Neglecting the coefficient of velocity, we have

$$v^2 = 2gH,$$

$$\therefore \quad wHA = \frac{wa}{g}.2gH,$$

or

$$A = 2a,$$

$$a = \tfrac{1}{2}A.$$

This mouthpiece has therefore a coefficient of contraction of 0·5.

Thomson's principle of similar flow. Prof. James Thomson's principle of similarity is of importance in dealing with the flow through orifices and over weirs; it may be stated as follows. Supposing we have a drawing of a vessel containing a frictionless liquid up to a fixed level, and that the liquid is flowing out through an orifice, the stream lines being shown on the drawing. The principle states that this drawing will serve for the discharge from any similar vessel containing the same liquid, the vessel having been constructed by merely altering the scale of the drawing; the stream lines in the similar vessel will also have the form shown in the original drawing to the altered scale. Further,

Let d = any linear dimension on the drawing.

v = velocity of flow at any point.

a = sectional area of a stream line at this point.

q = discharge through this stream line.

Then

$$v \propto d^{\frac{1}{2}},$$

and

$$a \propto d^2.$$

Also,

$$q = va;$$

$$\therefore \quad q \propto d^{\frac{1}{2}}.d^2;$$

$$\therefore \quad q \propto d^{\frac{5}{2}}.$$

A convenient linear dimension to choose is the head of liquid H from the point considered up to the free surface level. Then

$$q \propto H^{\frac{5}{2}}.$$

Gauge notches. A gauge notch is a device used for measuring the quantity of water flowing along a stream. The stream is dammed by vertical boards, and a notch is cut in the dam to permit the water to flow through. The usual form is triangular (Fig. 658) or rectangular (Fig. 659). In the case of a **triangular notch,** it will be evident that the streams flowing through the notch will be similar, whatever may be the head, hence Thomson's principle may be applied to any filament such as *ab*. Thus,

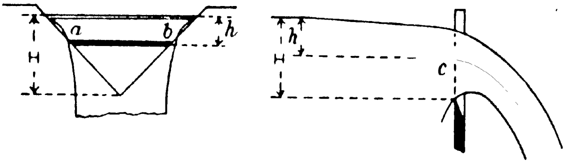


FIG. 658.—Triangular gauge notch.

$$q \propto h^{\frac{5}{2}},$$

$$\therefore\ q = ah^{\frac{5}{2}},$$

where *a* is an experimental coefficient.

The same expression will serve for any other filament in the stream. Hence the total quantity flowing will be given by

$$Q = aH^{\frac{5}{2}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

The value of *a* for a notch having an angle of 90° may be taken as 2·635; if H be measured in feet, we have for such a notch:

$$Q = 2{\cdot}635H^{\frac{5}{2}} \text{ cubic feet per second.}$$

Care must be taken in measuring H; this must be the head from the bottom of the notch to the level of still water. As the water in the stream approaching the notch is increasing its velocity, its kinetic energy is increasing, and consequently its potential energy is diminishing. Hence there will be a gradual fall of surface level in the water in the vicinity of the notch. The still water level will be found at some distance from the notch.

The water flowing over a **rectangular notch** under different heads will not present the same similarity which would exist in a triangular notch in the same circumstances. Reference to Fig. 659 shows that the water may be divided into three portions, one in the middle of the notch, in which the stream lines as seen in front elevation are moving parallel and vertically, together with two side portions in which the water is moving partly inwards, due to the contraction produced by the sharp vertical edges of the notch. The breadth of the middle portion evidently will increase if the quantity of water flowing diminishes by reason of a reduction in head. Hence the lack of similarity when the entire section of the stream is considered.

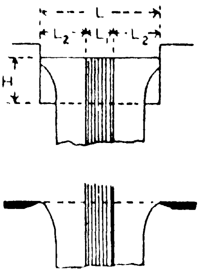


FIG. 659.—Rectangular gauge notch having two side contractions

The side contracted portions may be got rid of by having sides fitted on the upstream face. In Fig. 660 (*a*) and (*b*) such a notch is shown discharging water under a head H_1 in (*a*) and under a smaller head H_2 in (*b*). Let the whole section in (*a*) be divided into N_1 portions by vertical sections, and let that of (*b*) be divided in the same manner into N_2 portions. The discharges through all the portions in (*a*) will be equal, as will also be the discharges through all the portions in (*b*). Further, any one portion in (*a*) will be similar to any one portion in (*b*) provided the following proportion is complied with:

$$N_1 : N_2 = H_2 : H_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

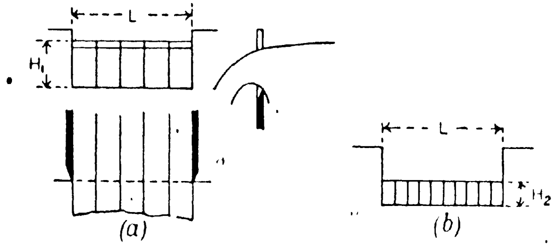


FIG. 660.—Rectangular gauge notches having both side contractions suppressed

Let q_1 and q_2 be the discharges per portion in (*a*) and (*b*) respectively. Then, by Thomson's principle of similarity, we have

$$q_1 : q_2 = H_1^{\frac{5}{2}} : H_2^{\frac{5}{2}},$$

or

$$\frac{q_1}{q_2} = \left(\frac{H_1}{H_2}\right)^{\frac{5}{2}} \quad \ldots\ldots(3)$$

The total discharges Q_1 and Q_2 in (*a*) and (*b*) respectively may be obtained by multiplying q_1 and q_2 by N_1 and N_2 respectively. Hence,

$$\frac{Q_1}{Q_2}=\frac{q_1N_1}{q_2N_2}=\left(\frac{H_1}{H_2}\right)^{\frac{5}{2}}\cdot\frac{H_2}{H_1}\quad\text{(from (2) and (3) above)}$$

$$=\left(\frac{H_1}{H_2}\right)^{\frac{3}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

It is therefore apparent that Q varies as $H^{\frac{3}{2}}$ in a notch of this type, and is also proportional to the width L. Hence,

$$Q=c_1LH^{\frac{3}{2}},\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where c_1 is an experimental coefficient.

Suppose that, in the rectangular notch shown in Fig. 659, the relation of L and H is such that there is no central portion showing parallel flow, the side contracted portions filling the whole section of the stream. It is evident that this drawing would serve for a notch of this type, having any dimensions by merely altering the scale. Hence, Thomson's principle of similarity may be applied at once, and we may write

$$Q=c_2H^{\frac{5}{2}}.\ \ldots\ldots\ldots\ldots(6)$$

Consider now the notch shown in Fig. 659, having side contractions. Let L_1 be the width of the middle portion which is unaffected by the contractions, and let L_2 be the width of each side contracted portion. L_2 will depend upon H, and may be written *a*H, where *a* is a constant. If there be *n* side contracted portions,

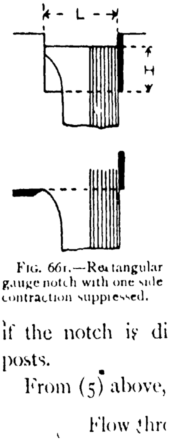


Fig. 661.—Rectangular gauge notch with one side contraction suppressed.

$$L_1=L-nL_2$$

$$=L-naH.$$

The values of *n* will be 2, 1 and 0 in the notches shown in Figs. 659, 661 and 660 respectively; the value may be greater than 2 if the notch is divided into several portions by means of vertical posts.

From (5) above,

$$\text{Flow through the middle portion}=c_1L_1H^{\frac{3}{2}}$$

$$=c_1(L-naH)H^{\frac{3}{2}}.\ \ldots(7)$$

From (6), Flow through each side portion $\propto H^{\frac{5}{2}}$;

$$\therefore \text{ flow through } n \text{ side portions} = nc_2 H^{\frac{5}{2}}.$$

Hence,
$$\begin{aligned} \text{Total flow} &= c_1(L - naH)H^{\frac{3}{2}} + nc_2 H^{\frac{5}{2}} \\ &= c_1 L H^{\frac{3}{2}} - n(ac_1 - c_2)H^{\frac{5}{2}} \\ &= \{c_1 L - n(ac_1 - c_2)H\}H^{\frac{3}{2}}. \end{aligned}$$

c_1, a and c_2 being constant coefficients, this result may be simplified by using other coefficients α and β, giving

$$\text{Total flow} = \alpha(L - n\beta H)H^{\frac{3}{2}}. \quad \ldots\ldots (8)$$

This is the **Francis** or **Lowell formula.** Experiments show that the value of α is 3·33, and that of β is 0·1. L and H being in feet, we have

$$\text{Total flow in cubic feet per second} = 3{\cdot}33(L - 0{\cdot}1nH)H^{\frac{3}{2}}. \quad ..(9)$$

Pitot tube The Pitot tube may be used for determining the velocity of flow in a stream; the principle may be understood by reference to Fig. 662. A, B and C are similar tubes, each having a small hole at the end of the horizontal limb. A points up stream, C points down stream and B is at right angles to the stream. On account of impact, the head shown in A will be greater than in B or C; a certain amount of suction occurs in C, and B will show the pressure head nearly. Experiment shows that, if h_1, h_2 and h_3 are the heads in feet shown by A, B and C respectively, and if v is the velocity of the stream in feet per second, then

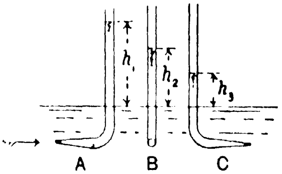


FIG. 662.—Pitot tubes

$$h_1 - h_2 = \frac{v^2}{2g} \text{ very nearly.} \quad \ldots\ldots (1)$$

Also,
$$h_1 - h_3 = \frac{v^2}{g} \text{ very nearly.} \quad .(2)$$

In Fig. 662, the gauges are supposed to be inserted in a pipe, with the mouths of the tubes in the axis of the pipe. If the velocity be calculated from (2), the result will be the maximum velocity in the pipe; the velocity diminishes near the boundary of the pipe and the average velocity may be taken as 0·84 of the calculated result for the maximum velocity.

Flow through a uniform pipe. Let A and B (Fig. 663) be two tanks or reservoirs connected by a pipe of uniform bore through which the water is flowing from A into B. Let the free surface levels at *a* and *b* be preserved at constant heights H_a and H_b respectively above the datum level OX, and let the difference in levels be $H = (H_a - H_b)$ feet. The diminution in energy of a pound of water in passing from *a* to *b* will be H foot-lb. (p. 595), and this quantity represents the total energy wasted per pound of water.

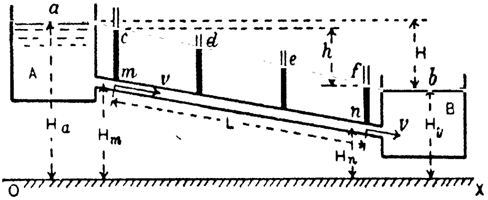


FIG. 663.—Flow through a uniform pipe.

The wasted energy is made up of three quantities:

(*a*) The kinetic energy possessed by the water flowing in the pipe is wasted when the water enters B by the production of surging and eddies. It may be noted that, as the pipe is uniform in bore and is assumed to be filled completely throughout its length, the velocity, and hence the kinetic energy of the water, will be constant while it is in the pipe.

(*b*) The water entering the pipe from A loses energy by the production of eddies in the pipe, especially if the pipe entrance is sharp-edged.

(*c*) Energy is wasted in overcoming frictional resistances to motion in the pipe.

Of these sources of waste, (*a*) and (*b*) are of importance in a short pipe, but become negligible by comparison with (*c*) in a pipe of great length.

If a number of glass tubes be inserted in the top of the pipe, it will be found that the water stands in these tubes at levels as shown at *c*, *d*, *e* and *f* (Fig. 663). These heights above the pipe indicate the pressure heads of the water in the pipe. A line joining *cdef* is called the **hydraulic gradient**. The slope of the hydraulic gradient, between two points *c* and *f* say, measured by dividing the difference in level of *c* and *f*, say *h* feet, by the total length of the pipe between m and n, say L feet, is called the **virtual slope** of the pipe, and is written i. Hence,

$$i = \frac{h}{L} \qquad \ldots\ldots \qquad (1)$$

Note that, in measuring the virtual slope, the actual length of the pipe, whether straight or curved, must be taken.

We have the following expressions for the total energy of one pound of water at m and n.

Let H_m = elevation of m over OX in feet.
H_n = ,, n ,, ,,
P_m = pressure at m, in lb per square foot
P_n = ,, n, ,, ,, ,,
cm = height of column at m, in feet.
fn = ,, ,, n, ,,
w = weight of one cubic foot of water.
v = the constant velocity, in feet per sec.

Then

$$\text{Total energy per lb. of water at } m = H_m + \frac{P_m}{w} + \frac{v^2}{2g}$$

$$= H_m + cm + \frac{v^2}{2g}. \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{Total energy per lb. of water at } n = H_n + \frac{P_n}{w} + \frac{v^2}{2g}$$

$$= H_n + fn + \frac{v^2}{2g}; \qquad \ldots\ldots\ldots\ldots (3)$$

∴ reduction of energy between m and n

$$(H_m + cm) - (H_n + fn)$$

$$= h \text{ foot-lb. per lb of water.} \quad (4)$$

This reduction is clearly owing to frictional resistances having to be overcome in the pipe between m and n. Hence we may write, from (1):

> Virtual slope of a pipe = head lost in overcoming frictional resistances divided by the total length of the pipe.

In Fig. 664 (*a*) and (*b*) are shown the forms of the hydraulic gradient *ab* and *a′b′* for a bell-mouthed and for a sharp-edged entrance respectively. The more rapid drop near the entrance in both is owing to some of the pressure head being utilised in giving velocity to the water entering the pipe, and is complicated further in (*b*) by the contraction and eddies set up by the sharp-edged entrance. In a very long pipe, the drop at entrance may be disregarded, and the virtual slope may be measured by dividing

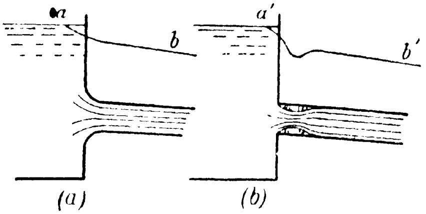


FIG. 664.—Hydraulic gradients for bell-mouthed and sharp edged entrances.

the difference in free surface levels in the two reservoirs by the total length of the pipe.

Frictional resistance in a uniform pipe. Consider a uniform horizontal pipe (Fig. 665) in which water is flowing steadily with velocity v feet per second. As both the velocity and the elevation over datum level are constant, it follows that the water will possess constant kinetic energy and also constant potential energy; the

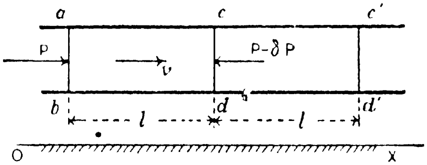


FIG. 665.—Frictional resistance in a uniform horizontal pipe.

pressure energy alone will show variation. Let the portion of water between cross sections ab and cd flow until ab reaches cd, and cd reaches $c'd'$; let l feet be the length of $a-a'$. During this movement, the energy required to overcome frictional resistances will be obtained at the expense of the pressure energy of the water, which accordingly will show a diminution. The pressure at cd will therefore be less than that at ab.

Let Area of section of stream = A square feet.
Wetted perimeter of pipe = B feet.
Weight of water per cubic foot = w lb.
Pressure at ab per square foot = P lb.
Pressure at cd per square foot = (P − δP) lb.
Frictional resistance per square foot of wetted surface = F lb.

Then

Net pressure urging the water along the pipe $= P - (P - \delta P)$

$= \delta P$ lb. per sq. foot.

Hence,

Resultant pressure acting on *abdc* $= \delta P . A$ lb.

Work done through a length l feet $= \delta P . Al$ foot lb. (1)

Again,

Total frictional resistance on *abdc* $= F \times$ wetted surface

$= FBl$ lb.

Work done against this resistance $= FBl^2$ foot lb. ...(2)

Equating (1) and (2), we have

$$\delta P . Al = FBl^2 \qquad \ldots \ldots \ldots (3)$$

In this equation, the whole weight of water in the portion *abdc* has been included. To reduce it to the form for one pound of water, divide each side by the weight of water in *abdc*. Thus,

Weight of *abdc* $= Alw$ lb.

Hence, from (3), $$\frac{\delta P . Al}{Alw} = \frac{FBl^2}{Alw},$$

or $$\frac{\delta P}{w} = \frac{FBl}{Aw} \text{ foot lb.} \qquad \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

Now $\dfrac{\delta P}{w}$ is the pressure energy lost by one pound of water in flowing through a distance l feet along the pipe. The pressure head h' feet lost will therefore be

$$h' = \frac{\delta P}{w}$$

$$= \frac{FBl}{Aw} \text{ feet.} \qquad \ldots \ldots \ldots \ldots \ldots \ldots (5)$$

It will be noted that this quantity is simply proportional to the length l. If the pipe has a total length L feet, the total pressure head lost will be given by

$$h = \frac{FBL}{Aw} \text{ feet.} \qquad \ldots \ldots \ldots \ldots \ldots \ldots (6).$$

The **hydraulic mean depth** of a pipe or channel is defined as the result of dividing the cross-sectional area of the stream by the wetted perimeter. The idea is obtained by substitution of a stream of rectangular section for the actual stream. The breadth being B and the depth being made equal to the hydraulic mean depth m, the cross-sectional area will remain unaltered, and we have

$$A = Bm,$$

or

$$\frac{B}{A} = \frac{1}{m}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Substitution of this in (6) gives

$$h = \frac{FL}{mw} \text{ feet.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The case of a sloping pipe may be examined by reference to Fig. 666. The pipe is still of uniform cross section, and the water

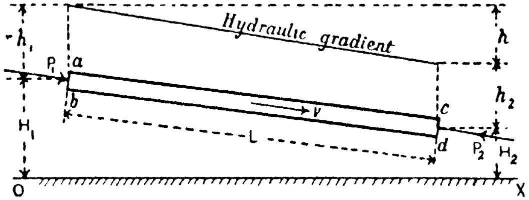


Fig. 666.—Frictional resistance in a sloping uniform pipe

will therefore possess uniform velocity and also uniform kinetic energy, provided the bore of the pipe is filled completely everywhere with water. Hence,

$$\text{Total energy at } ab = \left(H_1 + \frac{P_1}{w} + \frac{v^2}{2g}\right) \text{ ft.-lb. per lb. of water.}$$

$$\text{Total energy at } cd = \left(H_2 + \frac{P_2}{w} + \frac{v^2}{2g}\right) \text{ ft.-lb. per lb. of water.}$$

Energy expended in overcoming frictional resistances

$$= \left(H_1 + \frac{P_1}{w}\right) - \left(H_2 + \frac{P_2}{w}\right),$$

or,

$$\frac{FL}{mw} = \left(H_1 + \frac{P_1}{w}\right) - \left(H_2 + \frac{P_2}{w}\right)$$
$$= (H_1 + h_1) - (H_2 + h_2)$$
$$= h \text{ feet,} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

an equation having the same form as (8) above for a horizontal pipe.

The result may be written in terms of the virtual slope, giving

$$i = \frac{h}{L} = \frac{F}{mw}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Practical formulae for the calculation of the flow through pipes may be devised by making various assumptions regarding F.

Chezy formula. In this well-known formula it is assumed that F is proportional to the square of the velocity. Hence, for a given liquid such as water, for which w is constant, we may write, from equation (10),

$$i \propto \frac{v^2}{m},$$

or

$$i = k\frac{v^2}{m}, \tag{11}$$

where k is a coefficient. Let $k = \frac{1}{c^2}$. Then

$$i = \frac{v^2}{c^2 m},$$

$$v^2 = c^2 mi,$$

$$v = c\sqrt{mi}. \tag{12}$$

This is the Chezy formula. In using it, v will be in feet per second if m is in feet. The value of c varies considerably, increasing with the diameter of the pipe, and diminishing if the pipe surface becomes roughened by incrustation.

EXAMPLE. Find the velocity of flow in an old cast-iron pipe, 24 inches bore and 10,000 feet long, connecting two reservoirs in which the free surface levels differ by 120 feet. Take $c = 100$.

$$m = \frac{d}{4} = \frac{1}{2}; \quad i = \frac{120}{10{,}000};$$

$$v = c\sqrt{mi}$$

$$= 100\sqrt{\frac{1}{2} \cdot \frac{120}{10{,}000}}$$

$$= \underline{7{\cdot}74} \text{ feet per second.}$$

The pressure head lost in overcoming frictional resistances may be expressed conveniently in terms of the kinetic energy of the water. Taking equation (11) and multiplying both numerator and denominator by $2g$, we obtain

$$i = \frac{h}{L} = k\frac{v^2}{m}\frac{2g}{2g},$$

or

$$h = \frac{fL}{m} \cdot \frac{v^2}{2g}, \tag{13}$$

in which the coefficient f is written instead of $2kg$. To obtain the relation of f with the coefficient c in equation (12), we may proceed as follows:

$$v = c\sqrt{mi}\,;$$

$$\therefore\ i = \frac{h}{L} = \frac{v^2}{c^2m},$$

$$h = \frac{v^2L}{c^2m}.$$

Equating this result to the right-hand side of (13) gives

$$\frac{v^2L}{c^2m} = \frac{fLv^2}{m\,.\,2g},$$

$$\frac{1}{c^2} = \frac{f}{2g},$$

$$c = \sqrt{\frac{2g}{f}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

An equation suitable for a round pipe running full bore may be obtained from (13). Thus,

$$\text{Hydraulic mean depth}\quad m = \frac{\text{area of section}}{\text{wetted perimeter}}$$

$$= \frac{\pi d^2}{4} \div \pi d = \frac{d}{4},$$

where d is the diameter of the pipe. Hence, from (13),

$$h = \frac{fL}{m}\cdot\frac{v^2}{2g}$$

$$= \frac{4fL}{d}\cdot\frac{v^2}{2g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

In using equations (13) and (15) h, L, m and d should be in feet, v in feet per second, and g may be taken as 32·2. In choosing values of c and f it is safer to assume that the pipe is encrusted, or will be very soon after it is put into service.

Darcy formula. The experiments of Darcy show that the value of f in equation (15) can be expressed in the form

$$f = a\left(1 + \frac{1}{bd}\right). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

In this formula d is the diameter of the pipe in feet, and a and b are coefficients. For clean pipes, a may be taken as 0·005 and b as 12; for old pipes, the values are 0·01 and 12 respectively.

Inserting these values in (16), we have

For clean pipes, $f = 0{\cdot}005\left(1 + \dfrac{1}{12d}\right)$. .(17)

For old pipes, $f = 0{\cdot}01\left(1 + \dfrac{1}{12d}\right)$ .. .. (18)

Values of f calculated thus may be used in equation (15). As has been stated already, it is better to employ the value of f for old or encrusted pipes, and inserting this in (15) gives

$$h = 0{\cdot}01\left(1 + \frac{1}{12d}\right)\frac{4L}{d} \cdot \frac{v^2}{2g} \quad \ldots\ldots (19)$$

EXAMPLE A pipe 18 inches in diameter and 4 miles long connects two reservoirs, in which the difference in level is 200 feet. Find the velocity of flow and the quantity discharged per hour.

Inserting the given quantities in (19), we have

$$200 = 0{\cdot}01\left(1 + \frac{1}{12 \times 1{\cdot}5}\right)\frac{4 \times 4 \times 5280}{1{\cdot}5} \cdot \frac{v^2}{64{\cdot}4}$$

$$= \left(\frac{1}{100} \times \frac{19}{18} \times \frac{16 \times 5280}{1{\cdot}5 \times 64{\cdot}4}\right)v^2$$

$$= 9{\cdot}23\, v^2,$$

$$v = \underline{4{\cdot}65} \text{ feet per second.}$$

Cross sectional area of stream $= \dfrac{\pi d^2}{4} = \dfrac{22}{28} \times 1{\cdot}5 \times 1{\cdot}5$

$= 1{\cdot}77$ square feet.

Volume flowing per hour $= 1{\cdot}77 \times 4{\cdot}65 \times 3600$

$= 29{,}600$ cubic feet

$= \underline{185{,}000}$ gallons

Flow through a pipe having two different diameters. In Fig. 667 (*a*) are shown two pipes of different diameters joined in series. The length L of a pipe of uniform diameter d (Fig. 667 (*b*)), which would discharge the same quantity per second, may be found in the following manner.

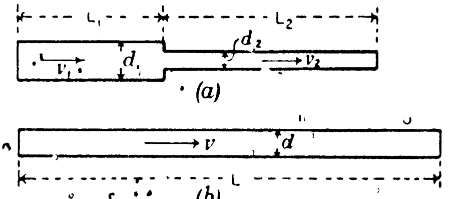


FIG 667 — Flow through a pipe having two different diameters.

The virtual slope of the separate parts of (*a*) will be given by

$$i_1 = \frac{h_1}{L_1}, \qquad i_2 = \frac{h_2}{L_2};$$

$$\therefore \ h_1 = i_1 L_1, \quad h_2 = i_2 L_2.$$

$$\therefore \text{ total loss of head in } (a) = h = h_1 + h_2$$

$$= i_1 L_1 + i_2 L_2. \qquad \ldots\ldots\ldots\ldots \ (20)$$

From (13) above,

$$h_1 = \frac{fL_1}{m_1} \cdot \frac{v_1^2}{2g}$$

$$= \frac{4fL_1}{d_1} \cdot \frac{v_1^2}{2g}, \qquad \ldots\ldots (21)$$

and a corresponding expression for h_2. Assuming that *f* has the same value for both portions of the pipe in Fig. 667 (*a*),

$$h = h_1 + h_2$$

$$= f\left(\frac{4L_1}{d_1} \cdot \frac{v_1^2}{2g} + \frac{4L_2}{d_2} \cdot \frac{v_2^2}{2g}\right). \qquad \ldots\ldots \ (22)$$

Assuming that the uniform pipe in Fig. 667 (*b*) will have the same total loss of head and the same value of *f*, then

$$\text{Total loss of head in } (b) = \frac{4fL}{d} \cdot \frac{v^2}{2g}. \qquad \ldots\ldots\ldots (23)$$

Equating (22) and (23) gives

$$\frac{4fL}{d} \cdot \frac{v^2}{2g} = f\left(\frac{4L_1}{d_1} \cdot \frac{v_1^2}{2g} + \frac{4L_2}{d_2} \cdot \frac{v_2^2}{2g}\right),$$

or

$$\frac{Lv^2}{d} = \frac{L_1 v_1^2}{d_1} + \frac{L_2 v_2^2}{d_2}. \qquad \ldots\ldots\ldots\ldots (24)$$

Let Q be the quantity flowing per second in both (*a*) and (*b*). Then

$$Q = \frac{\pi d_1^2}{4} v_1 = \frac{\pi d_2^2}{4} v_2 = \frac{\pi d^2}{4} v;$$

$$\therefore \ v = \left(\frac{d_1}{d}\right)^2 v_1 = \left(\frac{d_2}{d}\right)^2 v_2,$$

or

$$v_1 = \left(\frac{d}{d_1}\right)^2 v \quad \text{and} \quad v_2 = \left(\frac{d}{d_2}\right)^2 v. \qquad \ldots\ldots\ldots\ldots (25)$$

Inserting these values in (24) gives

$$\frac{Lv^2}{d} = \frac{L_1}{d_1}\left(\frac{d}{d_1}\right)^4 v^2 + \frac{L_2}{d_2}\left(\frac{d}{d_2}\right)^4 v^2;$$

$$\therefore \ \frac{L}{d} = d^4\left(\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5}\right),$$

$$L = d^5\left(\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5}\right). \qquad \ldots\ldots\ldots\ldots (26)$$

If the value chosen for d be the same as d_1, we may write

$$L = L_1 + \left(\frac{d_1}{d_2}\right)^5 L_2 \qquad \ldots\ldots\ldots \quad \ldots\ldots\ldots (27)$$

EXAMPLE. Two pipes are arranged in series, the first being 1 foot in diameter and 1000 feet long and the second pipe being 9 inches in diameter and 500 feet long. Find the length of an equivalent pipe of diameter 12 inches.

From (27),
$$L = 1000 + \left(\frac{12}{9}\right)^5 500$$
$$= 1000 + 2107 = \underline{3107} \text{ feet.}$$

Impact of inelastic bodies When two inelastic bodies collide, deformation will occur during the impact, and, as there will be no effort whatever to recover the original shapes, the bodies will move on together after the impact as one body.

In Fig. 668 (*a*) a body having a mass m_1 and a velocity v_1 is travelling in the same straight line as another body having a mass m_2 and a velocity v_2. After m_1 has overtaken m_2 and impact is complete, the bodies will move as shown at (*b*) and will possess a common velocity v. During impact, it is evident that equal forces have acted forwards on m_2 and backwards on m_1, and these forces have acted during the same interval of time; hence the total change of momentum during impact will be zero (p. 411), or we may say that the total momentum before collision is equal to the total momentum after collision. Hence,

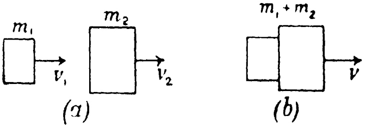


FIG. 668 Impact of inelastic bodies

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v,$$

$$v = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \quad \ldots \ldots \ldots (1)$$

Energy will be wasted during the collision, and the amount of this may be calculated as follows:

$$\text{Before impact, total energy} = \frac{m_1v_1^2}{2g} + \frac{m_2v_2^2}{2g} \quad \ldots\ldots\ldots\ldots (3)$$

$$\text{After impact, total energy} = \frac{(m_1 + m_2)v^2}{2g}$$

$$= \frac{(m_1 + m_2)}{2g}\left(\frac{m_1v_1 + m_2v_2}{m_1 + m_2}\right)^2$$

$$= \frac{(m_1v_1 + m_2v_2)^2}{2g(m_1 + m_2)} \quad \ldots \quad (4)$$

Subtracting (4) from (3) gives

$$\text{Energy wasted} = \frac{m_1v_1^2}{2g} + \frac{m_2v_2^2}{2g} - \frac{(m_1v_1+m_2v_2)^2}{2g(m_1+m_2)}$$

$$= \frac{m_1m_2(v_1^2+v_2^2-2v_1v_2)}{2g(m_1+m_2)}$$

$$= \frac{m_1m_2}{2g(m_1+m_2)}(v_1-v_2)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now $(v_1 - v_2)$ is the relative velocity of the two bodies. Hence,

$$\text{Energy wasted} = \frac{m_1m_2}{2g(m_1+m_2)} \times \text{the square of the relative velocity.}$$

For example, a jet of water falls into a pool of water (Fig. 669) with velocity relative to the earth or pool of v feet per second.

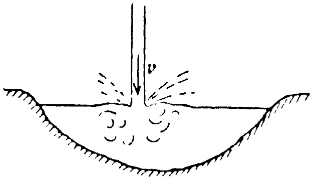


Fig. 669

Consider a mass m_1 pounds of water in the jet, this will be very small as compared with the mass m_2 of the water in the pool, backed as it is by the mass of the earth. In this case,

$$\text{Energy wasted} = \frac{m_1m_2}{2g(m_1+m_2)}v^2$$

$$= \frac{m_1}{2g\left(\frac{m_1}{m_2}+1\right)}v^2$$

$$= \frac{m_1}{2g(0+1)}v^2$$

$$= \frac{m_1v^2}{2g} \text{ ft. lb.}$$

That is, the energy wasted is simply the kinetic energy of the water in the jet.

Waste of energy at a sudden enlargement in a pipe. This case is illustrated in Fig. 670. Water flowing from the small into the larger pipe has its velocity diminished from v_1 to v_2 and mingles with the water already in the larger pipe. Eddies will be set up in the water as indicated in the illustration, and there will be a waste of energy produced in a somewhat similar manner to the case of one body overtaking another which has been discussed above. The relative velocity will be $(v_1 - v_2)$, and the waste of energy will be given by

$$\text{Energy wasted} = \frac{(v_1 - v_2)^2}{2g} \text{ ft.-lb. per pound of water.}$$

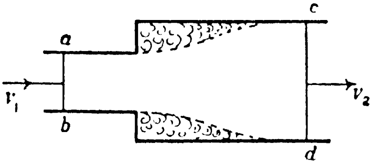


FIG. 670.—A sudden enlargement in a pipe.

St. Venant calculates the wasted energy by adding to the above result $\frac{1}{9} \cdot \frac{v_2^2}{2g}$.

Waste of energy at a sudden contraction in a pipe. Referring to Fig. 671, the water flows in the larger pipe with velocity v_1, and will contract as shown at *cd* on entering the smaller pipe. The velocity v at *cd* will be greater than v_1; hence, up to this section, the water has not overtaken any water moving in front of it, and therefore

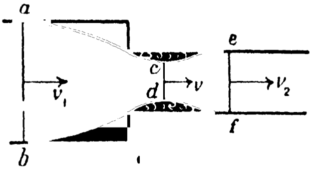


FIG. 671.—Sudden contraction in a pipe.

there has been no impact and consequently no waste of energy from this cause between *ab* and *cd*. Between *cd* and *ef*, the velocity of the water is diminishing again, and it will therefore be between these sections that waste of energy will occur. Hence,

$$\text{Energy wasted} = \frac{(v - v_2)^2}{2g} \text{ ft.-lb. per pound of water.}$$

If a value for the coefficient of contraction be assumed, it becomes possible to calculate v. Thus, let the sectional areas of the stream at *cd* and *ef* be A and A_2 square feet respectively, and let c_c be the coefficient of contraction. Then

$$A = c_c A_2.$$

The quantities of water flowing through *cd* and *ef* per second will be equal. Hence,

$$\frac{v}{v_2}=\frac{A_2}{A}=\frac{A_2}{c_c A_2}=\frac{1}{c_c};$$

$$\therefore\ v=\frac{v_2}{c_c}.$$

$$\therefore\ \text{energy wasted}=\frac{\left(\frac{v_2}{c_c}-v_2\right)^2}{2g}$$

$$=\left(\frac{1}{c_c}-1\right)^2\frac{v_2^2}{2g}\ \text{ft.-lb. per pound of water.}$$

The coefficient is not well known; the waste of energy at the pipe entrance from a reservoir or tank, if sharp-edged, may be taken as

$$\text{Waste of energy}=0\cdot 5\,\frac{v^2}{2g}\ \text{ft.-lb. per pound of water,}$$

where v is the velocity in the pipe in feet per second.

Energy is wasted also in pipe lines at bends and elbows. This is a subject for experimental investigation, see pp. 671 and 672, and there is not much definite information available. The energy thus wasted is small compared with the frictional waste in a long pipe line.

Resistance of ships. The principal causes of resistance to the passage of a ship through water are (*a*) the friction of the wetted surface, (*b*) the formation of waves owing to the vessel pushing the water laterally near the bow, and the return of the water near the stern to fill the space left by the vessel, (*c*) the formation of eddies, owing principally to swirls set up in the water closing in at the stern.

The first of these, viz. skin friction, may be taken as proportional to the area of the wetted surface and to the square of the speed. Wave formation resistances may be taken to be proportional to the sixth power of the speed. Eddy resistances are proportional to the area of the wetted skin and to the square of the speed.

The relation of the resistances of a ship and of those of a scale model of the ship may be deduced from these laws. Let the scales be such that any linear dimension of the ship is D times the corresponding dimension of the model. Then if A_s and A_m are the wetted surfaces of the ship and model respectively, we have

$$A_s : A_m = D^2 : 1. \qquad \text{(1)}$$

The frictional resistances will be given by

$$F_s = fA_sV_s^2,$$

$$F_m = fA_mV_m^2,$$

where f is a coefficient which may be assumed to have the same value for both ship and model, and V_s and V_m are the speeds of the ship and model respectively. We may write

$$\frac{F_s}{F_m} = \frac{A_s}{A_m}\cdot\left(\frac{V_s}{V_m}\right)^2,$$

or, from (1), $$= D^2\left(\frac{V_s}{V_m}\right)^2. \quad \ldots \quad (2)$$

The relation of the wave-forming resistances may be written

$$\frac{W_s}{W_m} = \left(\frac{V_s}{V_m}\right)^6 \ldots\ldots\ldots\ldots (3)$$

The relation of the eddy-forming resistances will be

$$\frac{E_s}{E_m} = \frac{A_s}{A_m}\left(\frac{V_s}{V_m}\right)^2$$

$$= D^2\left(\frac{V_s}{V_m}\right)^2. \ldots\ldots\ldots\ldots (4)$$

Now, supposing $D = \left(\frac{V_s}{V_m}\right)^2$, or $V_s = V_m\sqrt{D}$, the results (2), (3) and (4) may be written

$$\frac{F_s}{F_m} = D^2 . D = D^3. \ldots\ldots\ldots (5)$$

$$\frac{W_s}{W_m} = \left\{\left(\frac{V_s}{V_m}\right)^2\right\}^3 = D^3 \ldots\ldots\ldots (6)$$

$$\frac{E_s}{E_m} = D^2 . D = D^3. \ldots\ldots\ldots (7)$$

That is, the relation of each of the three kinds of resistance is proportional to D^3; hence their sum, which gives the total resistance, will also be proportional to D^3. The law has been expressed by Mr. Froude as follows, and is known by his name: If the ship be D times the linear dimensions of the model, and if at the speeds V_1, V_2, V_3, etc., the measured resistances of the model are R_1, R_2, R_3, etc., then for speeds $V_1\sqrt{D}$, $V_2\sqrt{D}$, $V_3\sqrt{D}$, etc., of the ship, the resistances will be D^3R_1, D^3R_2, D^3R_3, etc. The speeds of the model and of the ship so related are called **corresponding speeds**.

EXERCISES ON CHAPTER XXIII.

1. A straight horizontal pipe 6 inches in diameter gradually becomes 2 inches in diameter, and then diverges again to 6 inches diameter. Water is flowing steadily through it with a velocity of 4 feet per second in the larger portions. Find the velocity in the throat of the pipe, and hence calculate the difference in pressures at the largest and smallest sections of the pipe, neglecting friction.

2. In a pipe having the same dimensions as that in Question 1, the difference in head is observed to be 4 feet of water at a certain velocity of flow. Calculate the flow of water in cubic feet per second, neglecting friction.

3. Water is discharged through a circular sharp-edged orifice 1 inch in diameter in the vertical side of a tank The water-level in the tank is 3 feet above the level of the centre of the orifice Calculate the discharge in cubic feet per second and also in gallons per hour.

4. Answer Question 3 if a trumpet mouthpiece is fitted giving a jet 1 inch in diameter. Take the coefficient of velocity to be 0·95.

5. A circular sharp-edged orifice is situated in a tank bottom and discharges a jet vertically downwards The water level in the tank is 2 feet above the plane of the orifice Calculate the velocity at a section of the jet 6 inches below the plane of the orifice, neglecting friction.

6. A reservoir contains air at a pressure of 15·7 lb. per square inch absolute, and one pound weight of it may be taken to occupy a volume of 13 cubic feet. The air is being discharged through a sharp-edged orifice 2 inches in diameter into the atmosphere, the pressure of which is 14·7 lb. per square inch. Calculate the flow in cubic feet per second

7. A jet of water is discharged from a trumpet orifice 3 inches in diameter under a head of 200 feet. Calculate the reaction of the jet in lb, neglecting any reduction in velocity owing to friction

8. Taking the coefficient of velocity to be 0·96, calculate the flow in cubic feet per second through a Borda mouthpiece 2 inches in diameter under a head of 10 feet.

9. Water is flowing over a triangular gauge notch of 90° under a head of 10 inches. Calculate the flow in gallons per hour.

10. A rectangular gauge notch has a width of 2 feet, and water is flowing through it under a head of 6 inches. Calculate the flow in cubic feet per second, (*a*) for two side contractions, (*b*) for one side contraction, (*c*) for no side contractions.

11. Calculate the virtual slope of a pipe 2·5 miles long connecting two reservoirs, in which the difference in levels is 35 feet.

12. A circular pipe 4 feet in diameter is running half full of water. Find the hydraulic mean depth.

13. Use the Chezy formula to calculate the velocity of flow for a pipe 30 inches diameter and 10 miles long connecting two reservoirs, for which the fall in level is 150 feet. Take $C = 105$. Calculate also the quantity flowing per day of 24 hours.

14. Suppose in Question 13 that a sluice valve in the pipe close to the lower reservoir is closed partially and that it is found that water rises in a tube connected to the pipe side of the valve to a level 40 feet below the water level in the upper reservoir. Calculate the velocity of flow, using $c = 100$.

15. Use the Darcy formula to calculate the velocity of flow and the discharge in cubic feet per second in a pipe 24 inches in diameter and having a virtual slope of 0·01. Give the answers both for a new pipe and for an old one.

16. A pipe 18 inches in diameter and 2 miles long is connected to a second pipe 15 inches in diameter and 0·5 mile long. Find the length of an equivalent pipe of diameter 18 inches.

17. A pipe is enlarged suddenly from 4 inches to 6 inches diameter. The velocity of flow in the smaller portion is 4 feet per second. Calculate the energy wasted per pound of water.

18. Suppose in Question 17 that the direction of flow is reversed and that the velocity in the smaller portion of the pipe is still 4 feet per second. Calculate the energy wasted per pound of water, assuming that the coefficient of contraction is 0·7.

19. Describe, with a sketch, the Venturi water meter, and state the principle of its action. (B.E.)

20. A particle of water is at a place A, where the pressure is 0, its height above datum is 50 feet, its velocity is 5 feet per second, and it finds its way without friction to a place B, where the pressure is 0 and height above datum 30 feet. What is its velocity at B? (B.E.)

21. A particle of air flows without friction. Its pressure p (in lb. per square foot) and its speed v (in feet per second) may both alter, but the sum

$$\frac{v^2}{2g} + \frac{p}{w}$$

remains constant. If w is the average weight of a cubic foot of air, g is 32·2. At a place A, p is 1·1 atmospheres and v is 0; a particle finds its way from A to B. At B the pressure is 1 atmosphere; what is the speed at B? Take w as 0·075 lb. per cubic foot. (B.E.)

22. A fan drives air vertically downwards through a horizontal circular opening 8 feet in diameter and so exerts a lifting force of 200 lb. What is the average downward velocity of the air in the opening? The weight of 1 cubic foot of the air is 0·08 lb. (B.E.)

23. A pipe, $1\frac{1}{2}$ inches in diameter, is enlarged gradually to 3 inches in diameter; the pipe has a falling gradient of 1 in 15. At the point where the diameter begins to enlarge, the velocity of flow is 4 feet per second and the pressure is 30 lb. per square inch. Find the velocity and pressure in the 3 inch portion of the pipe at a point distant 45 feet, measured along the horizontal from the point where the enlargement began. Neglect all frictional losses. (B.E.)

24. A Venturi meter is fitted on to a 50-inch diameter main. The diameter of the throat of the meter is 20 inches. It is found that at a certain instant the pressure in the main at the point of entry to the meter is equal to a head of 110 feet of water, and in the throat of the meter it is equal to a head of 97 feet of water. How many gallons of water is this 50-inch main delivering per hour under these conditions? (B.E.)

25. Froude's law of corresponding speeds, which is used in model experiments upon ships, applies also to flying machines. A certain flying machine is to be constructed, weighing 2 tons and travelling at 100 miles per hour. The model is $\frac{1}{14}$ full size Find its weight and corresponding speed (See p. 621 for Froude's law.) (B E.)

26. In the horizontal floor of a tank, in which the water is 3 feet deep and in which the water-surface is 15 square feet in area, a sharp-edged circular orifice 2 inches in diameter is opened. In what time will the water-level in the tank sink 1 inch, if there be no supply of water to the tank? In answering make no attempt at correction for the time spent in starting the flow, that is, upon initial acceleration of the mass of the first parts of the flow. (I C E)

27. Water flows down a sloping pipe (running full) from a point where the velocity is 10 feet per second and the pressure 15 lb per square inch absolute to a point 100 feet lower, where the pipe diameter is three times as great. Calculate the pressure at this point, neglecting losses (I.C E)

28. A straight pipe of circular section is 400 feet long and 4 inches internal diameter Its down gradient is 1 in 200 It discharges full bore into the atmosphere at the level of the lower end of the pipe, and draws by a bell mouth from a reservoir with 12 feet head of water over the mouth of the pipe. The gradient of the frictional loss of head in the pipe is 0·00012 times the square of the velocity divided by the mean hydraulic depth, the units being feet and seconds Find the linear velocity through the pipe and the discharge in gallons per hour (I.C.E.)

29. A pipe of circular section of 30 inch diameter has a fall of 16 feet in a straight run of 2000 feet In the formula for loss of head in feet per foot run $i = fv^2/m$, its coefficient is 0·00009, the units being feet and seconds. What volumetric flow in cubic feet per second must there be through this pipe to maintain throughout its length the level of the water at the centre of the circular section? (I C E)

30. State the nature of the losses that occur when a stream of water in a pipe running full encounters, (*a*) a sudden enlargement, (*b*) a sudden contraction of the channel area A surface condenser contains 530 tubes 9 ft 3 in long and 0·65 inch internal diameter, through which 450 gallons of water are pumped per minute The water flows through half the number of tubes and then back again through the other half. Calculate the total loss of head in the condenser, assuming that the coefficient of contraction at entrance to the tubes is 0·59 and $f = 0·007$. What horse-power is required to force the water through the condenser? (L U)

31. Two reservoirs are connected by a straight pipe 1 mile long. For the first half of its length the pipe is 6 inches diameter; its diameter is then suddenly reduced to 3 inches The surface of the water in the upper reservoir is 100 feet above that in the lower. Tabulate the losses of head which occur, including that at a sharp-edged entry, and determine the flow in gallons per minute. (Assume $f = 0·01$). (L U)

32. A Venturi meter is placed in a horizontal 6 inch pipe, the diameter of the throat being 2 inches If the difference of head is equivalent to 10 inches of mercury, find the flow in gallons per minute, and the velocities. Assume the coefficient of the meter to be 1·00 (L.U.)

CHAPTER XXIV.

HYDRAULICS: PRESSURES OF JETS. TURBINES. CENTRIFUGAL PUMPS

Pressure caused by a jet impinging on a fixed plate. A jet of water impinging on a fixed plate or vane will arrive at the plate with a certain velocity, and will leave the plate with a velocity changed in direction and also reduced in magnitude, owing to friction and eddies. Disregarding the latter, we may examine the examples shown in Fig. 672. The pressure in each case may be found by first estimating the change in velocity, and hence the change of momentum per second. The jet section is assumed to be A square feet, and v_1 is the velocity of the jet in feet per second.

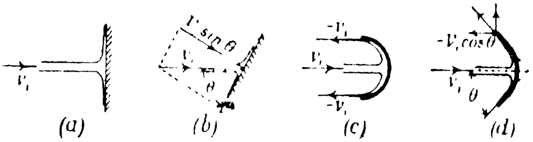


FIG. 672.—Pressures of jets on plates; impact cases.

(*a*) The jet impinges on a plate fixed at 90° to the axis of the jet. The water spreads and leaves the plate tangentially in all directions. Hence the whole of v_1 has been eliminated.

$$\text{Mass of water reaching plate per second} = Av_1w \text{ pounds.}$$

$$\text{Change of momentum per second} = Awv_1^2$$

$$\text{Pressure on plate} = \frac{Awv_1^2}{g} \text{ lb.}$$

(*b*) The plate is fixed at an angle θ to the axis of the jet. As in (*a*), the water will spread and leave the plate tangentially in all directions. The change in velocity will be the component of v_1 perpendicular to the plate, viz. $v_1 \sin\theta$.

$$\text{Mass of water reaching plate per second} = Av_1w \text{ pounds.}$$

$$\text{Change of momentum per second} = Av_1wv_1 \sin\theta = Awv_1^2 \sin\theta.$$

$$\text{Pressure on plate} = \frac{Awv_1^2 \sin\theta}{g} \text{ lb.}$$

(*c*) The jet impinges on a hemispherical cup fixed so as to oppose the jet axially. The leaving water will be directed backwards with a velocity equal to v_1. Hence the total change in velocity will be $2v_1$.

$$\text{Mass of water reaching plate per second} = Av_1w \text{ pounds.}$$

$$\text{Change of momentum per second} = 2Awv_1^2.$$

$$\text{Pressure on plate} = \frac{2Awv_1^2}{g} \text{ lb.}$$

(*d*) A similar cup to that in case (*c*), but the tangent at the cup exit makes an angle θ with the jet. The total change in velocity will be $(v_1 + v_1 \cos\theta)$.

$$\text{Mass of water reaching plate per second} = Av_1w \text{ pounds.}$$

$$\text{Change of momentum per second} = Av_1w(v_1 + v_1\cos\theta) = Awv_1^2(1 + \cos\theta).$$

$$\text{Pressure on plate} = \frac{Awv_1^2}{g}(1 + \cos\theta).$$

In each of the above cases there has been **impact**, which may be eliminated by the device of so shaping the vane as to allow of the jet sliding on to it, *i.e.* the jet enters the vane tangentially. Fig. 673

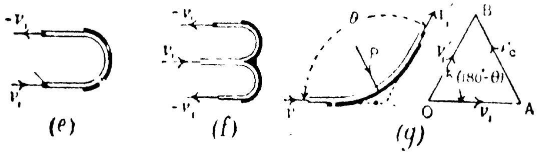


FIG. 673.—Pressures of jets on plates; impact eliminated.

(*e*), (*f*) and (*g*) are examples. The pressures in the cases of (*e*) and (*f*) will be given by the same expression as that found in case (*c*), viz.

$$\text{Pressure on plate} = \frac{2Awv_1^2}{g} \text{ lb.}$$

(*g*) Here the water enters a curved vane with a velocity v_1 and leaves it with an equal velocity v_1' (neglecting friction). The change in velocity is found from the diagram OAB, in which OA represents v_1 and OB represents v_1'. AB will give the change in velocity v_c. Let θ be the angle between the entering and leaving jets. Then the angle

AOB will be ($180° - \theta$), and as the triangle AOB is isosceles, each of the angles OAB and OBA will be equal to $\frac{1}{2}\theta$. Hence,

$$\frac{v_r}{v_1} = \frac{AB}{OA} = \frac{\sin(180-\theta)}{\sin\frac{1}{2}\theta} = \frac{\sin\theta}{\sin\frac{1}{2}\theta}$$

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 2\cos\frac{1}{2}\theta;$$

$$\therefore\ v_r = 2v_1\cos\frac{1}{2}\theta.$$

Mass of water reaching the vane per second $= Av_1w$ pounds.

$$\text{Change of momentum per second} = Av_1w \times 2v_1\cos\tfrac{1}{2}\theta$$
$$= 2Awv_1^2\cos\tfrac{1}{2}\theta$$
$$\text{Pressure on vane} = \frac{2Awv_1^2}{g}\cos\tfrac{1}{2}\theta.$$

This pressure acts in the same line as v_r.

Pressure of jets on moving vanes In Fig 674 is shown a vane AB making 90° with the axis of a jet of water having a velocity v_1 feet per second and a sectional area of A square feet. The vane is travelling in the same direction as the jet and has a velocity v_2 feet per second.

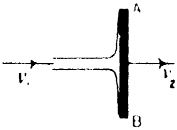


FIG. 674 Pressure of a jet on a moving plate

It is evident that the spreading water in contact with the plate still possesses a forward velocity v_2 equal to that of the plate, hence the change in velocity in the direction of motion of the plate will be $(v_1 - v_2)$ feet per second Further, owing to the plate moving away from the jet, the length of jet arriving at the plate per second will be $(v_1 - v_2)$. Hence,

Mass of water reaching the vane per second $= A(v_1 - v_2)w$ pounds.

$$\text{Change of momentum per second} = A(v_1 - v_2)w(v_1 - v_2)$$
$$= Aw(v_1 - v_2)^2.$$
$$\text{Pressure on vane} = \frac{Aw(v_1 - v_2)^2}{g}\ \text{lb.}\dots(1)$$

This case as stated above has no practical value, but becomes of importance if we have a succession of vanes brought perpendicularly into the jet one after another, such as might be realised by having a number of vanes mounted on the circumference of a rotating wheel (Fig. 675). In one second a length of jet equal to v_1 feet will reach the wheel now. Hence,

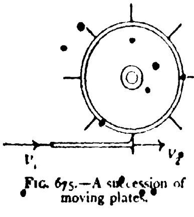


FIG. 675.—A succession of moving plates.

Mass reaching wheel per second = Av_1w pounds.

$$\text{Change of momentum} = Av_1w(v_1 - v_2).$$

$$\text{Pressure on vanes} = P = \frac{Aw}{g}\cdot(v_1 - v_2)v_1 \text{ lb.} \quad \ldots\ldots(2)$$

Work will be done on the wheel by P to the amount of Pv_2 foot-lb. per second, or

$$\text{Work done per second} = \frac{Aw}{g}(v_1 - v_2)v_1v_2 \text{ ft.-lb.} \quad \ldots\ldots\ldots(3)$$

The efficiency of the arrangement may be found by evaluating the energy available in the jet. Since Av_1w pounds of water reach the wheel per second with a velocity v_1,

$$\text{Kinetic energy available per second} = \frac{Av_1wv_1^2}{2g}$$

$$= \frac{Awv_1^3}{2g} \text{ ft.-lb.} \quad \ldots\ldots\ldots(4)$$

$$\text{Hence,} \qquad \text{Efficiency} = \frac{\text{work done}}{\text{energy supplied}}$$

$$= \frac{\frac{Aw}{g}(v_1 - v_2)v_1v_2}{\frac{Aw}{2g}v_1^3}$$

$$= \frac{2(v_1 - v_2)v_2}{v_1^2} \quad \ldots\ldots(5)$$

With a constant supply of water, the work done may be varied by varying the velocity v_2 of the vanes.

Using (3), the work done per second will be greatest when $(v_1 - v_2)v_1v_2$ attains its maximum value. Assuming a steady velocity of supply, v_1 will be constant, and maximum work will occur when $(v_1 - v_2)v_2$ has its maximum value. To obtain this value, differentiate $(v_1 - v_2)v_2$ and equate the result to zero, giving

$$v_1 - 2v_2 = 0,$$

$$\text{or} \qquad v_2 = \tfrac{1}{2}v_1 \quad \ldots\ldots(6)$$

The efficiency under these conditions may be obtained by substitution in (5), giving

$$\text{Maximum efficiency} = \frac{2(v_1 - \frac{1}{2}v_1)\frac{1}{2}v_1}{v_1^2}$$

$$= \tfrac{1}{2} = 50 \text{ per cent.} \quad \ldots\ldots\ldots\ldots(7)$$

In connection with hydraulic machines, the term **hydraulic efficiency** is employed often, and may be defined thus: Let H = the energy supplied per pound of water, and let h = the energy carried away in the water leaving the machine. Then

$$\text{Hydraulic efficiency} = \frac{H - h}{H}.$$

If it is known that there are hydraulic losses in the machine which produce a waste of h_f foot lb. of energy per lb. of water, then

$$\text{Hydraulic efficiency} = \frac{H - h - h_f}{H}.$$

Poncelet wheel blade. An efficiency much better than that possible with a flat vane can be obtained by using vanes suitably curved. Thus, in Fig. 676, AB is one of a number of vanes attached to the circumference of a revolving wheel and receiving a jet of water at A. The velocity of the jet being v_i and that of the vane at A being V_i, the relative velocity may be obtained from the parallelogram ADEC, in which DA represents V_i and CA represents v_i. The diagonal EA then gives the relative velocity v_r. If the tangent to the vane at A coincides with EA, the water will slide on to the vane without impact, and hence there will be no wasted energy owing to shock, a waste which must occur with the vane shown in Fig. 675.

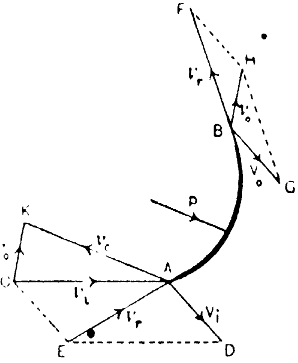


FIG. 676.—Blade of a Poncelet water wheel.

The water travels round the curve of the vane, preserving unaltered its relative velocity v_r. At the point of exit, B, therefore it will possess velocities represented by BF = v_r and BG = V_o, the latter being the velocity of the vane at B relative to the earth. The absolute velocity v_o of the leaving water may be found by completing the parallelogram BGHF, giving v_o = BH.

Since the water enters with the velocity v_i and leaves with a velocity v_o, the change of velocity may be found by making CK equal and parallel to BH, when AK = v_c will be the change in velocity. The resultant pressure P on the vane will be in a line parallel to v_c, and may be estimated by taking the change in momentum per second.

The energy supplied per pound of water in the jet is $\frac{v_i^2}{2g}$ foot-lb.; assuming the pressure throughout the stream to be constant and equal to that of the atmosphere, and disregarding any change in potential energy, the energy carried away in the water leaving the vane will be $\frac{v_o^2}{2g}$ foot-lb. per pound. The difference between these, viz. $\left(\frac{v_i^2 - v_o^2}{2g}\right)$, may be converted into work done on the wheel. This difference would be $\frac{v_i^2}{2g}$ if v_o were zero, *i.e.* if v_r and V_o were equal, opposite, and in the same straight line; in this case the whole of the energy supplied would be converted into mechanical work, and the efficiency would be 100 per cent. This, of course, neglects frictional losses and the inevitable lashing of the water, which must occur as the vanes successively enter the jet. In the **Poncelet water wheel**, constructed after this method, an efficiency of nearly 70 per cent. has been obtained.

Pelton wheel. In the **Pelton wheel** (Fig. 677), the vanes are similar to those shown in Fig. 673 (*f*). Let the velocities of the jet and of the vane be v_1 and V respectively, and let the shape of the vane be such as to cause the direction of the leaving water to be parallel to the entering jet; usually v_2 is not quite parallel to v_1, the blade being shaped so as to throw the leaving water clear of the entering jet as shown in Fig. 677.

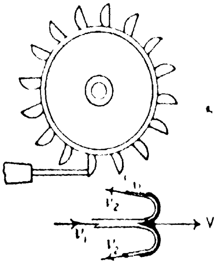


FIG. 677.—Action in a Pelton wheel.

Relative velocity of entering jet and vane $= v_1 - V$. This relative velocity will be preserved unaltered as the water travels round the curved vane. At exit, the water will have therefore a velocity $(v_1 - V)$ towards the left, together with a velocity V towards the right, and its absolute velocity will be

$$v_2 = V - (v_1 - V) = 2V - v_1.$$

As the initial velocity is v_1, we have

$$\begin{aligned} \text{Change in velocity} &= v_1 - v_2 \\ &= v_1 - (2V - v_1) \\ &= 2(v_1 - V). \quad \ldots\ldots\ldots\ldots\ldots\ldots(1) \end{aligned}$$

Mass of water reaching the wheel per second $= wAv_1$.

Change in momentum per second $= wAv_1 \times 2(v_1 - V)$.

$$\text{Pressure on vane} = P = \frac{2wAv_1(v_1 - V)}{g} \text{ lb.}$$

Work done on the wheel per second $= PV$

$$= \frac{2wAv_1V(v_1 - V)}{g} \text{ ft.-lb.} \quad \ldots (2)$$

Assuming a constant velocity of supply, v_1, the work done will have a maximum value when $V(v_1 - V)$ is a maximum. Differentiating and equating to zero, we have

$$v_1 - 2V = 0,$$

$$V = \tfrac{1}{2}v_1. \quad \ldots\ldots \quad \ldots\ldots \quad (3)$$

Substitution of this value in (2) gives

$$\text{Maximum work per second} = \frac{2wAv_1}{g} \times \tfrac{1}{2}v_1(v_1 - \tfrac{1}{2}v_1)$$

$$= wAv_1 \cdot \frac{v_1^2}{2g} \text{ ft.-lb.} \quad \ldots\ldots\ldots (4)$$

Since wAv_1 is the mass of water reaching the wheel per second, it follows from (4) that the whole of the kinetic energy available in the jet has been converted into work done on the wheel, *i.e.* the efficiency in these circumstances is 100 per cent. Actual efficiencies of from 70 to 90 per cent. have been obtained.

The hydraulic efficiency may be obtained by dividing the result given in (2) for the work done per second by the energy supplied in the jet. Thus,

$$\text{Hydraulic efficiency} = \frac{2wAv_1V(v_1 - V)}{g} \div wAv_1\frac{v_1^2}{2g}$$

$$= \frac{4V(v_1 - V)}{v_1^2} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

This will become unity when

$$V = \tfrac{1}{2}v_1.$$

Other types of water wheels, in which gravity plays the most important part, have been much used. In the **over-shot water wheel** (Fig. 678), the water is led to the top of the wheel, and there partially fills a number of buckets fixed to the wheel. The weight of these descending buckets enables work to be done on the wheel. In the **breast-shot**

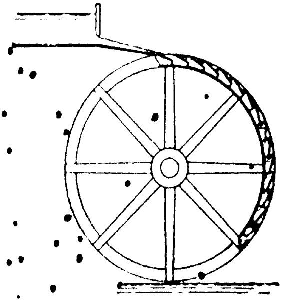


Fig. 678.—Over-shot water wheel.

wheel, the water enters the buckets on a level about the same as that of the wheel centre. The Poncelet wheel is an improved form of **under-shot wheel**.

Hydraulic turbines. A **hydraulic turbine** is a machine in which the energy of a supply of water is converted into mechanical work by passage of the water through a wheel furnished with blades or vanes. In general, the action consists of causing the water to whirl before it enters the wheel. In this condition it possesses angular momentum, and the function of the wheel blades is to abstract this angular momentum and to discharge the water without whirl. A couple will thus act on the revolving wheel (p. 430), and will produce mechanical work.

Turbines may be classed generally as **impulse turbines** and **reaction or pressure turbines**. In impulse turbines, the energy of the water is practically entirely in the kinetic form before the water enters the wheel, a result obtained by reduction of the pressure head to that of the atmosphere and by giving the water a corresponding velocity. In reaction turbines, the energy of the water at the wheel entrance is partly in the pressure form and partly kinetic.

Turbines may be classed further with reference to the principal direction of flow of the water. In **axial-flow turbines**, the flow is parallel to the axis of rotation of the wheel: in **inward-flow turbines**, the flow is radial towards the wheel centre, in **outward flow turbines**, the flow is radial from the centre to the circumference of the wheel; in **mixed-flow turbines**, the flow is partly radial and partly axial.

The energy available in a given stream or river depends on the quantity of water flowing per second and on the height of the available fall.

Let W = flow of water per second, in pounds.
H = height of fall, in feet.
E = the efficiency of the turbine, all sources of waste being included.

Then Energy available = WH foot lb. per second.

Energy produced by the turbine = WHE foot-lb. per second.

$$\text{H.P. delivered by the turbine} = \frac{WHE}{550}.$$

If a given horse power has to be delivered by the turbine, then the quantity of water which must be passed through the wheel will be given by

$$W = \frac{550 \times \text{H.P.}}{HE} \text{ pounds per second.}$$

The turbine must be proportioned so as to permit of this quantity of water being dealt with per second. If the efficiency is not known with precision, a value of from 70 to 75 per cent. may be assumed.

The **general conditions of efficiency** in any hydraulic turbine include the entrance of the water into the wheel without shock, and its leaving the wheel without whirl and with as little residual velocity as possible. Further, the guide passages and the passages through the wheel must be shaped so as to offer the minimum frictional and eddy-forming resistance to the flow of water.

Impulse turbines. The **Girard turbine** shown in outline in Fig. 679 is an example of an impulse turbine. The water supply flows from a chamber AA through a ring of guide orifices BB. These orifices are furnished with guide blades which cause the water to leave the orifices with whirling velocity. The water then passes through a revolving wheel CC fixed to a vertical shaft DD, and is discharged into the tail water EE. The wheel passages are furnished with blades having such a shape as to abstract the velocity of whirl from the water. The guide passages BB run full of water, and at their discharge edges the pressure of the water is equal to that of the atmosphere. The maintenance of atmospheric pressure in the wheel passages is obtained (*a*) by having thin streams of water on the concave sides of the blades only, (*b*) by having side openings in the wheel as shown at EE in Fig. 680; these openings permit of free access of air to the streams of water passing through the wheel.

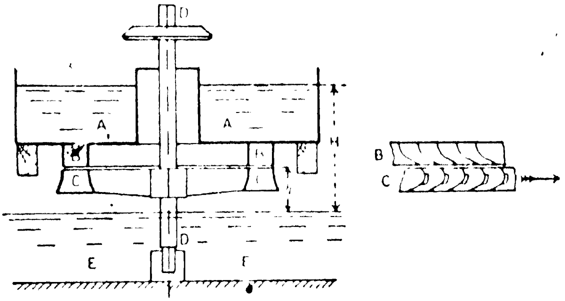


Fig. 679.—Action of a Girard impulse turbine

It will be evident that the wheel of a turbine of this type must be situated above the level of the tail water, so as to permit of the access of air. If *h* feet is the height of the lower edges of the guide orifices above the tail water (Fig. 679), then the head available for giving velocity to the water at these edges will be (H − h) feet, where H is the difference in levels of the supply and the tail water.

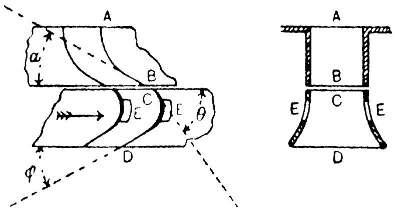


Fig. 680 Blades of an axial flow impulse turbine.

Assuming a coefficient of velocity of 0·95, we have for the velocity v_i of the water leaving the guide blade:

$$v_i = 0{\cdot}95\sqrt{2g(H - h)} \quad \ldots\ldots\ldots\ldots \quad (1)$$

In Fig. 680, AB and CD are the guide blade and the wheel blade respectively. Let α be the angle to the horizontal made by the tangent to the guide blade at B. v_i will have horizontal and vertical components, represented by $w_i = da$ and $u_i = ba$ respectively in Fig. 681 (a), the parallelogram of velocities being $abcd$. Hence,

$$u_i = v_i \sin \alpha, \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$w_i = v_i \cos \alpha, \quad \ldots\ldots\ldots\ldots \quad (3)$$

Let V be the circumferential velocity of the wheel at its mean radius. Then the relative velocity v_{ri} of the inlet water and the wheel

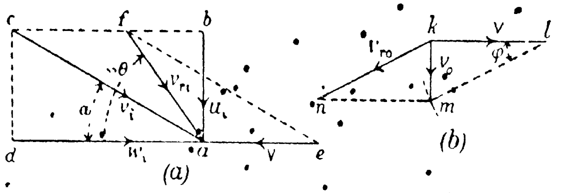


Fig. 681.—Velocity diagrams for an axial flow impulse turbine.

may be found by making ea represent −V (Fig. 681 (a)) and completing the parallelogram of velocities $aefc$; fa then represents v_{ri}. Let θ be the angle which fa makes with the horizontal. Then the angle with the horizontal made by the tangent to the wheel blade at

C (Fig. 680) should be made equal to θ; the inlet water will then slide on to the wheel blade without shock. Referring again to Fig. 681 (*a*), we have

$$\cot\theta = \frac{fb}{ba}$$

$$= \frac{w_i - V}{u_i} \quad \ldots\ldots\ldots\ldots (4)$$

Also, $$\frac{v_{ri}}{u_i} = \frac{fa}{ba} = \operatorname{cosec}\theta,$$

$$\therefore\ v_{ri} = u_i \operatorname{cosec}\theta. \quad \ldots (5)$$

The relative velocity remains unchanged in magnitude by passage over the wheel blades; hence, at the exit D of the blade (Fig. 680), the water will have an absolute velocity v_o, which is the resultant of component velocities $v_{ro} = v_{ri}$ and V, the mean circumferential velocity of the wheel at the outlet. In Fig. 681 (*b*), make $kl = V$, draw km vertical (as is required for the condition of no whirl at the outlet), and describe an arc with centre l and radius $lm = v_{ro}$; the parallelogram of velocities for the exit water will then be $klmn$. v_o is at an angle ϕ to the horizontal, and the tangent to the wheel blade at D (Fig. 680) should be at the same angle. In Fig. 681 (*b*), we have

$$\frac{V}{v_{ro}} = \frac{kl}{lm} = \cos\phi$$

$$\cos\phi = \frac{kl}{lm} = \frac{V}{v_{ro}}. \quad \ldots\ldots\ldots\ldots (6)$$

Also, $$\frac{v_o}{v_{ro}} = \frac{km}{lm} = \sin\phi,$$

$$\therefore\ v_o = v_{ro}\sin\phi = v_{ri}\sin\phi. \quad \ldots\ldots\ldots\ldots (7)$$

Having found α, θ and ϕ, the guide and wheel blades may be constructed by drawing easy curves tangential to these directions.

Since the pressure at D (Fig. 680) is atmospheric, we may write

Total energy carried away by one pound of exit water $= \dfrac{v_o^2}{2g}$ ft.-lb.

Total energy available in the fall, per pound of water $=$ H ft.-lb.

Hence the hydraulic efficiency, *i.e.* the efficiency disregarding all sources of waste excepting the energy carried away in the leaving water, is given by

$$\text{Hydraulic efficiency} = \frac{H - \dfrac{v_o^2}{2g}}{H}. \quad \ldots\ldots\ldots\ldots (8)$$

EXAMPLE. Taking practical values as follows:

$$u_i = 0{\cdot}45\sqrt{2g(H-h)},$$
$$V = 0{\cdot}5\sqrt{2g(H-h)},$$

find the other velocities and angles of an axial-flow impulse turbine. Find also the hydraulic efficiency.

From (1) and (2), $\sin\alpha = \frac{u_i}{v_i} = \frac{0{\cdot}45}{0{\cdot}95} = 0{\cdot}474;$

$\therefore\ \alpha = 29^\circ$ nearly.

From (3), $w_i = v_i\cos\alpha = (0{\cdot}95\times0{\cdot}875)\sqrt{2g(H-h)}$
$= 0{\cdot}83\sqrt{2g(H-h)}.$

From (4), $\cot\theta = \frac{w_i - V}{u_i} = \frac{0{\cdot}83 - 0{\cdot}5}{0{\cdot}45} = 0{\cdot}735;$

$\therefore\ \theta = 54^\circ$ nearly.

From (5), $v_{ri} = v_{ro} = u_i\operatorname{cosec}\theta = \left(0{\cdot}45\times\frac{1}{0{\cdot}809}\right)\sqrt{2g(H-h)}$
$= 0{\cdot}557\sqrt{2g(H-h)}.$

From (6), $\cos\phi = \frac{V}{v_{ro}} = \frac{0{\cdot}5}{0{\cdot}557} = 0{\cdot}898;$

$\phi = 26^\circ$ nearly.

From (7), $v_o = v_{ro}\sin\phi = (0{\cdot}557\times0{\cdot}438)\sqrt{2g(H-h)}$
$= 0{\cdot}244\sqrt{2g(H-h)}.$

Taking $v_o = 0{\cdot}24\sqrt{2gH}$, we have, from (8),

$$\text{Hydraulic efficiency} = \frac{H - \frac{v_o^2}{2g}}{H} = \frac{H - 0{\cdot}0576H}{H}$$
$$= 0{\cdot}9424$$
$$= 94 \text{ per cent. nearly.}$$

The actual efficiency after all sources of waste are taken account of in this type of turbine is about 75 per cent.

Impulse wheel passages. It will be noted from the above example that the ratio

$$\frac{u_i}{v_o} = \frac{0{\cdot}45}{0{\cdot}244} = 1{\cdot}84.$$

Now u_i is the component velocity of the water entering the wheel taken at 90° to the inlet surfaces of the wheel; the quantity of water flowing into the wheel will be given by

$$Q_i = u_iA_i \text{ cubic feet per second,}$$

where A_i is the total inlet surface in square feet.

Again v_o is the leaving velocity of the water, perpendicular to the exit surfaces of the wheel; hence the quantity of water leaving the wheel will be

$$Q_o = v_o A_o \text{ cubic feet per second,}$$

A_o being the exit surface in square feet.

It is evident that Q_i and Q_o must be equal. Hence,

$$u_i A_i = v_o A_o,$$

or

$$\frac{A_o}{A_i} = \frac{u_i}{v_o} = 1 \cdot 84.$$

In practice, this increase of outlet surface is obtained by splaying the wheel passages as shown in the section in Fig. 680. If Q, the quantity of water in cubic feet per second which must be passed through the wheel, be known, the dimensions of wheel required may be calculated from

$$\begin{aligned} Q &= u_i A_i \\ &= u_i \pi (R_1^2 - R_2^2), \end{aligned} \qquad (9)$$

where R_1 and R_2 feet are the external and internal radii respectively of the wheel at the inlet surface. A correction must be applied to this for the area abstracted owing to the thickness of the wheel blades at entry.

Work done and horse-power developed. In Fig. 682 is shown roughly the actual path, or the path relative to the earth, of a particle of water passing along the guide blade AB and then pursuing the curve CPD through the wheel. At any point P, the resultant velocity v of the water is made up of components w in the direction of the motion of the wheel, and u in a direction parallel to the axis of rotation. The water enters the wheel with a velocity v_i at an angle α_i to the inlet surface, and leaves with a velocity v_o making an angle α_o with the outlet surface. The whirl components at the inlet and outlet respectively will be

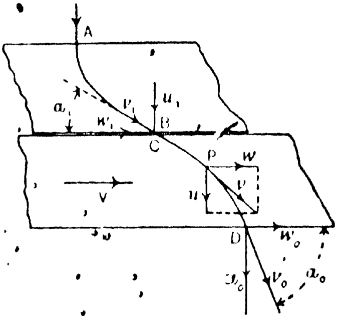


FIG. 682.—Change of angular momentum in an axial flow turbine.

$$w_i = v_i \cos \alpha_i,$$
$$w_o = v_o \cos \alpha_o.$$

Let R = the mean radius of the wheel, in feet.

V = the circumferential velocity of the wheel at its mean radius, in feet per second.

$\omega = \frac{V}{R}$ = the angular velocity of the wheel, in radians per second.

W = the flow of water passing through the wheel per second, in pounds.

Then, considering the momentum of the water in the direction of motion of the wheel, we have

Momentum of one pound of water at the inlet $= 1 \times w_i = v_i \cos \alpha_i$.
" " " " outlet $= 1 \times w_o = v_o \cos \alpha_o$.

Angular momentum per lb. of water at the inlet $= Rv_i \cos \alpha_i$.
" " " " outlet $= Rv_o \cos \alpha_o$.

Change in angular momentum per lb. of water

$$= R(v_i \cos \alpha_i - v_o \cos \alpha_o).$$

Total change in angular momentum per second

$$= WR(v_i \cos \alpha_i - v_o \cos \alpha_o).$$

Couple required to effect this change (p. 431)

$$= \frac{WR(v_i \cos \alpha_i - v_o \cos \alpha_o)}{g} \text{ lb.-ft.}$$

Work done per second by this couple

$$= \frac{WR(v_i \cos \alpha_i - v_o \cos \alpha_o)}{g}$$

$$= \frac{W(v_i \cos \alpha_i - v_o \cos \alpha_o)}{g} V \text{ ft.-lb.}$$

If v_o is parallel to the axis of the wheel, as is required by the condition of no whirl at the outlet, then

$$\text{Work done per second on the wheel} = \frac{WVv_i \cos \alpha_i}{g} \text{ ft.-lb.} \quad \ldots (10)$$

$$\text{H.P. developed} = \frac{WVv_i \cos \alpha_i}{550g} \quad \ldots\ldots\ldots (11)$$

In this result, no allowance has been made for wasted energy while the water is passing through the wheel, nor for the frictional waste in the wheel bearings, etc.

Regulation of the power developed in this type of turbine is effected by means of gates or sluices, which close as many as may be required of the guide orifices.

Fig. 683 shows in outline part of an outward flow impulse turbine suitable for high falls. The water flows from a passage A through nozzles B, and thence through the wheel passages. As before, the water flows in streams along the concave sides of the wheel blades, and the pressure is kept equal to that of the atmosphere by means of side openings.

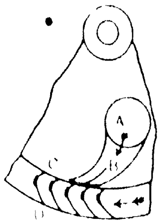


Fig. 683.—Outward flow impulse turbine for high falls.

Reaction turbines. The **Jonval turbine** may be taken as an example of an axial-flow reaction turbine, and is shown in outline in Fig. 684. The water flows from AA through a ring of guide orifices BB, and thence through the wheel passages CC. Both guide passages and wheel passages run full of water, and the turbine wheel may be below the level of the tail water, as shown. Hence the whole head H is available in this type of turbine. For greater accessibility, the turbine may be enclosed in a casing and placed above the level of the tail water; the whole head H will be available still if the discharge from the turbine casing is arranged through a suction tube which has its mouth opening below the level of the tail water. The suction tube must run full bore.

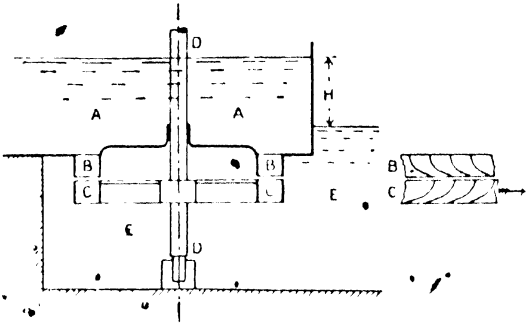


Fig. 684.—Action in a Jonval reaction turbine.

The diagrams showing the velocities of the water and the angles of the guide and wheel blades are given in Figs. 685 (*a*) and (*b*). The constructions are the same as for the axial-flow impulse turbine already described (p. 634), and the diagrams are lettered in the same manner. The guide and wheel blades are shown separately in Fig. 686.

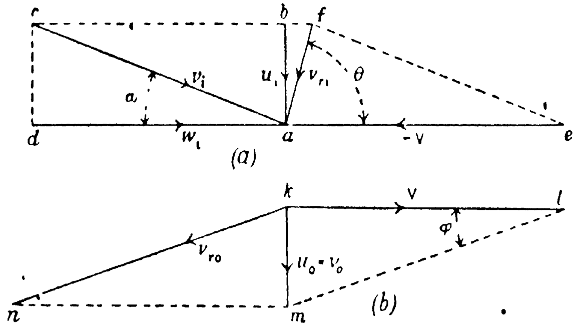


FIG. 685.—Velocity diagrams for an axial flow reaction turbine.

Pressure variation in an axial-flow reaction wheel. In this type the inlet and discharge areas of the wheel are equal; hence the inlet and outlet velocities of flow u_i and u_o in Fig. 685 (*a*) and (*b*) are equal. It will be evident therefore from these diagrams that the relative velocity at the outlet, v_{ro} represented by *kn*, must be considerably larger than the relative velocity at the inlet, v_{ri} represented by *fa*.

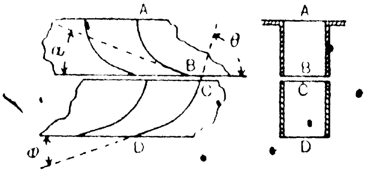


FIG. 686.—Blades of an axial flow reaction turbine.

In Fig. 687, CPD is the wheel blade, and EPF shows approximately the path of the water through the wheel, the discharge velocity $u_o = v_o$ at F being axial. At any point P, the actual velocity v of the water is made up of a constant component V, equal to the circumferential velocity of the wheel and a component v_r tangential to the blade, and increasing from a value v_{ri} at the inlet to v_{ro} at the outlet. In consequence of this increase in velocity, there will be a decrease in pressure from the inlet to the outlet.

Let OX be taken as a datum level, the top of the wheel being at an elevation h_w above OX, and apply Bernoulli's equation, neglecting friction. Let P_i and P_o be the pressures in lb. per square foot at the

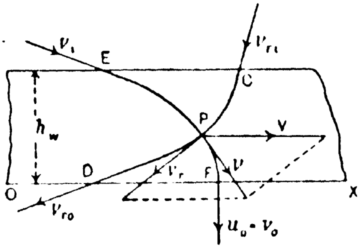


FIG. 687 Pressure variation in an axial flow reaction wheel

inlet and outlet respectively, and let w be the weight of a cubic foot of water. Then

$$\frac{P_i}{w}+\frac{v_{ri}^2}{2g}+h_w=\frac{P_o}{w}+\frac{v_{ro}^2}{2g}.$$

Let h_i and h_o be the pressure heads corresponding to P_i and P_o respectively. Then the above equation may be written:

$$h_i+\frac{v_{ri}^2}{2g}+h_w=h_o+\frac{v_{ro}^2}{2g},$$

or
$$h_i-h_o=\frac{v_{ro}^2-v_{ri}^2}{2g}-h_w \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(h_i-h_o) represents the portion of the total available head H, which is utilised in overcoming the pressure resistance in the wheel; the remainder of H may be used for giving the velocity v_i to the water at the discharge edge of the guide blade. In Fig. 685 (*b*)

$$v_{ro}^2=V^2+u_o^2=V^2+u_i^2.$$

As may be seen from inspection of Fig. 685 (*a*), no great error will be made by assuming that u_i and v_{ri} are equal. Hence,

$$v_{ro}^2=V^2+v_{ri}^2 \text{ nearly.}$$

We may therefore write (1),

$$h_i-h_o=\frac{V^2+v_{ri}^2-v_{ri}^2}{2g}-h_w$$

$$=\frac{V^2}{2g}-h_w \text{ nearly.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence the head available for producing the velocity v_i is given by

$$\text{Velocity head} = \mathrm{H} - (h_i - h_o)$$

$$= \mathrm{H} - \frac{\mathrm{V}^2}{2g} + h_o \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

And
$$v_i = \sqrt{2g\left\{\mathrm{H} - \frac{\mathrm{V}^2}{2g} + h_o\right\}}.$$

Practical values for the velocities are

$$u_o = u_i = 0{\cdot}18\sqrt{2g\mathrm{H}} \text{ feet per second.}$$

$$\mathrm{V} = 0{\cdot}64\sqrt{2g\mathrm{H}} \text{ feet per second.}$$

$$v_i = 0{\cdot}7\sqrt{2g\mathrm{H}} \text{ to } 0{\cdot}8\sqrt{2g\mathrm{H}} \text{ feet per second.}$$

The relations of the velocities and angles may be deduced from Fig. 685 (*a*) and (*b*), giving equations similar to those already found for the Girard turbine (p. 635). If there is no whirl at the outlet, the work done per second and the horse-power may be found in the same manner as for the impulse turbine, and will lead to equations similar to (10) and (11) (p. 638).

$$\text{Work done per second on the wheel} = \frac{\mathrm{WV}w_i}{g} \text{ foot-lb.} \quad \ldots\ldots\ldots\ (4)$$

$$\text{H.P. developed} = \frac{\mathrm{WV}w_i}{550g} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Also, Work done per lb. of water $= \dfrac{\mathrm{V}w_i}{g}$ foot lb.

Energy available per lb. of water = H foot-lb.;

$$\therefore \text{ hydraulic efficiency} = \frac{\mathrm{V}w_i}{g\mathrm{H}} \quad \ldots\ldots\ldots\ldots(6)$$

Inward flow reaction turbine. The **Thomson turbine**, of which an outline diagram is given in Fig. 688, is an example of the inward-flow reaction type. The water enters a large casing furnished with guide blades, which cause the water to whirl as it moves towards the centre of the casing. The water flows through the wheel, and is discharged through central orifices which open sideways. Velocity diagrams are given in Fig. 689, and a part cross section of the wheel is shown also in this illustration. AB is one of the guide blades, and the water is delivered from it at B with a velocity v_i. By means of the parallelogram of velocities B*dcb*, v_i is resolved into a radial component u_i and a tangential component w_i. The guide blade at B is tangential to *c*B, which makes an angle α with *d*B, the latter being tangential to the inlet surface of the wheel at C. The circumferential velocity of the wheel at C is V_i; by making *e*C to represent $-V_i$ and completing the parallelogram *efd*C, the relative velocity v_{r_i} is found and is repre-

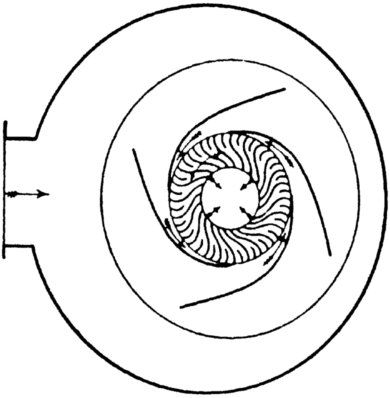


Fig. 688 Thomson inward flow reaction turbine.

sented by *f*C, making an angle θ with C*d* The wheel blade CD is drawn so as to be tangential to *f*C at C.

At the wheel blade outlet D, the water possesses a velocity v_{r_o} relative to the blade, together with a velocity V_o, the latter being the

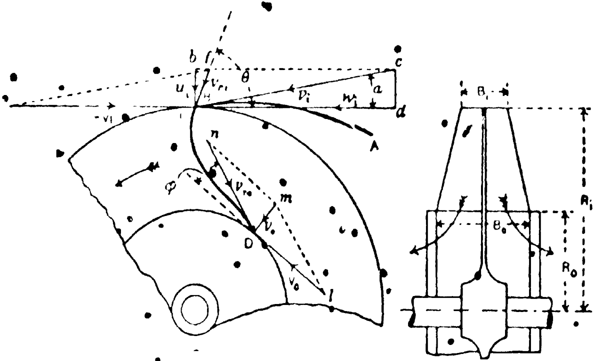


Fig. 689.—Velocity diagrams for an inward-flow reaction wheel.

circumferential velocity of the wheel at D. The resultant of these will give v_o, the absolute velocity of the water discharged from the wheel. To meet the condition of no whirl at the wheel outlet, v_o

should be radial. Draw Dm radial and equal to v_o, make lD equal to V_o and complete the parallelogram $Dlmn$: v_{ro} will be given then by nD, making an angle ϕ with the tangent lD to the wheel at D. The wheel blade at D should be made tangential to nD.

The dimensions of a wheel of this type are determined from the required flow of water and the area of the outlet passages required to accommodate this flow. In Fig. 689, let

R_i and R_o = the inlet and outlet radii respectively, in feet.
B_i and B_o „ „ breadths „ „

Then, since the velocity of the water perpendicular to the exit circumference of the wheel is v_o, we have

$$\text{Flow of water} = v_o \times 2\pi R_o B_o \text{ cubic feet per second.}$$

It is usual to have the same values for the radial components of the inlet and outlet water; hence u_i and v_o will be equal. Now, u_i is the velocity of the water perpendicular to the wheel inlet surface. Hence,

$$\begin{aligned}\text{Flow of water} &= u_i \times 2\pi R_i B_i\\ &= v_o \times 2\pi R_i B_i \text{ cubic feet per second}\end{aligned}$$

As the inlet and outlet flows must be equal, it follows that

$$2\pi R_o B_o = 2\pi R_i B_i,$$

or

$$R_o B_o = R_i B_i.$$

It is advantageous in the Thomson turbine to make R_i double of R_o; hence B_o will be double of B_i In making the above calculations of the flow of water, no allowance has been made for the area abstracted by reason of the thickness of the wheel blades. The wheel shown in section in Fig. 689 is double, and has two side discharge orifices at the centre, each of radius R_o; assuming that the velocity of the water through these is v_o, we have

$$\text{Flow of water} = v_o \times 2\pi R_o^2 \text{ cubic feet per second,}$$

a result which enables R_o, and hence the other wheel dimensions, to be calculated when v_o is known.

Variation of pressure in a radial-flow reaction wheel. It will be noted from the diagrams of velocity in Fig. 689 that the outlet relative velocity v_{ro} is considerably larger than the inlet relative velocity v_{ri}. Hence, as explained on p. 640, there will be a fall in pressure as the water passes through the wheel, the fall being given by

$$\text{Fall in pressure due to the change in relative velocity} = \frac{v_{ro}^2 - v_{ri}^2}{2g} \text{ feet.}$$

There is also centrifugal action on the water which is passing through the wheel, the effect being also to cause the inlet pressure to be greater than the outlet pressure.

In Fig. 690, the actual path of the water through the wheel is BPE, and CD is the wheel blade. At any point P, the absolute velocity v of the water may be resolved into components v_r tangential to the blade, and V in the direction of motion of the wheel. The magnitude of V will be proportional to the wheel radius at P, and the centrifugal effects owing to V may be examined by reference to Fig. 691 (*a*). ABCD is a closed tube full of water and rotates with angular velocity ω about a vertical axis XY. The cross section of the water is taken as one square foot in area, and the radii are measured in feet. At any radius r, the pressure on the section EF may be found from the general equation:

$$P = \frac{M\omega^2 R}{g} \quad \text{(p. 425).}$$

In the present case,

M = mass of water in AEFD

$= 1 \times r \times w$ pounds.

$R = 0{\cdot}5r$.

Hence, $$P = \frac{rw\omega^2 \frac{r}{2}}{g} = \frac{w\omega^2 r^2}{2g} \text{ lb. per square foot.}$$

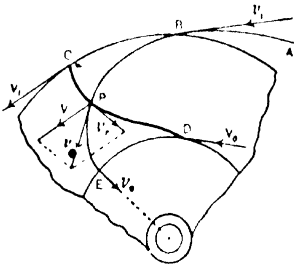


FIG. 690.—Pressure variation in a radial flow reaction wheel.

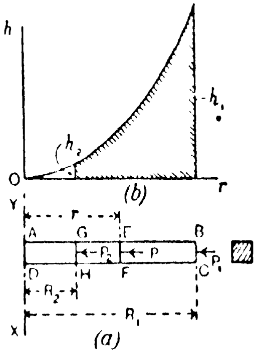


FIG. 691. Pressure variation in a whirling tube.

Now, the circumferential velocity of the tube at radius r is given by

$$v = \omega r;$$

$$\therefore \; P = \frac{wv^2}{2g} \text{ lb. per square foot.}$$

Let h be the pressure head corresponding to P. Then

$$h = \frac{P}{w} = \frac{v^2}{2g} \text{ feet.}$$

At sections BC and GH, where the circumferential velocities are v_1 and v_2 respectively, the pressure heads will be

$$h_1 = \frac{v_1^2}{2g} \text{ feet}, \quad h_2 = \frac{v_2^2}{2g} \text{ feet.}$$

The difference in pressure heads at BC and GH will be

$$h_1 - h_2 = \frac{v_1^2 - v_2^2}{2g} \text{ feet.}$$

In Fig. 691 (b), h and r have been plotted; it is evident that, as h varies as v^2, and v varies as r, the shape of the curve is parabolic.

Application of this result to the radial flow turbine (Fig. 690) gives for the difference in pressure caused by centrifugal action

$$h_1' - h_2' = \frac{V_i^2 - V_o^2}{2g} \text{ feet}$$

Taking account of the change in pressure head due to the change in relative velocity, we have for the total difference in pressure heads at inlet and outlet

$$h_1 - h_2 = \frac{V_i^2 - V_o^2}{2g} + \frac{v_{ro}^2 - v_{ri}^2}{2g}$$

This expression gives the portion of the total available head H which is required in order to overcome the pressure in the wheel; the remainder is available for giving the velocity v_i to the water leaving the guide blade. The result is applicable to both inward and outward radial-flow turbines.

It will be noticed that any increase in the speed of rotation of the wheel will cause $\left(\frac{V_i^2 - V_o^2}{2g}\right)$ to increase, and hence the difference in the pressures at the inlet and the outlet also will increase. In inward-flow turbines, the effect of this increase in pressure will be to diminish the flow of water through the wheel; the water is held back, as it were, by the increase in centrifugal force. Now, a diminished flow of water results in less power being developed, and thus will tend to lower the speed of the wheel; hence inward-flow turbines are to a certain extent self-governing. In the Thomson wheel, advantage is taken of the self-regulating centrifugal action by making the outer diameter of the wheel double of the inner diameter.

In outward-flow turbines, the centrifugal action causes an increase in flow if the speed of rotation of the wheel becomes greater, and thus is entirely opposed to self-regulation. For this reason, outward-flow turbines have the wheel diameters at outlet and inlet more nearly equal; the outer diameter in practice may be from 1·2 to 1·25 times the inner diameter. Practical values of the velocities may be as follows:

$$u_i = u_o = 0·125\sqrt{2gH} \text{ feet per second}$$

$$V_i = 0·66\sqrt{2gH} \text{ feet per second.}$$

$$v_i = 0·73\sqrt{2gH} \text{ feet per second.}$$

Efficiencies in practice of from 75 to 80 per cent. have been obtained with Thomson turbines.

Power developed in radial-flow reaction wheels. Let w_i and w_o be the inlet and outlet velocities of whirl respectively, and consider one pound of water passing through the wheel.

Angular momentum of inlet water $= w_iR_i$.

" " outlet " $= w_oR_o$.

Change in angular momentum per pound of water $= w_iR_i - w_oR_o$.

Let W be the flow of water in pounds per second. Then

Total change in angular momentum, per second $= W(w_iR_i - w_oR_o)$.

$$\text{Couple acting on the wheel} = \frac{W(w_iR_i - w_oR_o)}{g} \text{ lb.-feet.}$$

$$\text{Work done by this couple} = \frac{W(w_iR_i - w_oR_o)}{g}\omega \text{ ft.-lb. per sec.}$$

$$\therefore \text{H.P. developed} = \frac{W(w_iR_i - w_oR_o)\omega}{550g}.$$

If there is no whirl at the outlet, w_o will be zero, and ω may be written $\frac{V_i}{R_i}$. Hence,

$$\text{H.P. developed} = \frac{Ww_iR_i}{550g} \cdot \frac{V_i}{R_i}$$

$$= \frac{Ww_iV_i}{550g}.$$

Also, Work done per pound of water $= \frac{w_iV_i}{g}$ ft.-lb. per sec.

If H is the total fall available, then in each pound of water supplied there is energy represented by H foot-lb. Hence,

$$\text{Hydraulic efficiency} = \frac{w_1 V_1}{g\text{H}}.$$

These results are applicable to both inward and outward-flow turbines.

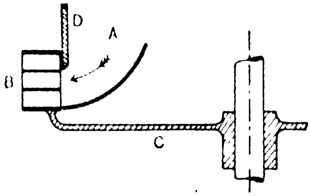


Fig. 692.—Section of an outward-flow reaction wheel.

Outward-flow reaction turbine. A part section of an outward-flow turbine is given in Fig. 692. The water flows through a passage A furnished with guide blades, and passes through the wheel C, which has blades at B. In this wheel, the passages at B are divided into three compartments by means of horizontal partitions, the effect being to produce three wheels. Regulation of the power is effected by means of a cylindrical gate or sluice, shown in section at D, and capable of vertical movement. By adjusting D, one, two or all of the wheel compartments may be used, thus giving considerable variation in the power.

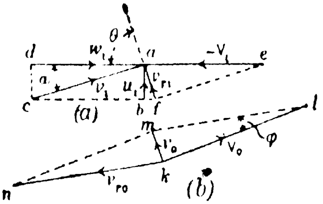


Fig. 693.—Velocity diagrams for an outward-flow reaction wheel.

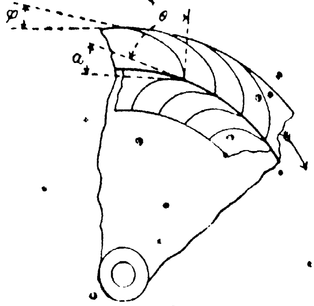


Fig. 694.—Blades for an outward-flow reaction turbine.

The velocity diagrams are given in Fig. 693 (*a*) and (*b*), and the guides and wheel blades are shown separately in Fig. 694. These diagrams are lettered to correspond with those already given, and will be understood readily. The following practical values may be assumed:

$$u_1 = 0\cdot25\sqrt{2gH} \text{ feet per second.}$$

$$V_1 \quad 0\cdot55\sqrt{2gH} \text{ feet per second.}$$

$$v_1 = 0\cdot6\sqrt{2gH} \text{ feet per second.}$$

Centrifugal pumps. Water may be raised from a lower to a higher level by means of a centrifugal pump, in which the water flows through a revolving wheel driven from some source of power; the function of the wheel is to impart additional energy to the water, and this is converted into potential energy by allowing the water to flow up a discharge pipe into an elevated reservoir. The general arrangement will be understood by reference to Fig. 695, in which the wheel is situated at B and takes its supply of water from A. The water is discharged at C into a pipe which opens at D into a tank E. A back-pressure valve at A prevents the water flowing back into the lower tank when the wheel is at rest. The pump may be situated below the level of the supply water, or above this level, as shown in Fig. 695; in the latter case, the pressure of the atmosphere causes a flow up the pipe AB. Before any centrifugal pump will start discharging, it must be charged fully with water.

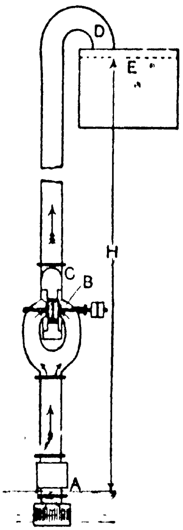


FIG. 695.—Arrangement of a centrifugal pump

In Fig. 696 is shown a centrifugal pump in more detail; water enters at A, and is led to central orifices situated on both sides of the wheel B. The wheel is furnished with curved blades, and is driven in the contra-clockwise direction. The water is discharged at the outer circumference of the wheel into a chamber C; the discharge pipe is connected to this chamber at D. The action is very similar to that of a reversed inward-flow reaction turbine, and the theory is also similar.

Velocity diagrams for a centrifugal pump. Velocity diagrams are given in Fig. 697. At the inlet edge A of the wheel blade AB, the absolute velocity v_1 of the inlet water may be assumed to be radial, and hence is equal to u_i. The circumferential velocity of the wheel at A is V_i; if cA be made equal to $-V_i$ and tangential to the circumference at A, and aA drawn radial and equal to v_i, then the diagonal bA will represent the relative velocity of the water and the wheel blade at the inlet. The tangent to the blade at A should coincide with bA in order that the water may enter the wheel without shock.

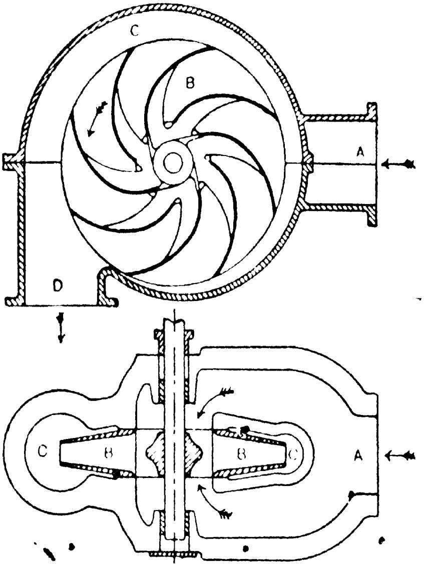


FIG. 696.—Sections of a centrifugal pump.

Let Q = flow of water through the wheel, in cubic feet per second.

R_i and R_o = the inlet and outlet radii respectively of the wheel, in feet.

B_i and B_o = the inlet and outlet widths respectively of the wheel, in feet.

Then $Q = 2\pi R_i B_i u_i$

$= 2\pi R_i B_i v_i$ for radial velocity of inlet. .. ....(1)

The wheel is made less in breadth at the outlet surface usually, and as the same quantity Q is discharged, we have

$$Q = 2\pi R_o B_o u_o$$

Hence, $$2\pi R_i B_i u_i = 2\pi R_o B_o u_o,$$

or $$R_i B_i u_i = R_o B_o u_o.$$

It is usual to preserve the velocity of flow constant, *i.e.*

$$u_i = u_o;$$

$$\therefore \; R_i B_i = R_o B_o. \qquad (2)$$

Also, $$\frac{V_o}{V_i} = \frac{R_o}{R_i};$$

$$\therefore \; V_o = \frac{R_o}{R_i} V_i. \qquad (3)$$

The velocity diagram may be constructed for the outlet B (Fig. 697) as follows. Make B*d* equal to V_o and tangential to the wheel circumference; draw *cf* parallel to B*d* and at a distance $u_o = u_i$, represented by B*g* from it.

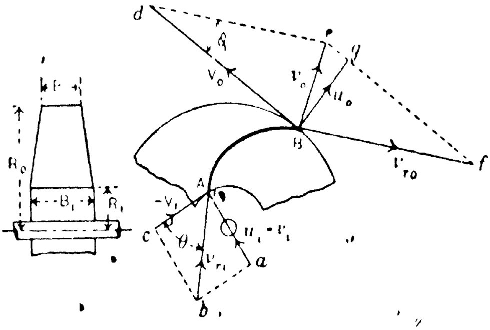


FIG. 697.—Velocity diagrams for a centrifugal pump.

A value is then chosen for φ, the angle which the tangent to the blade at B makes with the outer circumference of the wheel, and *de* is drawn at the angle φ to *d*B. Completion of the parallelogram B*def* will give the relative velocity v_{ro}, represented by B*f*, and the absolute velocity v_o of the water leaving the wheel, represented by B*e*. The wheel blade at B is drawn tangentially to B*f*. v_o has a radial component $u_o = $ B*g* and a whirl component w_o, represented by *ge*.

Work done on the wheel.

Let W = the flow of water, in pounds per second.

„ ω = the angular velocity of the wheel, in radians per second.

Then, as there is no whirl, and therefore no angular momentum in the inlet water, we have

Change in angular momentum per second $= Ww_oR_o$.

Couple which must be applied to the wheel $= \frac{Ww_oR_o}{g}$ lb.-feet.

Work done per second by this couple $= \frac{Ww_oR_o}{g}\omega$ foot-lb.

Also,
$$\omega = \frac{V_o}{R_o};$$

$$\therefore \text{ work done per second by the couple } = \frac{Ww_oV_o}{g} \text{ foot lb. } \quad \text{..} \quad (4)$$

$$\text{H.P. required} = \frac{Ww_oV_o}{550g}. \quad \text{.. ......} \quad (5)$$

It will be noticed that these results take no account of any sources of waste.

If H_m represents the maximum height through which the water could be raised, neglecting wasted energy, then

$$\text{Work done per second} = WH_m = \frac{Ww_oV_o}{g} \text{ (from 4)};$$

$$\therefore \; H_m = \frac{w_oV_o}{g} \text{ feet. } \quad \text{.........} (6)$$

Let H feet be the actual height through which the water is raised (Fig. 695), and let v_D be the velocity in the discharge pipe. The discharge energy at D will be

$$\text{Energy of discharge} = H + \frac{v_D^2}{2g} \text{ foot-lb. per pound of water.}$$

It will be evident that the kinetic energy $\frac{v_D^2}{2g}$ at discharge is entirely wasted in surging and eddies in the upper tank. Other sources of wasted energy external to the pump casing are owing to frictional resistances in the suction and discharge pipes. Let H_F feet represent the head equivalent to the frictional waste of energy in these pipes. Then the **gross lift** of the pump will be defined as

$$\text{Gross lift} = H_G = H + H_F + \frac{v_D^2}{2g} \text{ feet. } \quad \text{.........} (7)$$

The total disposal of energy outside the pump casing will be represented by

$$\text{Energy disposed of outside the pump casing} = WH_G \text{ foot-lb. per sec.} (8)$$

The ratio of this quantity to the work done per second in the wheel, given by (4), is the **hydraulic efficiency of the pump.** Hence, for radial flow at the wheel inlet, we have

$$\text{Hydraulic efficiency} = E_H = WH_a \div \frac{Ww_0V_0}{g} = \frac{H_ag}{w_0V_0}. \quad \ldots (9)$$

Within the wheel casing, in addition to frictional waste in the passages, there is waste from shock due to the velocity v_0 at the wheel exit being greater than the velocity v_D in the discharge pipe. It is rarely the case that any effective means are taken for ensuring that v_0 shall diminish to v_D, in such a manner as to produce an effective increase in the pressure head. More usually the waste is complete, and the pressure head at the connection of the discharge pipe to the wheel case is equal to that at the exit circumference of the wheel. As has been noted already, the kinetic energy of the water in the discharge pipe is wasted entirely in the upper tank; it therefore follows that the waste, in addition to that due to friction, will be the whole kinetic energy of the water leaving the wheel exit, viz. $\frac{v_0^2}{2g}$ foot-lb. per pound, and the maximum height of lift given in (6) will be reduced to this extent.

Again, a centrifugal pump, having been designed for a constant speed of rotation and for a given flow of water, will have a definite angle θ (Fig. 697) for the wheel blade at the entrance A. If any change be made in the speed of the wheel or in the velocity of flow, the fixed angle θ will not suit the altered conditions, and there will be waste of energy by reason of shock at A. Further, the assumption of radial velocity of the entering water is approximate only, as there are no guides in the central orifices of the wheel to ensure constant direction in the flow of water.

In obtaining the actual efficiency E_A of the whole arrangement, the energy wasted in overcoming the frictional resistances of the wheel bearings also must be considered. Taking, as before, a flow of W pounds per second raised through an actual height H feet, we have

$$\text{Useful work done} = WH \text{ foot-lb. per sec.}$$

The energy which must be supplied to the pump shaft will be given by

$$\text{Energy supplied to pump} = \frac{WH}{E_A} \text{ foot-lb. per sec.} \quad \ldots\ldots (10)$$

$$\text{Horse-power required to drive the pump} = \frac{WH}{550E_A}. \quad \ldots\ldots (11)$$

Variation in pressure in the wheel. The change in pressure head of the water passing through the wheel may be deduced in the same manner as for the inward-flow turbine discussed on p. 644. Referring to Fig. 697, it will be noted that the centrifugal action is to cause the pressure to increase in flowing from A to B; v_{ro} being greater than v_{ri}, there will be a decrease in pressure owing to the change in relative velocity. Hence, the total difference in pressure heads at A and B respectively will be

$$h_1 - h_2 = \left(\frac{V_o^2 - V_i^2}{2g}\right) - \left(\frac{v_{ro}^2 - v_{ri}^2}{2g}\right) \text{ feet. } \ldots\ldots\ldots\ldots(12)$$

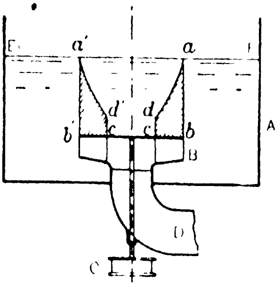


FIG. 698.—Pressure variation in a revolving wheel; no flow of water.

The changes of pressure which occur in the whirling mass of water passing through the wheel and casing of a centrifugal pump may be studied in more detail. In Fig. 698 is shown a tank A containing a pump wheel B, which may be rotated by means of a shaft and a pulley at C. Water may flow into the wheel through the pipe D. Suppose that the wheel is revolving at a given speed, sufficiently large to cause the water to rise to the level EF, and to remain constant at this level. No more water will flow into the wheel; the water contained in the wheel passages will whirl with the wheel, and the water in the tank will be at rest, save for the frictional drag given to it by the revolving wheel. As there is no flow, the changes in pressure head inside the wheel will be owing entirely to centrifugal action, and these will follow the law already explained on p. 645. The pressure heads are shown by the shaded diagrams *abcd* and *a'b'c'd'*; if the water were whirled in a revolving open cup instead of in the wheel, the form of the vertical section of its surface would be that of the curves *ad* and *a'd'*, and the state of motion is called a **forced vortex**. It will be noted from what has been said on p. 645 that the condition for a forced vortex is that the whole of the whirling water possesses the same angular velocity.

The conditions illustrated in Fig. 698 are realised in a centrifugal pump often if the speed of the wheel be insufficient to cause the water to reach the level of the mouth of the discharge pipe. In such a case, there is, of course, no flow, and hence no work is done other than that against frictional resistances.

Free vortex. A free vortex may be produced easily by filling with water a circular vessel having a central plug, stirring the water and then drawing the plug. The shape assumed by the water will be found to resemble that shown in Fig. 699, which has been plotted from calculated results. In such a vortex, the condition is that although interchanges of kinetic, pressure, and potential energies may take place freely throughout the mass of water, the total energy per pound of water remains constant.

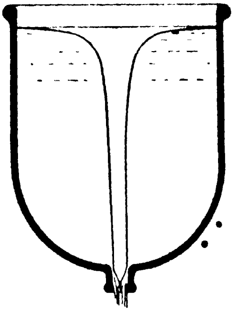


Fig. 699.—A free vortex

In Fig. 700 (*a*), AB is a portion of the surface of a free vortex. At any point P, at a distance y feet from the free surface level OX, and at radius x feet from the axis of the vortex, the resultant R of the centrifugal force F and the weight W of a particle of water must cut the surface of the vortex normally. This is owing to the fact that the surface at P is exposed to the pressure of the atmosphere, which is a force acting normally on the surface.

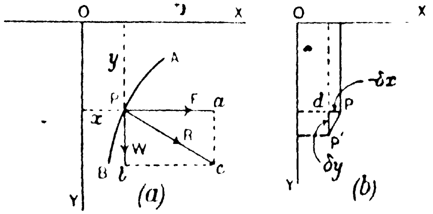


Fig. 700.—Forces acting in a free vortex

Consider a mass of m pounds of water at P. The centrifugal force F will be $\frac{mv^2}{x}$ poundals, where v is the velocity of whirl in feet per second; the weight W will be mg poundals. Hence,

$$\frac{W}{F} = \frac{mg}{\frac{mv^2}{x}} = \frac{gx}{v^2}.$$

Now, owing to the change of potential energy into kinetic energy caused by the water descending through a height y at constant atmospheric pressure, we have

$$v^2 = 2gy;$$

$$\therefore \frac{W}{F} = \frac{gx}{2gy} = \frac{x}{2y}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In Fig. 700 (*b*) another point P′ has been taken very close to P; y is increased now by a small amount δy, represented by P′*d*, and x is diminished by a corresponding small amount $(-\delta x)$, represented by P*d*. The triangles *a*P*c* in Fig. 700 (*a*) and *d*P′P in Fig. 700 (*b*) are similar. Hence,

$$\frac{W}{F} = \frac{ac}{aP} = \frac{Pd}{P'd};$$

and, from (1),

$$\frac{W}{F} = \frac{x}{2y} = \frac{\delta x}{\delta y};$$

or

$$\frac{\delta y}{2y} = -\frac{\delta x}{x}.$$

Let δx and δy become very small, and integrate both sides, giving

$$\tfrac{1}{2}\int \frac{dy}{y} = -\int \frac{dx}{x},$$

$$\tfrac{1}{2} \log_e y = -\log_e x + c,$$

where c is a constant to be determined by the given conditions. The result may be written

$$y^{\frac{1}{2}} = \frac{c}{x},$$

or

$$y = \frac{a}{x^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where a is an arbitrary constant.

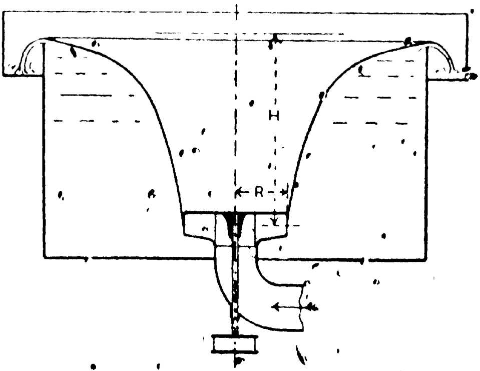


Fig. 701.—A free vortex formed outside a revolving wheel.

In Fig. 701, water is discharged from a rotating wheel into a circular tank, and allowed to form a free vortex. If H is the height in feet of the free surface level above the wheel centre, R the radius of the wheel in feet and V its rim velocity in feet per second, then

$$V^2 = 2gH,$$

or

$$H = \frac{V^2}{2g}.$$

Hence, from (2),

$$H = \frac{a}{R^2} = \frac{V^2}{2g},$$

or

$$a = \frac{V^2R^2}{2g}.$$

Hence, equation (2) becomes, in this case,

$$y = \frac{V^2R^2}{2gr^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This is the equation used in plotting the vortices shown in Figs. 699 and 701.

In Fig. 702 is shown a wheel A surrounded by a chamber B, which is open to an ordinary volute chamber C. The water whirled out of the wheel moves through the chamber B in spiral stream lines, and the changes in pressure follow the free vortex law approximately. This is shown in the pressure diagrams by the curves *cd* and *c'd'*.

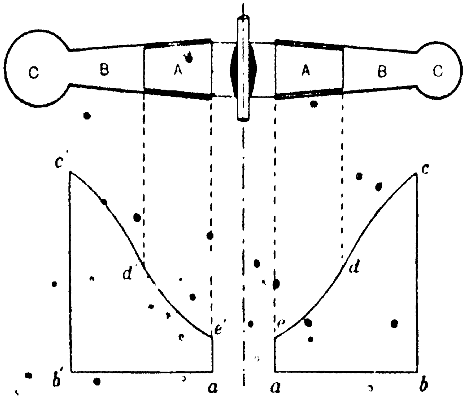


FIG. 702.—Pressure variation in a wheel having a whirlpool chamber.

The water in the wheel itself follows the forced vortex law approximately, shown by the curves *de* and *d′e′*. The chamber B, called the whirlpool chamber, was suggested by Prof. James Thomson; it has not been used to any great extent in practice on account of the very large dimensions required if the chamber is to be effective in permitting the changes from kinetic to pressure energy to be effected gradually.[*]

High lift centrifugal pumps. The lift which can be obtained by use of an ordinary centrifugal pump is limited by the speed at which the wheel can be run without exceeding safe limits of stress in the material. By the use of several wheels in series, so that the water leaving one wheel is delivered to the next wheel of the series, almost any lift can be obtained. Fig. 703 shows such a pump constructed by Messrs. Mather & Platt, for the purpose of mine drainage.[†] There are seven wheels in series, the first being that on the right-hand end of the shaft. This pump is capable of delivering 2500 gallons per minute to a height of 2000 feet when running at 1450 revolutions per minute, and absorbs over 1900 horse-power at the spindle.

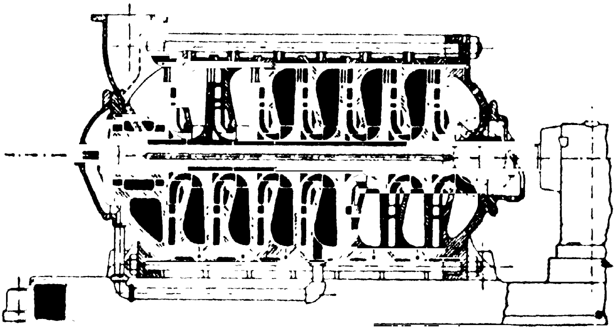


FIG. 703.—Multiple wheel centrifugal pump for high lifts.

Students desirous of further information in hydraulics are referred to the following standard works:

(*a*) *Hydraulics*, Prof. A. H. Gibson (Constable); (*b*) *Hydraulics*, Prof. W. C. Unwin (Black); (*c*) *Hydraulics*, F. C. Lea (Arnold).

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[*] See "Experiments on the Efficiency of Centrifugal Pumps," by Dr. Stanton. *Proc. Inst. Mech. Eng.*, 1903, p. 726.

[†] "The Evolution of the Turbine Pump"; Hopkinson & Chorlton. *Proc. Inst. Mech. Eng.*, January 1912.

EXERCISES ON CHAPTER XXIV.

1. A jet of water having a sectional area of 0·5 square inch and moving with a velocity of 40 feet per second impinges on a fixed flat plate. Find the pressure on the plate when the jet makes angles of 0°, 30°, 45°, 60° and 90° with it. Plot a diagram showing the relation of pressure and angles.

2. The jet of water in Question 1 slides tangentially on to a fixed vane. Find the pressures on the following vanes, given that the tangents to the vanes at the leaving edges make angles with the jet of 0°, 30°, 60°, 90°, 120°, 150°, 180°. Plot a diagram showing the relation of pressures and angles.

3. A jet of water issues horizontally from a trumpet orifice one inch diameter under a head of 4 feet. It impinges normally on a flat plate; calculate the pressure on the plate (*a*) when the plate is receding from the orifice with a velocity of 2 feet per second, (*b*) when the plate is approaching the orifice with a velocity of 2 feet per second.

4. In Question 3, calculate the work done in each case. Under what conditions of plate velocity would there be zero work done? Sketch a diagram showing approximately the work done between the limiting velocities of plate (*a*) equal to and of the same sense as the velocity of the jet, (*b*) equal to and of opposite sense to that of the jet.

5. An overshot water wheel is supplied with 2 cubic feet of water per second. If the fall available is 30 feet and if the efficiency is 60 per cent., what horse-power will be developed?

6. In an undershot water wheel, the water is supplied under a head of 3 feet and passes through a sluice 4 feet in width and 6 inches in height. Calculate the flow through the sluice, taking the coefficient of discharge to be 0·6. Hence calculate the horse-power developed by the wheel if the efficiency is 40 per cent.

7. In a Poncelet water wheel, the velocity of the jet is 25 feet per second and the velocity of the rim of the wheel is 15 feet per second. If the direction of the jet makes an angle of 15° with the tangent to the wheel circumference at the entrance, what angle should the tip of the wheel vane make with the same tangent?

8. In a Poncelet water wheel, the velocity and direction of the jet are the same as those given in Question 7. The tip of the wheel vane makes an angle of 28° with the tangent to the wheel circumference. Find the circumferential velocity of the wheel.

9. A wheel 3 feet in diameter makes 250 revolutions per minute and has radial vanes. Water flows through the wheel from the outside to the centre, and has a radial component of 6 feet per second at the inlet. If shock is to be avoided, find and show in a diagram the direction angle of the entering water.

10. A Pelton wheel is 30 inches in mean diameter and is supplied with water under a head of 1500 feet. If the jet is 1 inch in diameter, find the best speed in revolutions per minute of the wheel and also the horse-power developed if the efficiency is 80 per cent.

11. In a Girard axial-flow impulse turbine having a horizontal wheel, the internal and external diameters of the wheel at the inlet are 33 and 39 inches respectively, the mean diameter being 36 inches. The total available fall is 20 feet and the delivery edges of the guide vanes are 18 inches above the tail race.

(*a*) Find the velocity of the water leaving the delivery edges of the guide vanes, taking the coefficient of velocity as 0·95.

(*b*) The axial component of the velocity (*a*) is to be 0·45 of the result you have found in (*a*). Find the angle made by the tips of the guide blades with the horizontal at the mean diameter.

(*c*) Calculate the flow of water through the wheel in lb. per second, neglecting the sectional area abstracted by the guide and wheel vanes.

(*d*) What is the total energy per second available in the fall? If the efficiency of the wheel is 75 per cent, what is its probable horse-power?

(*e*) The mean circumferential velocity of the wheel is to be 0·5 of the velocity calculated in (*a*), and the water leaves the wheel with axial velocity only. The relative velocities at inlet and outlet are equal. Find the angles to the horizontal of the vanes at the mean wheel circumference at inlet and outlet respectively. Draw the curve of the vane. Find also the speed of the wheel in revolutions per minute.

(*f*) Find the absolute velocity of the leaving water, and hence calculate the hydraulic efficiency.

(*g*) From (*b*), find the velocity of whirl of the entering water. Assuming no whirl at the outlet, calculate the couple in lb.-feet acting on the wheel, the work done by this couple in foot lb. per second and the horse-power developed by it.

12. In an axial-flow reaction turbine having a horizontal wheel, the mean circumference velocity V of the wheel vane is 16 feet per second, the inlet velocity v_i of the water is 18 feet per second and the axial components u_i and u_o of the velocity of the water at inlet and outlet are each equal to 4·5 feet per second. The water is discharged from the wheel without whirl. Find the angle of the guide blade; also the angles of the wheel vane at inlet and outlet. Find the relative velocities v_{ri} and v_{ro} at inlet and outlet, and calculate the change in pressure head in the wheel passages if the depth of the wheel is 8 inches.

13. If a plate is inserted in a stream of fluid and held at right angles to the direction of the stream, the total pressure on it is equal to the momentum per second which would pass through its area if the plate were absent, multiplied by a factor which, for the present purpose, may be taken to be 0·65. Find the pressure upon a plate 4 feet square due to a steady wind of 20 miles per hour. (One cubic foot of air weighs 0·08 lb.)
(B E.)

14. The radial vanes of an under-shot water wheel are acted upon by water moving at a velocity of 45 feet per second. If the quantity of water which reaches the wheel per second is 2 cubic feet, and if the vanes of the wheel are moving at a velocity of 22·5 feet per second, find (*a*) the pressure of the water on the vanes, (*b*) the theoretical efficiency of the wheel and (*c*) the horse-power the wheel is capable of developing.

15. An experiment with a small Pelton water wheel gave results shown in the annexed table:

Mean revs. per min	Net load on brake wheel, lb.	Water passing through the turbine, in lb. per sec.	Speed of jet, in ft. per sec., v	Peripheral speed of wheel vanes, in feet per second, V	Ratio, $\frac{v}{V}$	Total kinetic energy of jet, foot lb. per sec., E	Energy taken up by brake wheel, foot lb. per sec., e	Efficiency of wheel, per cent, $\frac{e}{E}\times 100$
1450	0	3·87						
770	12·5	4·20						
660	14·1	4·10						
620	15·1	4·10						
395	19·2	4·12						
121	21·9	4·17						

The cross-sectional area of the nozzle was 0·000764 square feet. The mean diameter of the wheel (centre to centre of buckets) was 10·7 inches. The mean diameter of the brake wheel was 7·25 inches. Fill in the columns left blank in the table. Plot a curve showing the variation of efficiency of wheel with variation of ratio $\frac{v}{V}$. According to simple theory, that is, neglecting all losses, what is the value of $\frac{v}{V}$ which should give the maximum efficiency, and what value would this maximum efficiency be? (B.E.)

16. An axial-flow impulse turbine is to be designed. Determine, from the data given below, the angles of tips of the guide and of the moving blades:

(1) Original head of water, 169 feet;
(2) Head wasted in friction in the guide blades, 1 foot;
(3) Head wasted in the tail race, 8·5 feet.

The horizontal velocity of the wheel blade is to be taken as half the horizontal component of the velocity of the water as it enters the wheel. (B.E.)

17. An inward-flow turbine, when running steadily, uses 40 cubic feet of water per second. The water enters the wheel with an absolute velocity of 45 feet per second, and its direction makes an angle of 23 degrees with the direction of motion of the wheel rim; it leaves the wheel with an absolute velocity of 8 feet per second, and its direction makes an angle of 112 degrees with the direction of motion of the wheel rim. The internal and external radii of the wheel are 1·6 and 2·8 feet respectively. What horse-power is the turbine receiving? State the principle on which you calculate. (B.E.)

18. An outward-flow turbine wheel has an internal diameter of 5·35 feet, an external diameter of 6·50 feet, and it makes 250 revolutions per minute. The wheel has 34 vanes, which may be taken as 0·75 of an inch thick at the inlet, and 1·25 inches thick at the outlet. The head available above the centre of the wheel is 145 feet, and the wheel exhausts into the atmosphere. The effective width of the wheel face at inlet and outlet is 12 inches, the quantity of water supplied per second is 250 cubic feet.

Determine the angles of the tips of the vanes at inlet and outlet, so that the water shall leave the wheel radially. You may neglect all friction losses. (B.E.)

19. The wheel of a centrifugal pump is delivering very little water; show in a curve how the pressure varies from inside the wheel to a point some distance outside the rim. Show this when there is a considerable delivery of water. Draw the shape of the vane of the wheel. (B.E.)

20. The radial speed of the water in the wheel of a centrifugal pump is 6 feet per second; the vanes are directed backwards at an angle of 35 degrees to the rim; what is the real velocity of the water relatively to the vanes? What is the component of this which is tangential to the rim? When the water has left the wheel, what is its velocity in the tangential direction if the rim of the wheel moves at 20 feet per second? (B.E.)

21. The rim of a turbine is travelling at 60 feet per second; 200 lb. of fluid enter the wheel per second with a velocity of 70 feet per second in the tangential direction, leaving it at the same radius with no velocity in the direction of the wheel's motion. Find the momentum lost per second by the fluid, and calculate the horse power given to the turbine. (I.C.E.)

22. In an inward-flow turbine the speed of the wheel at inlet is $0{\cdot}65v$; the velocity of flow, assumed constant throughout, is $0{\cdot}15v$; the outer radius is twice the inner radius, and the flow at outlet is radial. $v=\sqrt{2gh}$, where h is the head above the turbine centre. Find, in terms of h, the centrifugal head in the turbine, the velocity at inlet and the hydraulic efficiency. (L.U.)

23. A Pelton wheel runs at 900 revs. per min., its diameter being d feet. 750 gallons of water per minute are supplied from a hydraulic main at 250 lb. per sq. in. at the nozzle. Find the actual horse-power, allowing a reasonable efficiency for the wheel. Find also a suitable value for d. Sketch the form of bucket and shape of nozzle you would use, showing how the sectional area of the jet may be varied. (L.U.)

CHAPTER XXV.

HYDRAULIC EXPERIMENTS.

Hydraulic experiments. Most laboratories equipped for applied mechanics now possess apparatus suitable for carrying out experiments in hydraulics. Generally speaking, these are on a small scale; to carry out experimental hydraulics on a practical scale requires very costly apparatus occupying much space, owing to the nature of the experiments, it is best to have a separate laboratory entirely if completeness is aimed at. In this section, the experiments described are on a small scale usually, and are such as may find a place in any mechanical laboratory with great benefit to the student. It must be borne in mind however, that the experimental results obtained by such apparatus cannot agree well with those given by larger apparatus. For example, the results for the flow through a drawn copper tube 0·5 inch in diameter could scarcely be compared with those for a cast-iron or steel water main 48 inches in diameter. This, however, does not detract seriously from the educational value of the experiments, and the student has the satisfaction of having the whole apparatus under his own control without the necessity for a multitude of assistants.

Apparatus for some general hydraulic experiments. In Fig. 704 is shown apparatus in a compact form for enabling experiments to be made on the flow of water through orifices and over gauge notches, and also on the power and efficiency of a small Pelton wheel. A is a closed vessel supplied with water from a pipe B having a regulating valve C. The pipe B inside the vessel has its lower end plugged, and is perforated with a number of small holes. This portion is surrounded by a gauze cage, which has the effect of stilling most of the eddies in the entering water. In cases where the head in the ordinary water main is insufficient for experimental purposes, a pump may be used for delivering water under pressure into A. In the author's laboratory, the pressure of supply may be 100 lb. per square inch. Any head up to this limit may be obtained in the vessel A by regulating the valve C, and this head will be maintained steadily by the action of the air cushion in the upper part of A. If very low heads are required, an air valve D is opened, thus reducing the pressure on the surface of the water in A to that of the atmosphere. The head is measured then by means of a glass tube gauge EF, which is connected to A near the top and bottom. For higher pressures, the pressure gauge G is used; this is graduated in feet of water.

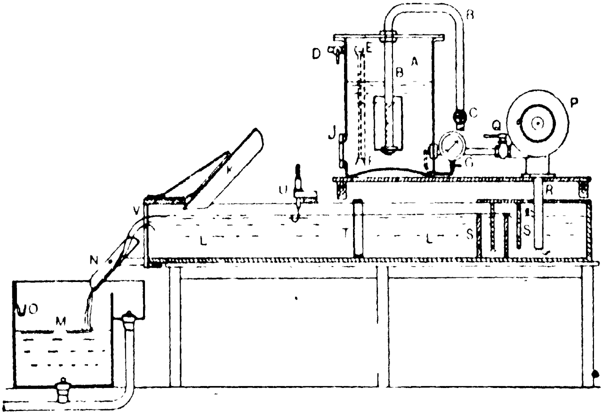

FIG. 704.—General arrangement of hydraulic apparatus.

For experiments on the flow through orifices, a flange J is secured to the tank A, and brass plates having various orifices may be bolted to it. The discharged jet is caught by a baffle plate K, and is directed by it into the trough L, and thence into the measuring tank M. The latter tank has a small compartment at one side, and the water may be directed into this or into the larger compartment as required, by means of a swinging shoot N. Each compartment has a plug and waste pipe, and the larger holds a known quantity of water when the surface is level with the point of the hook O.

A small Pelton wheel is situated at P, and is connected by means of a pipe to the tank A and has a regulating cock Q. The exhaust water is led by a pipe R into the trough L; baffle plates S, S, and a wire gauze screen T get rid of any turbulence and ensure that the water will flow quietly in the left-hand portion of

L. The Pelton wheel is fitted with a brake and a revolution counter for the horse-power estimations.

The left hand end of the trough L is fitted with a gauge notch V, which may be exchanged for others of different types. The head of water in L may be measured by means of the hook-gauge U, which consists of a graduated rod having a hook at its lower end and sliding vertically in a fixed tube.

EXPT. 50.—**Flow through various orifices.** Shut the cock Q, fix the required orifice at J and arrange the shoot N so as to discharge the water into the smaller compartment of M. Plug the larger compartment, which should be quite empty, and adjust the valve C so as to secure steady conditions at the required head. When steadiness has been secured, switch N over so as to discharge into the larger compartment of M, and note the time. When M is full up to O, again note the time, and record the duration of the test. The experiment may be repeated by switching the water into the smaller compartment, and then emptying the larger, which may be done without any interference with the actual flow from the orifice. The mean time of the two experiments should be taken. Several experiments should be made under different heads, and the results tabulated as follows

EXPERIMENT ON THE DISCHARGE FROM AN ORIFICE.

No. of Expt	Shape of orifice	Area of orifice, A square feet	Head of supply water, H feet.	Flow in time t secs., Q_1 cubic feet	Duration of test, t seconds	Flow per second, $Q = \frac{Q_1}{t}$ cubic feet	Coefficient of discharge, $c_d = \frac{Q}{A\sqrt{2gH}}$

Plot a curve showing the relation of H and Q. Useful shapes of orifice are circular, square, rectangular, triangular, trumpet mouthpiece and the Borda mouthpiece. Care must be taken in measuring the dimensions required for calculating the area A. In the case of the trumpet orifice, the coefficient of discharge found from the experiment will be the coefficient of velocity. In the Borda mouthpiece, assume that the coefficient of contraction is 0·5, and calculate the coefficient of velocity from your experimental value of the coefficient of discharge. Thus,

$$c_d = c_c c_v = 0{\cdot}5 c_v ;$$

$$\therefore c_v = 2c_d .$$

EXPT. 51.—**Measurement of c_v for a round orifice.** It is difficult to obtain the coefficient of contraction by direct measurement of the jet diameter in small scale experiments. The following method may be used to determine roughly the coefficient of velocity, and the coefficient of contraction may be calculated then from the known coefficient of discharge.

Use a circular orifice under low head, which should be maintained very steady. The curve of the jet is shown at APB in Fig. 705, and the dimensions H and D should be measured carefully. The orifice is at A. At any point P on the curve, the actual velocity v_1 will be tangential to the curve, and will have horizontal and vertical components v_x and v_y respectively. It may be assumed that v_x is equal to v, the velocity with which the water leaves the orifice, and that v_y follows the law of a body falling freely. Hence, if d and h are the coordinates of P, and t is the time taken to flow from A to P, we have

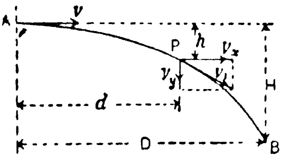

FIG. 705. Curve of a jet of water.

$$d = vt. \qquad (1)$$

$$h = \tfrac{1}{2}gt^2. \qquad (2)$$

If H_1 is the head under which the water leaves the orifice, then

$$v = c_v\sqrt{2gH_1}.$$

Hence (1) becomes

$$d = c_v t\sqrt{2gH_1},$$

and

$$d^2 = c_v^2 t^2 \times 2gH_1. \qquad (3)$$

Division of (3) by (2) gives

$$\frac{d^2}{h} = \frac{c_v^2 t^2 \times 2gH_1}{\frac{1}{2}gt^2}$$

$$= 4c_v^2 H_1. \qquad (4)$$

This result will be true for all points on the curve. Applying it to B, we have

$$\frac{D^2}{H} = 4c_v^2 H_1,$$

or

$$c_v^2 = \frac{D^2}{4H_1H},$$

$$c_v = \frac{D}{2}\sqrt{\frac{1}{H_1H}}. \qquad (5)$$

The experiment should be performed for three or four values of H_1, and D and H measured for each. The values of c_v calculated from each set of measurements should agree fairly well.

EXPT. 52.—**Flow over gauge notches.** Close the orifices at J (Fig. 704) by means of a blank flange. Secure the wheel of the

Pelton motor so as to prevent rotation. Get ready the measuring tank M as in the previous experiments. Fix the experimental gauge notch at V. By means of a long steel straight edge laid on the sill of the notch and on the point of the hook gauge U, and brought level by use of a spirit level, find and note the zero reading of the hook gauge. Open and regulate the cock Q until the desired head is obtained in the trough L and the water is flowing steadily. Start and stop the measurement of the water in M as directed previously. The experiment should be repeated for several heads and the results tabulated as follows:

EXPERIMENT ON THE FLOW OVER A GAUGE NOTCH.

No. of Expt.	Description of notch	Dimensions of notch	Head by hook gauge H feet	Flow in time t secs., Q_1 cubic feet	Duration of test t secs.	Flow per sec., $Q = \frac{Q_1}{t}$ cub. ft.	Coefficient of discharge	
							V notch, $a = \frac{Q}{H^{\frac{5}{2}}}$	Rectangular notch, $a = \frac{Q}{(l - 0.1nH)H^{\frac{3}{2}}}$

Plot a curve for each notch showing the relation of Q and H.

Useful notches for experimental purposes are triangular, having angles of 90° and 60°, rectangular, having two side contractions, another having a vertical plate fitted so as to give one side contraction and a third having two such plates in order to eliminate side contraction.

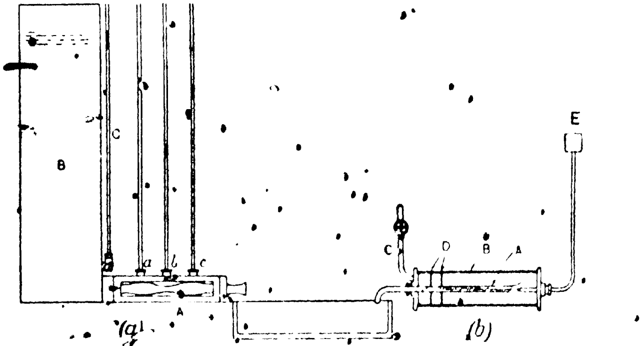

FIG. 706.—Apparatus for illustrating (*a*) Bernoulli's law; (*b*) Osborne Reynolds' colour band.

EXPT. 53.—**Bernoulli's law.** In Fig. 706 (*a*) is shown a box A of varying cross-sectional dimensions and fitted with a glass panel on the front side. Water under a given head may be supplied from a tank B fitted with a glass tube gauge C. Other three tube gauges are connected to A, and show the pressure heads at *a*, *b* and *c*. As the box A is horizontal, there will be no change in potential energy; hence, neglecting friction, the sum of the pressure and kinetic energies at *a*, *b* and *c* will be equal. Thus,

$$h_a + \frac{v_a^2}{2g} = h_b + \frac{v_b^2}{2g} = h_c + \frac{v_c^2}{2g}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where h_a, h_b and h_c are the heads in feet and v_a, v_b and v_c are the velocities in feet per second at *a*, *b* and *c* respectively.

h_a, h_b and h_c are given by the observed heads in the glass tube. Find the quantity of water flowing per second by measuring the time taken to discharge a given quantity; let this be Q cubic feet per second. Measure the cross-sectional areas A_a, A_b, A_c, in square feet, at *a*, *b* and *c* respectively. As the same quantity passes each section per second, we have

$$v_a = \frac{Q}{A_a}; \quad v_b = \frac{Q}{A_b}; \quad v_c = \frac{Q}{A_c}.$$

Substitution in (1) gives

$$h_a + \frac{Q^2}{2gA_a^2} = h_b + \frac{Q^2}{2gA_b^2} = h_c + \frac{Q^2}{2gA_c^2}. \quad \ldots\ldots\ldots\ldots(2)$$

Insert the observed quantities in equation (2) and note the divergence from strict equality in the results.

EXPT. 54.—**The Venturi meter.** A small meter of this type is illustrated in Fig 707. The axis of the meter is arranged horizontally in order that the potential energy of the water may remain constant.

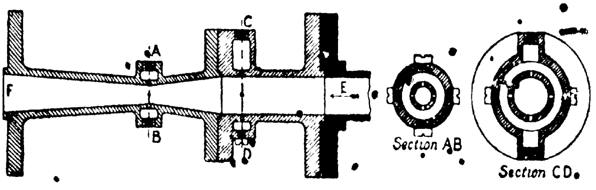

FIG. 707.—Sections of a Venturi meter.

The difference in pressure heads at the sections AB and CD is measured by means of glass tube gauges containing water or by means of a glass U-tube containing mercury. The meter is connected at E to a water supply, and the discharge pipe, connected to the meter at F, should have a regulating valve fitted. Discharge takes place into a measuring tank or into a trough fitted with a gauge notch. To make an experiment maintain steady the pressure-head difference as shown by the tube gauge, and measure the flow in cubic feet per second. Apply your results to equation (3) (p. 596), and so ascertain the value of the coefficient which must be applied to the right-hand side. Repeat the experiment with several different rates of flow, and obtain the coefficient for each. Plot curves showing (*a*) the relation of the flow in cubic feet per second and the pressure-head difference in feet, (*b*) the relation of the coefficient and the pressure-head difference in feet.

EXPT. 55.—**The critical velocity in a pipe.** Prof. Osborne Reynolds has shown that, when water flows in a pipe, the motion may be steady, *i.e.* free from eddies, or unsteady, *i.e.* sinuous or broken up in eddies. The critical speed is that at which the flow ceases to be steady; at higher speeds the water is broken up into eddies. In Fig. 706 (*b*) is illustrated a simple form of Reynolds' experiment, which serves to illustrate the point. A test tube of glass, A, having a bell-mouthed entrance, is inclosed in a larger closed glass tube B and discharges into a sink. B is supplied with water from a pipe C having a regulating valve; two gauze screens D serve to prevent turbulence in B. A small vessel E contains a coloured liquid (red ink serves well), and has a small glass tube which enters B and discharges a fine band of coloured liquid into the bell mouth of A. At low velocities of flow through A, it will be found that the coloured band will travel unbroken throughout A. If the velocity be increased slowly, it will be found that the stream becomes broken up at a certain velocity, thus showing the presence of eddies.

Reynolds also investigated the laws of resistance for these different methods of flow, and found that the resistance varied as the speed at speeds below the critical value. At speeds higher than the critical value, the resistance varies as some power of the speed, approximately the square.

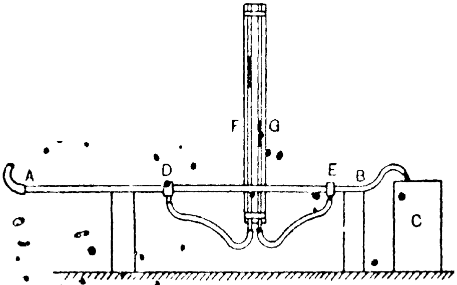

FIG. 708.—Apparatus for investigating frictional resistances in a pipe.

EXPT. 56—**Frictional resistances in a pipe.** The apparatus illustrated in Fig. 708 consists of a straight pipe AB of as nearly uniform bore as is possible (solid drawn copper or brass does well for a small scale experiment), connected to a tap at A and discharging into a measuring vessel C. The pipe is arranged horizontally, and is connected at D and E by means of rubber tubing to two open-topped glass tubes F and G. A scale permits the levels of the water in the glass tubes to be observed. The connections at D and E are shown in more detail in Fig. 709. A collar having a recess inside is soldered to the pipe, and has three screw plugs and a nozzle for connecting the rubber tube. Four very small holes, two of which are shown in Fig. 709, are bored through the pipe so as to connect the recess in the collar to the interior of the pipe; care should be taken that no arras is left round these holes on the inner skin of the pipe. The water rises to heights in the glass tubes proportional to the pressures at D and E, and the difference in these heights gives the loss in pressure head due to frictional resistances.

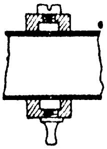

FIG. 709.—Section of pressure gauge connection.

Measure the diameter of the pipe; this may be done by use of a taper gauge inserted into the pipe end, or by filling a measured length of pipe with water and removing the water to be weighed; its volume may then be found and the diameter of the pipe may be calculated. Measure also the length of the pipe between the small holes at D and E. Turn on the water gently, and adjust the flow until steady conditions are attained with the desired difference in levels in the glass tubes. Care must be taken to get rid of air from the pressure gauge tube connections. At a noted time, turn the discharge into the measuring vessel and note the time taken to discharge a measured quantity of water. Repeat and take the average of the two times. Repeat the experiment several times, using different velocities of flow, and note the difference in heads and the times of flow for each. Tabulate these as follows.

EXPERIMENT ON THE FRICTIONAL RESISTANCES IN A PIPE.

No. of Expt.	Diam. of pipe, d feet.	Test length of pipe, L feet.	Difference in heads, h feet.	Flow in time t seconds, Q_1 cubic feet.	Duration of test, t seconds.	Virtual slope, $i=\frac{h}{L}$	Hydraulic mean depth, $m=\frac{d}{4}$ feet.	Flow per sec., $Q=\frac{Q_1}{t}$ cub. ft	Cross-sectional area of stream, $A=\frac{\pi d^2}{4}$ sq ft	Velocity of stream, $v=\frac{Q}{A}$ ft. sec	Value of c in the Chezy formula, $c=\frac{v}{\sqrt{mi}}$

It is useful to experiment on two pipes of different diameters. Very fair results may be obtained by use of solid drawn copper tubes of $\frac{3}{8}$ inch and $\frac{3}{4}$ inch diameters on a length L of 5 feet.

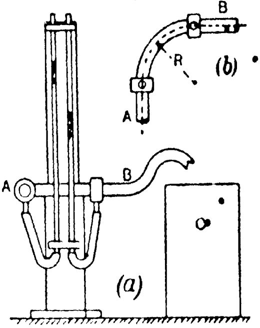

Fig. 710.—Apparatus for measuring the loss of head at a bend.

EXPT. 57.—**Loss of head at bends.** The apparatus is shown in Fig. 710. The right-angled bend AB is supplied with water at A from a tap and discharges into a measuring vessel C. Collars similar to that in Fig. 709 are soldered at the beginning and end of the curved part, and are connected to two glass tubes. The difference in level shows the loss in head. The pipe is arranged horizontally in order that there shall be no change in the potential energy of the water.

Measure the bore of the pipe and the radius of the bend, and find the flow in the same manner as that used in the previous experiment. Do this for several different velocities of flow, and tabulate. A coefficient for the head lost may be found from

$$h = c\frac{v^2}{2g},$$

or

$$c = \frac{2gh}{v^2}.$$

This result expresses the head lost in terms of the kinetic energy of the flowing water.

EXPERIMENT ON THE HEAD LOST AT A BEND.

No. of Expt.	Diameter of pipe, d feet.	Radius of bend, R feet.	Difference in pressure heads, h feet.	Flow in time t sec., Q_1 cub. ft.	Duration of test, t sec.

Flow per sec., $Q = \frac{Q_1}{t}$ cub. ft.	Cross sectional area of stream, $A = \frac{\pi d^2}{4}$ sq. ft.	Velocity of stream, $v = \frac{Q}{A}$ ft. sec.	Value of c, $c = \frac{2gh}{v^2}$.

EXPT. 58.—**Loss of head at an elbow.** This experiment is carried out in the same general manner as for a bend. If the pipe ABC (Fig. 711) is cut from a piece similar to that used in finding the frictional loss in a straight pipe, the frictional loss h_1 feet per inch length of pipe will be known. Measure AB and BC along the pipe axis in inches, and let the sum be l. Then the frictional loss apart from shock loss at the elbow will be $h_1 l$ feet. This should be deducted from the difference in head shown in the tube gauges, and the result will then be the shock loss. The calculations in reducing the results are similar to those for the bend, and the value of c is found in the same way.

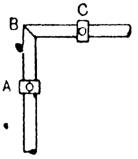

FIG. 711.

EXPT. 59.—**Losses at sudden enlargements and contractions in a pipe.** The apparatus shown in Fig. 712 consists of a pipe AB having both ends plugged and smaller pipes AC and BD fitted to the plugs. Water is supplied at D and is discharged into a measuring vessel at C. There is thus a sudden enlargement at B and a sudden contraction at A. The pressure heads are measured at E, F, G, H, K and L; collars resembling that in Fig. 709 are fitted at these places and are connected to six glass tubes having graduated scales. The following record of an actual experiment will illustrate the method of recording and reducing the results.

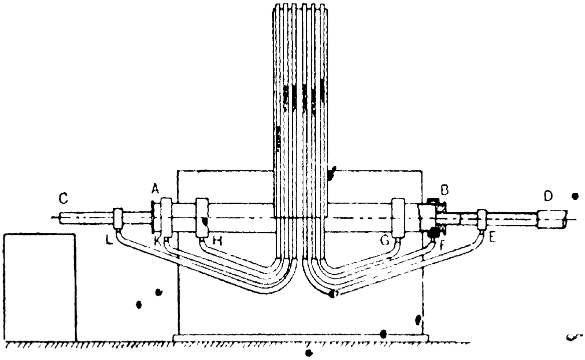

FIG. 712.—Apparatus for measuring the losses at sudden enlargements and contractions.

The pipe diameters were found by filling a measured length of pipe with water, and the areas A_1 and A_2 of the sections of the small and the large pipe respectively were calculated then. These were found to be $A_1 = 0{\cdot}00331$ and $A_2 = 0{\cdot}00386$ square feet. The pipe lengths are given in Fig. 713.

OBSERVED QUANTITIES.

No. of Test.	Heads in feet above pipe centre, shown by gauges.						Flow in t seconds, Q_t cubic feet.	Duration of test, t seconds.
	h_E	h_F	h_G	h_H	h_K	h_L		
20	2·875	2·633	2·775	2·775	2·775	1·217	0·08	33

It will be noted that the pressure is lower at F than at G; this is owing to the formation of violent eddies at the sudden enlargement.

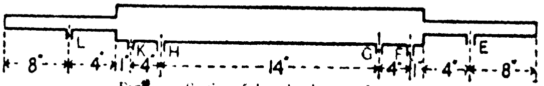

FIG. 713. Section of the tube shown in Fig. 712.

The pressures at G, H and K were equal so far as could be detected by means of the gauges employed. The absence of violent eddies at K is shown by the pressure there being equal to that in the body of the large pipe. The velocities of flow are as follows:

$$\text{Flow per second} \qquad = Q = \frac{Q_t}{t} = 0{\cdot}00242 \text{ cub. ft.}$$

$$\text{Velocity in the small pipe} = v_1 = \frac{Q}{A_1} = 7{\cdot}32 \text{ feet per sec.}$$

$$\text{Velocity in the large pipe} = v_2 = \frac{Q}{A_2} = 0{\cdot}628 \text{ feet per sec.}$$

The pipe being horizontal, there were no changes in potential energy, and pressure and kinetic energies alone are considered in the following calculations:

SUDDEN ENLARGEMENT.

No. of Test.	Velocities, ft. per sec.		Kinetic energies, ft. lb. per pound		Pressure energies, ft.-lb. per pound.		Total energies, ft.-lb. per pound		Energy wasted, by expt., ft.-lb. per pound.
	at E, v_1.	at G, v_2.	at E, $\frac{v_1^2}{2g}$.	at G, $\frac{v_2^2}{2g}$.	at E, h_E.	at G, h_G.	at E	at G.	
20	7·32	0·628	0·835	0·00615	2·875	2·775	3·71	2·781	0·93

The energy wasted may be calculated from the equation given on p. 619 and the result compared with that found by experiment. Thus,

$$\text{Energy wasted per lb. of water} = \frac{(v_1 - v_2)^2}{2g}$$

$$= \frac{6{\cdot}69 \times 6{\cdot}69}{64{\cdot}4} = 0{\cdot}695 \text{ foot-lb.}$$

$$\text{Ratio, } \frac{\text{experimental waste}}{\text{calculated waste}} = 1{\cdot}34.$$

SUDDEN CONTRACTION.

No. of Test.	Velocities, ft. per sec at K, v_k.	Velocities, ft. per sec at L, v_l	Kinetic energies, ft. lb. per pound. at K, $\frac{v_k^2}{2g}$.	Kinetic energies, ft. lb. per pound. at L, $\frac{v_l^2}{2g}$.	Pressure energies, ft.-lb per pound. at K, h_K.	Pressure energies, ft.-lb per pound. at L, h_L.	Total energies, ft.-lb per pound at K.	Total energies, ft.-lb per pound at L.	Energy wasted, by expt., ft.-lb. per pound.
20	0·628	7·32	0·00615	0·835	2·775	1·217	2·781	2·052	0·729

$$\text{Ratio, } \frac{\text{energy wasted, by expt.}}{\text{kinetic energy at L}} = \frac{0·729}{0·835} = 0·873.$$

$$\text{Hence, } \qquad \text{Energy wasted} = 0·873 \frac{v_l^2}{2g} \text{ foot-lb per lb. of water.}$$

Compare this result with the equation given on p. 620

EXPT. 60.—**Pressure of a jet impinging on a plate.** The apparatus used is illustrated in Fig. 714. A tank A is furnished with a trumpet mouthpiece B in the bottom, and discharges a vertical downward jet. The head may be observed by means of a glass

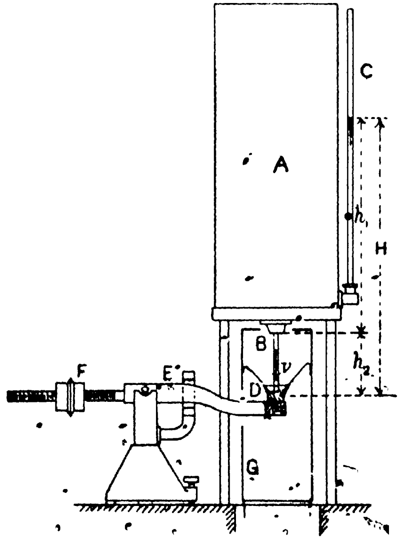

FIG. 714.—Apparatus for measuring the impact of jets.

tube C and a graduated scale. The head is maintained constant during the experiment. The jet impinges on a plate D, which is screwed to the end of a balance beam E. Equilibrium of E is secured by moving the counterpoise F. G is a cylinder of transparent celluloid, arranged so as to catch the waste water and deliver it into a sink.

In making an experiment, it is best to proceed as follows. Place a known weight, say 0·2 lb., centrally on the plate D, and move the counterpoise until balance is restored. Remove the weight and turn on the jet, gradually increasing the head until equilibrium of the balance beam is restored. The pressure on the plate will now be 0·2 lb., and the head H from the surface level in the tank to the level of the plate should be noted. The experiment should be repeated for several different pressures, and also for several plates.

It is useful to have a flat plate, a hemispherical cup and a cup having a lip angle of 45°. Equations for these are given on pp. 625 and 626, and the experimental values of the pressures should be compared with the calculated values.

The flow of water reaching the plate per second should be found by a separate series of tests on the tank orifice. This series should cover the range of heads used in the above experiments, and should be carried out in the same manner as for the discharge from an orifice. From the results of this series,

Let M = mass of water discharged under a head h_1 feet, in pounds per sec.

h_1 = the head over the plane of the orifice, in feet.

h_2 = the height from the plate to the orifice, in feet.

v = the velocity of the water reaching the plate, in feet per sec.

Then $$H = h_1 + h_2.$$

Also $$v = \sqrt{2gH}, \text{ nearly.}$$

$$\text{Momentum reaching the plate per second} = Mv$$
$$= M\sqrt{2gH}.$$

This result will take the place of (Av^2w) in the equations given on pp. 625 and 626, and will enable the pressures on the various plates to be calculated.

Expt. 61.—**Horse-power and efficiency of a Pelton wheel.** Fig. 715 illustrates a small Pelton wheel specially constructed for experimental work. There is one nozzle only, and the supply of water is controlled by means of a needle valve. The buckets are of the Doble type, shaped to receive the jet with as little shock as possible, and cut away at the entrance edge as shown in the side elevation, in order that the bucket entering the jet may do so with the minimum disturbance. The exhaust water is discharged from the lower part of the casing into a trough having a gauge notch, thus enabling the water consumption to be measured.

The brake horse-power is measured by a simple pattern of band brake, consisting of a band lapped round half the circumference of the brake wheel; the ends of the band are connected to a pivoted wooden lever which is under the control of a spring balance. A revolution counter driven by the shaft enables the wheel speed to be measured. Provision is made for running water into the interior of the brake wheel should it show any tendency to become hot.

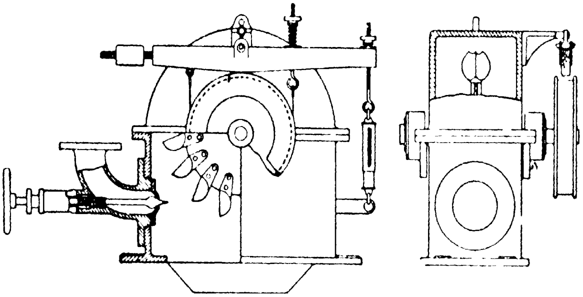

FIG. 715.—An experimental Pelton wheel.

The supply water may be taken from an overhead tank, or from a tank similar to that in Fig. 704. In the former case, the head H feet may be measured from the constant surface level in the tank to the level of the jet; each pound of water leaving the nozzle will possess H foot-lb. of energy, neglecting the frictional waste in the pipe and nozzle. In the latter case, the head H feet in the tank (Fig. 704). may be measured by means of a pressure gauge, and, as the water in the tank is practically at rest, this head will represent the energy supplied to the wheel per pound of water, again neglecting friction in the pipe connecting the tank to the wheel casing as well as the friction of the nozzle.

Sometimes the arrangement consists of a power-driven centrifugal pump, which delivers water to the nozzle of the Pelton wheel without any tank intervening. In this case, the pressure head H_1 feet of the supply water may be measured by means of a pressure gauge connected to the supply pipe. From a knowledge of the diameter of the pipe at the point of connection of the gauge and also of the flow of water, the velocity v feet per second may be calculated, and hence the kinetic energy $\frac{v^2}{2g}$ foot-lb. per pound of water may be found. Suppose also that the point of connection of the gauge is h feet above the nozzle level. The total energy of supply will then be

$$H = \text{pressure energy} + \text{kinetic energy} + \text{potential energy}$$

$$= \left(H_1 + \frac{v^2}{2g} + h\right) \text{ foot-lb. per pound of water.}$$

Constants required in reducing the results of the tests are the radius R_1 feet from the centre of the wheel to the axis of the jet and the radius from the wheel centre R_2 feet at which the brake spring balance exerts its pull. We also have

$$\text{Velocity of the jet} = V_1 = 0{\cdot}93 \text{ to } 0{\cdot}95\sqrt{2gH} \text{ feet per sec.}$$

Let N be the revolutions per minute. Then

$$\text{Velocity of the bucket} = V_2 = \frac{2\pi R_1 N}{60} \text{ feet per sec.}$$

If the brake is of the type shown in Fig 715, and if the spring balance is exerting a pull P lb., then

$$\text{B.H.P.} = \frac{P \times 2\pi R_2 \times N}{33000}.$$

Let W = the flow in pounds per minute.

Then Energy available in the supply water = WH foot lb. per min.

$$\text{Horse-power supplied} = \frac{WH}{33000}.$$

$$\text{Actual efficiency} = \frac{\text{B.H.P.}}{\frac{WH}{33000}}$$

$$= \frac{P \times 2\pi R_2 N}{WH}.$$

To make a series of tests, maintain constant the head of the supply water and keep constant the setting of the needle valve; the flow will then be constant, as indicated by the gauge notch readings. Put a given load on the brake and read the revolution counter every minute for, say, five minutes. This completes one test. Alter the brake load and again note the revolutions; this process should be repeated until information has been obtained regarding the revolutions per minute corresponding to brake loads ranging from zero up to a load which nearly stops the wheel from rotating. A new series of tests can then be made by altering the head of the supply water. The results should be tabulated as follows:

TESTS ON A PELTON WHEEL.

No. of Test	Speed of wheel in revs. per min., N	Net brake load, in lb., P	Total head available in feet, H	Water supplied, in lb. per minute, W.	Energy available, ft.-lb. per minute, WH	B.H.P.

Actual efficiency.	Velocity of jet, in feet per sec., V_1.	Velocity of buckets, in feet per sec., V_2	Ratio, $\frac{V_2}{V_1}$.

Plot the B.H.P. and the revolutions per minute; plot also the efficiency and ratios $\frac{V_2}{V_1}$.

EXERCISES ON CHAPTER XXV.

1. An experiment on the discharge through a round orifice in a thin vertical plate gave the following results: Diameter of orifice, 0·5 inch; head of water over the centre of the orifice, 25 feet; time taken to discharge 450 pounds of water, 216 seconds. Find the coefficient of discharge.

2. A vertical jet discharged through a trumpet orifice 1 inch in diameter. Under a head of 22·68 inches of water, the discharge was found to be 112 pounds in 31 seconds. Find the coefficient of velocity.

3. In an experimental Borda mouthpiece, the internal diameter of the tube was 0·89 inch. Experiments were made under heads of 1·6 and 2·12 feet, when the discharge was found to be 450 pounds in 314 and 278 seconds respectively. Assume the coefficient of velocity to be 0·98, and calculate the coefficient of contraction in each case.

4. Experiments were made on the discharge over a 90 degree V notch. At heads of 0·123 and 0·165 feet, the times taken to discharge 450 pounds were found to be 537 and 259 seconds respectively. Calculate the values of the coefficient of discharge.

5. An experimental rectangular gauge notch, 4 inches long, had both end contractions suppressed by means of vertical plates. At heads of 0·08 and 0·093 feet, the times taken to discharge 450 pounds were 296 and 240 seconds respectively. Calculate the values of the coefficient of discharge.

6. The following measurements were made in calibration tests on a small Venturi meter Diameter of tube, 0·75 inch; diameter at throat, 0·375 inch.

Total discharge, gallons.	Time taken, seconds	Difference in heads		Actual discharge, Q_a cubic feet per sec	Theoretical discharge, Q cubic ft. per sec.	$c = \frac{Q_a}{Q}$
		Inches of mercury	Feet of water			
30	775	1				
30	534	2				
30	378	4				
30	338	5				
30	307	6				
30	269	8				
46·31	365	10				
46·31	334	12				
46·31	318	14				

Fill in the columns left blank. Find the average value of the coefficient c. Plot a curve showing the relation of the actual discharge in cubic feet per second and the difference in pressure heads in inches of mercury.

7. Experiments were made on the frictional loss of head in a smooth solid drawn copper tube, 0·77 inch bore and 5 feet between the gauge branches The following observations were made:

Flow in gallons	Time taken		Loss of head, inches of water	Velocity, feet per sec.	$c = \frac{v}{\sqrt{mi}}$
	min	sec			
30	9	45	2·76		
30	5	35	7·56		
30	3	38	16·68		
46·31	4	19	27·2		
46·31	2	47	62·56		

Fill in the blank columns; c is the coefficient in the Chezy formula. Plot a curve showing the relation of the heads lost and the velocities.

8. Two pieces of smooth solid drawn copper tube, 0·337 inch bore, were soldered together so as to form a right-angled elbow. A gauge branch was attached in each portion at 4 inches from the junction In one experiment, the difference in head was 8 inches of water and 5 pounds of water passed in 30 seconds. It was known that $v = 80\sqrt{mi}$ for a straight pipe of the same bore and material. Express the head lost at the elbow in terms of the kinetic energy per pound of water flowing.

9. In the apparatus illustrated in Fig. 714 for determining the pressure on plates, preliminary tests showed that the coefficient of discharge through the trumpet orifice had an average value of 0·888. The orifice is 0·25 inch diameter. Tests were made on three plates: A, a flat plate at

90 degrees to the jet ; B, a hemispherical cup, the jet discharging into its centre ; C, a cup having its sides at 45 degrees to the horizontal, the jet discharging into its centre. The following observations were taken :

Expt. No.	Plate.	Head of water, to plane of orifice, h_1 feet.	Head of water, to plate from orifice, h_2 feet.	Observed pressure on plate, lb.
1	A	1·972	0·028	0·07
2	A	2·232	0·028	0·08
3	A	2·542	0·028	0·09
4	B	1·572	0·028	0·10
5	B	1·672	0·028	0·11
6	B	1·852	0·028	0·13
7	C	1·672	0·028	0·08
8	C	1·802	0·028	0·09
9	C	1·972	0·028	0·10

Find the pressures on the plates in each case by calculation.

TABLES.

Useful Constants.

1 inch	2·54 centimetres = 25·4 millimetres.
1 metre	39·37 inches.
5280 feet	1 mile.
6 feet	1 fathom.
1 Gunter's chain	66 feet
80 Gunter's chains	1 mile.
1 kilometre	0·621 mile
1 square inch	6·45 square centimetres
1 square metre	1550 square inches
1 cubic inch	16·39 cubic centimetres
1 cubic metre	61,025 cubic inches = 1·308 cubic yards
1 litre	= 1000 cubic centimetres = 1·762 pint.
1 gallon	0·1605 cubic foot = 4·541 litres
1 bushel	= 1·284 cubic feet
1 radian	57·3 degrees
π	3·1416.

1 knot = 6080 feet per hour.
60 miles per hour = 1 mile per minute = 88 feet per second.
The value of g at London = 32·182 feet per sec. per sec

One pound avoirdupois = 7000 grains = 453·6 grams
One kilogram = 2·205 pounds
One gallon of pure water at 62° F. weighs 10 lb.
One cubic foot of pure water at 62° F. weighs 62·3 lb.
Weight of 1 pound in London = 445,000 dynes.
One cubic foot of air at 0° C. and 1 atmosphere pressure weighs 0·0807 lb.
One cubic foot of hydrogen at 0° C. and 1 atmosphere pressure weighs 0·00559 lb.

1 atmosphere = 14·7 lb. per square inch
= 2116 lb. per square foot
= 10^6 dynes per square centimetre nearly.

1 kilogram per square centimetre = 14·22 lb. per square inch.

A column of mercury 760 millimetres (= 30 inches) high produces at its base a pressure of 1 atmosphere.

A column of water 2·3 feet high produces at its base a pressure of 1 lb. per sq. inch.

1 foot-lb. = $1·3562 \times 10^7$ ergs.
1 metre-kilogram = 7·235 foot-lb.
1 horse-power = 33,000 foot-lb. per minute = 746 watts.
1 horse-power-hour = 33,000 × 60 foot-lb.
Volts × amperes = watts.
1 electrical unit = 1000 watt-hours.

1 B.T.U. = $\frac{5}{9}$ lb.-degree-Cent unit
= 252 gram-calories.

Absolute temperature $\tau = t°$ C. + 273·7
$= t°$ F. + 461.

$$\text{Joule's equivalent} = \begin{cases} 778 \text{ ft.-lb.} = 1 \text{ B.T.U} \\ 1400 \text{ ft.-lb.} = 1 \text{ lb.-degree Cent. unit.} \end{cases}$$

To convert common into Napierian or hyperbolic logarithms, multiply by 2·3026.

The base of the Napierian logarithms is $e = 2·7183$.

Table of Coefficients of Linear Expansion.

(These are given as the increase in length which a bar of unit length undergoes when heated through one degree Fahrenheit.)

Steel alloyed with 36 % nickel	0 000000483
Wrought iron and mild steel	0·00000673
Cast iron	0 0000063
Copper	0·0000096
Zinc	0·0000162
Brass	0·0000105
Phosphor bronze	0·0000107

Table of Ultimate Strength and Elasticity of Materials.

Material.	Tensile strength, tons per sq. inch.	Compressive strength, tons per sq. inch.	Shearing strength tons per sq. inch.	Modulus of elasticity, tons per sq. inch. E.	C.	Values of m, Poisson's ratio $= \frac{1}{m}$.
Cast iron	5 to 15, average 8	25 to 65	6 to 13	6,000	2,200	3.7
Wrought iron—						
Tested in direction of rolling	20 to 29	16 to 20	22	13,000	5,200	3.6
Tested across direction of rolling	19 to 24					
Mild steel	27 to 32		21 to 25	13,500	5,500	3.25
Cast steel	55 to 70					
Copper, cast	8 to 12			5,500	2,100	
,, rolled	15			6,200	2,500	2.6
,, wire (hard drawn)	28			7,500		
Tin	2					
Zinc, cast	1 to 3					
,, rolled	8 to 10					
Lead	1				...	
Aluminium, cast	5					
,, rolled	6 to 10		5.5	4,000	1,600	
Brass, ordinary	11	5.5	10	5,700	2,200	3
,, wire	20 to 25		..			
Sterro metal	35					
Delta metal, cast	22	65		5,300		
,, forged	34					
,, wiredrawn	55			5,800		
Muntz metal	22					
Gun metal	15		15	5,000	2,000	
Aluminium bronze	40			6,500	2,500	
Phosphor bronze, annealed	25		19	6,000	2,300	
,, unannealed	up to 70					
Granite		6 to 10				
Sandstone		2 to 5		..		
Portland stone		2		..		
Brick, London stock		1				
,, Staffordshire blue		2 to 6				
Pine	5	2½	¼	700		
Oak	7	4½	1	650		
Leather	2					

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	[illegible]	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2583	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	[illegible]	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS.

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TRIGONOMETRICAL TABLE.

Angle.	Radians.	Sine.	Tangent.	Cotangent.	Cosine.		
0°	0	0	0	∞	1	1·5708	90°
1	·0175	·0175	·0175	57·2900	·9998	1·5533	89
2	·0349	·0349	·0349	28·6363	·9994	1·5359	88
3	·0524	·0523	·0524	19·0811	·9986	1·5184	87
4	·0698	·0698	·0699	14·3006	·9976	1·5010	86
5	·0873	·0872	·0875	11·4301	·9962	1·4835	85
6	·1047	·1045	·1051	9·5144	·9945	1·4661	84
7	·1222	·1219	·1228	8·1443	·9925	1·4486	83
8	·1396	·1392	·1405	7·1154	·9903	1·4312	82
9	·1571	·1564	·1584	6·3138	·9877	1·4137	81
10	·1745	·1736	·1763	5·6713	·9848	1·3963	80
11	·1920	·1908	·1944	5·1446	·9816	1·3788	79
12	·2094	·2079	·2126	4·7046	·9781	1·3614	78
13	·2269	·2250	·2309	4·3315	·9744	1·3439	77
14	·2443	·2419	·2493	4·0108	·9703	1·3265	76
15	·2618	·2588	·2679	3·7321	·9659	1·3090	75
16	·2793	·2756	·2867	3·4874	·9613	1·2915	74
17	·2967	·2924	·3057	3·2709	·9563	1·2741	73
18	·3142	·3090	·3249	3·0777	·9511	1·2566	72
19	·3316	·3256	·3443	2·9042	·9455	1·2392	71
20	·3491	·3420	·3640	2·7475	·9397	1·2217	70
21	·3665	·3584	·3839	2·6051	·9336	1·2043	69
22	·3840	·3746	·4040	2·4751	·9272	1·1868	68
23	·4014	·3907	·4245	2·3559	·9205	1·1694	67
24	·4189	·4067	·4452	2·2460	·9135	1·1519	66
25	·4363	·4226	·4663	2·1445	·9063	1·1345	65
26	·4538	·4384	·4877	2·0503	·8988	1·1170	64
27	·4712	·4540	·5095	1·9626	·8910	1·0996	63
28	·4887	·4695	·5317	1·8807	·8830	1·0821	62
29	·5061	·4848	·5543	1·8040	·8746	1·0647	61
30	·5236	·5000	·5774	1·7321	·8660	1·0472	60
31	·5411	·5150	·6009	1·6643	·8572	1·0297	59
32	·5585	·5299	·6249	1·6003	·8480	1·0123	58
33	·5760	·5446	·6494	1·5399	·8387	·9948	57
34	·5934	·5592	·6745	1·4826	·8290	·9774	56
35	·6109	·5736	·7002	1·4281	·8192	·9599	55
36	·6283	·5878	·7265	1·3764	·8090	·9425	54
37	·6458	·6018	·7536	1·3270	·7986	·9250	53
38	·6632	·6157	·7813	1·2799	·7880	·9076	52
39	·6807	·6293	·8098	1·2349	·7771	·8901	51
40	·6981	·6428	·8391	1·1918	·7660	·8727	50
41	·7156	·6561	·8693	1·1504	·7547	·8552	49
42	·7330	·6691	·9004	1·1106	·7431	·8378	48
43	·7505	·6820	·9325	1·0724	·7314	·8203	47
44	·7679	·6947	·9657	1·0355	·7193	·8029	46
45	·7854	·7071	1·0000	1·0000	·7071	·7854	45
		Cosine.	Cotangent.	Tangent.	Sine.	Radians.	Angle.

ANSWERS.

Chapter I. Page 17.

1. 19,500 lb. **2** 4400 square feet **3.** 1200 square feet.

4 (a) $15x^2$. (b) $6x - 35x^4$.

(c) $2\cos x + 3\sin x$ (d) $2\sin x\cos x - 2\cos x\sin x = 0$

(e) $3\sin^2 x\cos x - 3\cos^2 x\sin x$ (f) $3\sec^2 x + \sin x$

5 1. **6** $x = 2$; $y = 4$.

7 (a) x^3. (b) $x^4 - \frac{2}{3}x^3$ (c) $4x - 2x^2 + \frac{1}{3}x^3$.

(d) $\frac{2}{3}x^3 + \sin x$ (e) $0 + \log_e \theta$.

8. 1 − 50,550 **9.** 1·1707

Chapter II. Page 36.

1. (a) 12·6 lb. weight at 9° 6′ to the 9 lb. force.

(b) 12·15 ,, ,, 13° 27′ ,, ,,

(c) 7·8 ,, ,, 26° 21′ ,, ,,

2. (a) 5·87 ,, ,, 49° 54′ ,, ,,

(b) 6·77 ,, ,, −24° 35′ ,, ,,

(c) 11·5 ,, ,, −17° 30′ ,, ,,

3. 4·9 lb. weight. **4** 2·5 lb. weight.

5. 90·1 lb. weight; 112 lb. weight at 34° 36′ to AC.

6. 15·43 lb. weight. **7** 20·14 lb. weight.

8. Q = 19·24 tons weight; T = 7·76 tons weight. **9.** 1 lb. weight from O to D.

10

Angles, degrees	170	172	174	176	178	179	180
Q, lb. weight	232	285	381	572	1146	2299	Infinite

11. P = 28·[illegible]8 tons weight; S = 45·95 tons weight; V = 17·67 tons weight.

12. Tie BO, 5·02 tons weight pull; strut CO, 7 tons weight push.

13

Member	AB	BC	AC	Reaction at A	Reaction at C.
Force in member, tons weight	4·60	2·91	2·63	3·77	1·23

14. 32·9 tons weight push in each leg; 30 tons weight pull in the backstay.

15. Push in AO = 1·57 tons weight; push in BO = 0·59 ton weight; push in CO = 8·17 tons weight.
16. 25 lb. weight.
17. 38·8 lb. weight pull; 86·8 lb. weight push at 26° 40′ to the vertical.
18. 3·464 lb. weight; 1·732 lb. weight. 19. 4·08 lb. weight.

Chapter III. Page 56.

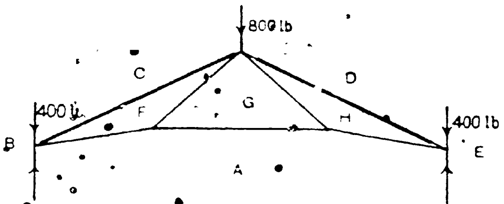
1. 10·29 inches from C, on the same side as D; 46 lb. weight.
2. 10·95 inches from A. 3. 45·3 lb. 4. 23·46 inches.
5. Reaction at A = 4·975 tons weight; reaction at B = 4·275 tons weight.
6. 6·55 tons weight; 7·44 tons weight.
7. Reaction at A = 4·571 tons; reaction at B = 3·429 tons.
8. Measure CD = 10 inches along CB; draw DG = 3·464 inches at 90° to CB; G is the centre of gravity.
9. $x = 0·583$ inches; $y = 1·083$ inches. 10. 50° 12′.
11. 30·5 lb.; 64·2 lb.; 17·3 lb. 13. A, 0·154 W; B, 0·402 W; C, 0·444 W.

Chapter IV. Page 74.

1. Top hinge, R = 100 lb., acting upwards away from the gate at 29° 48′ to the horizontal; bottom hinge, R = 100 lb., acting upwards towards the gate at 29° 48′ to the horizontal.
2. E = 2·828 lb., acting downwards towards the right, at an angle of 45° to the horizontal and at a perpendicular distance of 3·535 feet from the left lower corner of the square and to the left of it.
3. R = 1·732 lb., acting downwards towards the left, at an angle of 30° to the horizontal and at a perpendicular distance of 1·5 feet from the right-hand end of the base.
4. An anti-clockwise couple of 5·196 lb. feet must be applied.
5. Vertical reaction = 1638 lb.; inclined reaction = 3992 lb. acting at 37° 30′ to the horizontal.
6. Reaction at A = 711·1 lb.; reaction at B = 1227 lb. at 54° 35′ to the horizontal.
7. Reaction at B = 825·9 lb.; reaction at A = 950 lb. at 62° 8′ to the horizontal.
13. 4·47 lb. at 26° 34′ to the horizontal.
14. Reaction at A = 5·44 tons; reaction at B = 5·56 tons.
17. 42·42 lb. acting along DA from D towards A; 62·5 lb. at C, downwards towards the left at an angle of 36° 54′ to BC.
18. Let s = side of square; point required is outside the square, at distances from the 4 lb. force and 0·5 from the 3 lb. force.

Chapter V. Page 89.

1. See Fig. 716 (p. 691). 2. See Fig. 717 (p. 691). 3. See Fig. 718 (p. 692).
4. See Fig. 719 (p. 692). 5. See Fig. 720 (p. 693). 6. See Fig. 721 (p. 693).
8. See Fig. 722 (p. 694). 9. 2·308 tons.
10. The link polygon solution is shown in Fig. 723 (p. 695); reaction AB = 2400 lb. reaction KA = 3440 lb.
11. The substituted frame solution is shown in Fig. 723 (p. 695); reaction AB = 2400 lb.; reaction KA = 3440 lb.
12. Reaction AB = 2471 lb.; reaction KA = 3430 lb.



Member	Force in Member lb	
	Push	Pull
Reaction AB	800	—
" EA	800	—
CF, DH	1340	—
AF, AH	—	1225
FG, GH	—	300
AG	—	1000

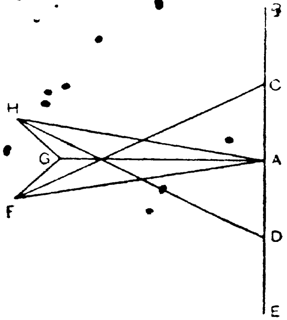
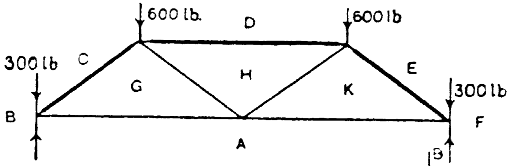


FIG. 716.



Member	Force in Member lb	
	Push	Pull
Reaction AB	900	—
" FA	900	—
CG, EK	970	—
DH	755	—
GH, HK	0	0
AG, AK	—	755

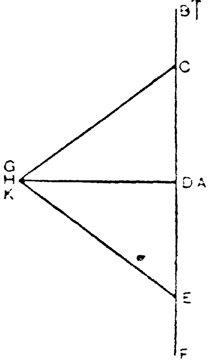
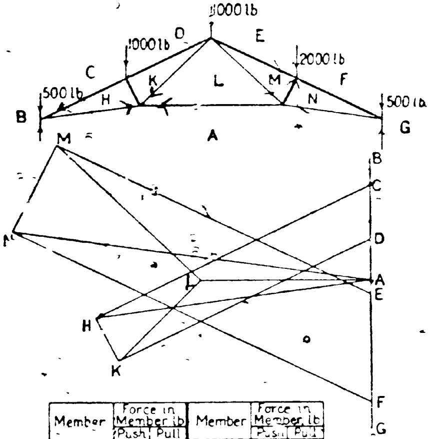
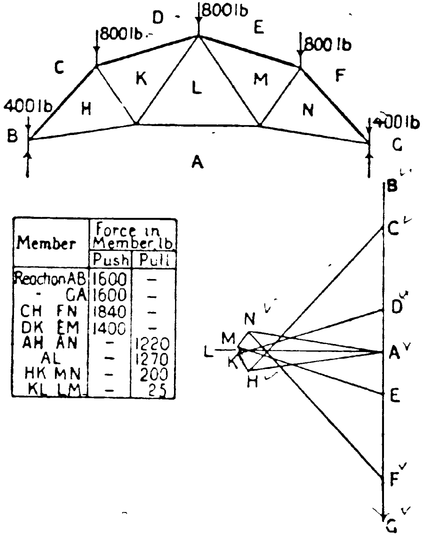


FIG. 717



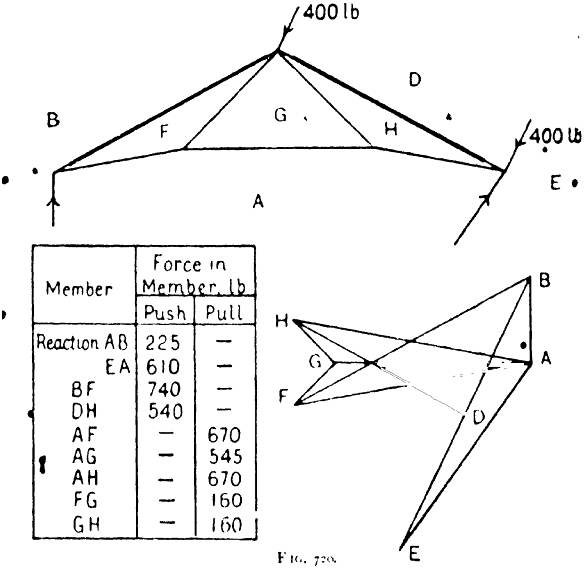
Member	Force in Member lb		Member	Force in Member lb	
	Push	Pull		Push	Pull
Reaction AB	2250	—	AH	—	4880
GA	2750	—	AL	—	3800
CH	5400	—	AN	—	6350
DK	4950	—	KL	—	2080
EM	6150	—	LM	—	3550
FN	030	—	HK	900	—
			MN	1700	—

Fig. 718



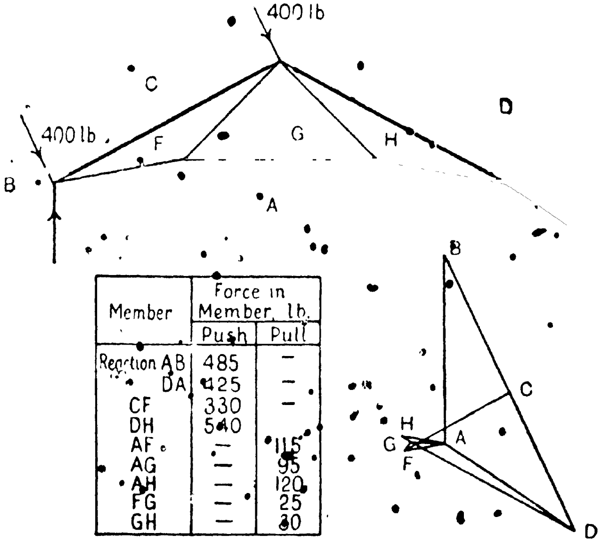
Member	Force in Member lb	
	Push	Pull
Reaction AB	1600	—
GA	1600	—
CH FN	1840	—
DK EM	1400	—
AH AN	—	1220
AL	—	1270
HK MN	—	200
KL LM	—	25

Fig. 719



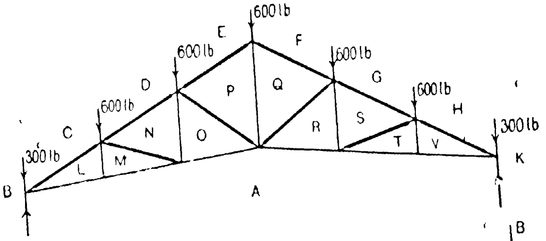
Member	Force in Member, lb	
	Push	Pull
Reaction AB	225	—
EA	610	—
BF	740	—
DH	540	—
AF	—	670
AG	—	545
AH	—	670
FG	—	160
GH	—	160

FIG. 720.

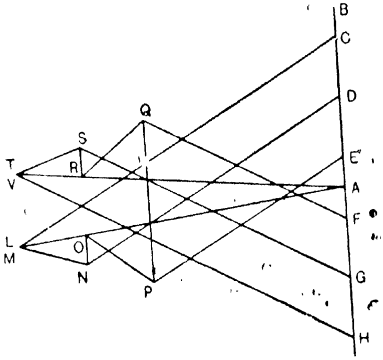


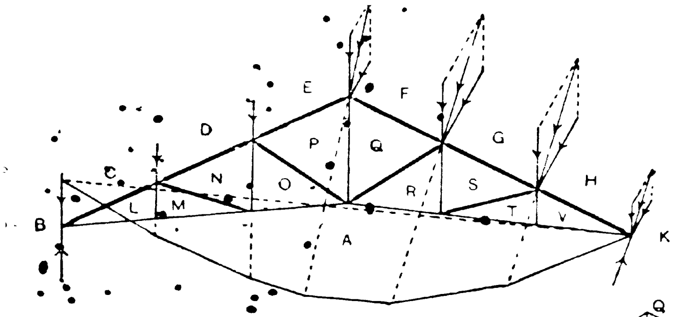
Member	Force in Member, lb.	
	Push	Pull
Reaction AB	485	—
DA	425	—
CF	330	—
DH	540	
AF	—	115
AG	—	95
AH	—	120
FG	—	25
GH	—	30

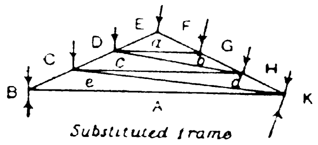
FIG. 721.



Member	Force in Member lb	
	Push	Pull
Reaction AB	1800	—
" KA	1800	—
CL HV.	4150	—
DN. GS.	3300	—
EP. FQ	2480	--
AL AM. AT. AV	—	3720
AO. AR.	—	2860
LM. TV.	0	0
MN, ST	780	—
NO.' RS.	—	300
OP. QR	900	
PQ		1620







Substituted frame

Member	Force in Member, lb		Member	Force in Member, lb	
	Push	Pull		Push	Pull
Reaction AB	2400	—	AR	—	5660
KA	3440	—	AT, AV	—	7520
CL	6000	—	LM TV	0	0
DN	5140	—	MN	780	—
EP	4300	—	NO	—	280
FQ	4080	—	OP	920	—
GS	5660	—	PQ	—	2760
HV	7360	—	QR	2160	—
AL AM	—	5400	RS	—	760
AO	—	4640	ST	1920	—

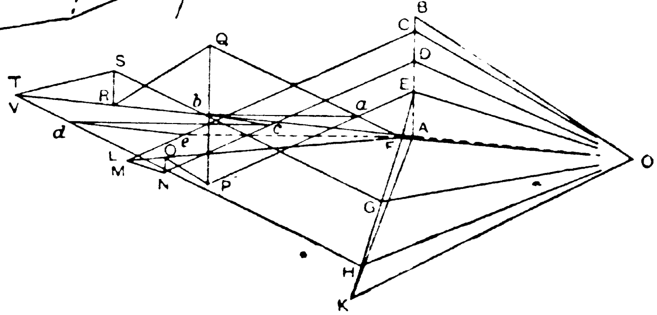
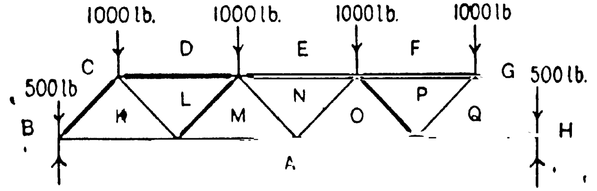


FIG. 723.



Member	Force in Member lb	
	Push	Pull
Reaction AB	2500	—
" HA	2500	—
AK, AQ	—	2000
AM, AO	—	4000
DL, FP	3000	—
EN	4000	—
CK, GQ	2800	—
KL, PQ	—	1410
LM, OP	1410	—
MN, NO	0	0

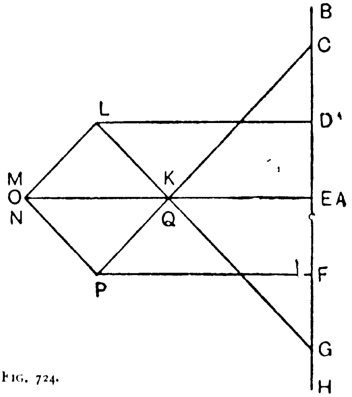
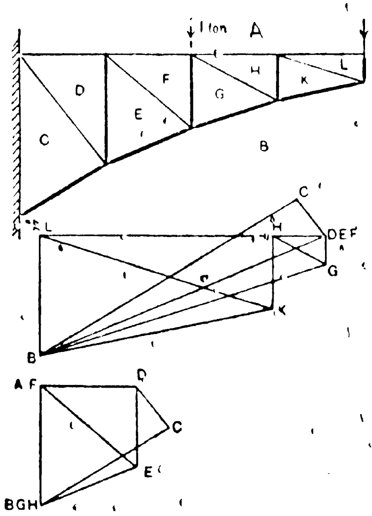


FIG. 724.



Member	Force in Member, tons			
	Load at end		Load at centre	
	Push	Pull	[illegible]	Pull
AL	0	0	0	0
AH	—	2·02	0	0
AF	—	2·48	0	0
AD	—	2·48	—	0·84
BK	2·08	—	0	0
BC	[illegible]	—	0	0
BE	2·67	—	0·9	—
BC	2·58	—	1·3	—
BL	1·00	—	0	0
KL	—	2·12	0	0
HK	0·6	—	0	0
GH	—	0·52	0	0
FG	0·23	—	1·00	—
EF	0	0	—	1·07
DE	0	0	0·66	—
CD	0·40		—	0·44

13. See Fig. 723 (p. 695). **14** See Fig 724 (p. 696).

15, 16. See Fig. 725 (p. 696).

17. Q = 1100 lb. pull; P = 12,350 lb. push; R 14,300 lb. push Taking the bars in order from the bottom, the forces are: 21,000 lb. pull; 9700 lb. push; 10,500 lb push; 21,000 pull

18. AB, 1·02 tons push; BC, 0·2 ton push; CD, 0·9 ton push; DA, 0·57 ton pull; AC, 0·46 ton push.

Chapter VI. Page 128.

1. 1·78 inches. **2.** 86·7 tons.

3. $d = \frac{7}{8}$ inch; $p = 2\frac{3}{4}$ inches; efficiency = 69·6 per cent.; bearing stress = 6·88 tons per sq. inch.

4. $d = 1\frac{1}{4}$ inches; $p = 4\frac{7}{16}$ inches; efficiency = 74·7 per cent.; bearing stress = 8·83 tons per sq. inch

5. 11 rivets on each side of the joint; 45·37 tons.

6. $\frac{11}{16}$ inch; 3·71 tons per sq inch; 1·85 tons per sq inch. **7.** 0·257 inch.

8. Extension in length 0·16 inch; contraction in width 0·000703 inch; contraction in thickness = 0·0000630 inch.

9. 0·0485 inch.

10. Change in diameter 0·001768 inch, change in length = 0·003635 inch; change in volume = 0·5712 cubic inch.

11 Pulls of 10·8 tons. **12** Pushes of 35·2 tons; 11·32 tons per sq. inch.

13 In the copper, 0·739 ton per sq inch; in the steel, 1·805 tons per sq. inch.

14. 161,900 lb; 53,025 lb.

15.

θ, degrees	0	30	45	60	90
p_n, tons per sq inch	0	1·25	2·5	3·75	5
p_t ,, ,,	0	2·165	2·5	2·165	0

16 67·5 tons, at 11·1 feet from the edge having the greatest stress

17. (*a*) 36·1 tons; 0·107 ton per sq inch (*b*) 23·9 tons; 1·905 tons per sq inch

18 28,000 lb. per sq foot; [illegible]500 lb per sq foot; 15,000 lb. per sq foot at 45° to the horizontal.

19.

Thickness of plate, inch		$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
Safe load per rivet, tons	Single shear	4·13	4·32	4·32	4·32	4·32
	Double shear	4·13	5·5	6·875	7·57	7·57

20. 3333 lb. per sq. inch; 1333 lb per sq. inch.

Chapter VII. Page 160.

1.

Section	At 6 tons load	At 4 tons load	At 2 tons load
M, ton-feet	29·6	38·0	27·6
S, left of load, tons	+7·4	+1·4	−2·6
S, right ,, ,,	+1·4	−2·6	−4·6

2.

Distance of section from A, feet	0	2	4	6	8
M, ton-feet	0	+8·33	+14·67	+19	+13·33
S, tons	+4·67	+3·67	+2·67	+1·67 −2·33	−3·33

Distance of section from A, feet	10	12	14	16
M, ton-feet	+5·67	4	−1	0
S, tons	−4·33	5·33 +2·00	+1	0

3.

Distance from A, feet	0	2	4	6	8	10
M, lb.-feet	0	13,440	21,120	22,080	15,360	0
S, lb.	+8,000	+5,360	+2,240	1,360	5,440	10,000

6. 1·5 inches from the 4·5 inches edge. **7.** 4 inches from the bottom edge.

10. 900 lb.

11. Bar on flat. (*a*) 2420 lb. per sq. inch; (*b*) 1936 lb. per sq. inch.
,, edge. (*a*) 1210 ,, ,, (*b*) 968 ,, ,,

12. (*a*) 2·56 tons per sq. inch; (*b*) 2·33 tons per sq. inch.; $q = 0·228$ ton per sq. inch.

13. 66·2 feet. **14.** $W_2 : W_1 = n^2 : 1$.

15. (*a*) Parabolic; $d = 1·335\sqrt{x}$, where x = the distance of the section from one end; d at centre = 5·67 inches. (*b*) Parallelogram; $b = 0·298x$; b at centre = 5·37 inches.

16. 25 lb. per sq. inch.

17.

Distance from N.A., inches	0	1	2	3	4	4⅜ In web	4⅜ In flange
Shear stress, tons per sq. in.	1·00	0·986	0·94	0·865	0·758	0·710	0·0887

18. $z = 1·357$ inch units; $k = 0·799$ inch.

19. $R_A = 9·4$ tons; $R_B = 14·6$ tons; $M = 84·5$ ton-feet.

20. $R = 7·75$ tons, passing through the column axis at 5·15 feet from the base and inclined at 14·5 degrees to the axis; $M = 9·7$ ton-feet.

Chapter VIII. Page 188.

1.

Distance from free end, inches	0	3	6	9	12	15	18	21	24
R, inches	∞	33,600	16,800	11,200	8,400	6,720	5,600	4,800	4,200

2. $i = 0·00286$ radian; $\Delta = 0·0457$ inch.

3. 0·0515 inch; 0·00129 radian; 23,200 inches.

4. 0·0429 inch; 0·001148 radian; 34,950 inches.

5. 2·314 tons; M at centre = +41·7 ton inches; M at ends = −41·7 ton inches; S for left half of beam = +1·157 tons; S for right half = −1·157 tons.

6. 3·472 tons, M at centre = 20·85 ton inches; M at ends = −41·7 ton inches; S at left end = +1·736 tons, S at right end = −1·736 tons; S at centre = 0.

7. 0·055 inch; 0·0415 inch. 8. Δ = 0·0761 inch; *i* = 0·001006 radian.

9. 23·9 lb.-inches; 1·875 inches. 10. 0·3704 inch.

11. 64,800 inches; 0·444 inch.

12. 3000 lb.; M at wall = −8000 lb. feet, maximum positive M = 4500 lb. feet; points of contraflexure are at the free end and at 6 feet from the free end; S at free end = −3000 lb.; S at wall = +5000 lb.

13. $P_A = P_B$ = 11·25 tons; P_C = 37·5 tons; the points of contraflexure occur at 5 feet on each side of C; maximum positive M = 42·2 ton feet; maximum negative M (at C) = −75 ton feet; S_A = +11·25 tons; S close to C on the left = −18·75 tons; S close to C on the right = +18·75 tons; S_B = −11·25 tons.

14. 0·06 inch.

15. 1·19 tons on each outer beam; 2·62 tons on inner beam; 5·36 tons per sq. inch.

16. (*a*) 9600 lb.; (*b*) 25,600 lb. feet, 10,000 lb.; (*c*) 2 feet from the wall.

17. 1·816.

18. Maximum M (at ends) = −23·625 ton feet; M at centre = +17·625 ton feet; points of contraflexure at 5·45 feet from each end.

19. $W_a : W_b$ = 1·165; $f_a : f_b$ = 0·984; $\Delta_a : \Delta_b$ = 1·18; maximum f_a = 5·66 tons per sq. inch.

20. At *x* feet from one end, M = $(13x - \frac{1}{2}x^2 - 70)$ ton feet for the end portions; M = $(10x - \frac{1}{2}x^2 - 40)$ ton feet for the centre portion; maximum M is at the ends and is 70 ton feet.

Chapter IX. Page 224.

1. 1·28 inch-tons; 0·00133 inch ton per cubic inch.

2. 12 tons per sq. inch; 0·213 inch. 3. 0·57 inch.

4. 7·083 tons per sq. inch; 11·29 sq. inches.

5. Maximum M (at centre) = 75 ton feet; maximum S (at ends) = ±10 tons.

6. ,, ,, = 168·7 ton feet; ,, ,, = ±22·5 tons.

7. Maximum combined M = 105 ton feet; at the left end, S varies from +14 to +4 tons; at the right end, S varies from −14 to −4 tons; the central 16·67 feet has shearing forces of both kinds.

8.

Distance of section from one end, feet	0	5	10	15	20	25	30	35	40
Maximum M at section, ton-feet	0	71	110	133	141	133	110	71	0

9. $M_A = 0$; M_B [illegible]6·67 ton feet; M_C = −11·6 ton feet; M_D = 0.

10. R_A = 23·3 tons; R_B = 29·6 tons; R_C = 7·8 tons; R_D = 8·3 tons.

11. Quantities required for the diagrams. Taking three simply supported spans, M at centre of AB = 100 ton-feet; M at centre of BC = 28·12 ton feet = M at centre of CD.

S_A = +23·3 tons; S_B (left) = −16·7 tons; S_B (right) = +3·0 tons;
S_C (left) = −11·1 tons; S_C (right) = +6·7 tons; S_D = −8·3 tons.

12. $A_t = 9·64$ sq. inches; $A_c = 11·25$ sq. inches; two plates in tension flange, each $\frac{7}{16}''$ thick; two plates in compression flange, inner plate $\frac{7}{16}''$ thick, outer plate $\frac{3}{8}''$ thick; web plate $\frac{3}{8}''$ thick; pitch of rivets, 3 inches.

13. Moments of resistance:—Compression flange: angles, 39·6 ton-feet; 1st plate, 50·5 ton-feet; 2nd plate, 45·6 ton-feet; total, 135·7 ton-feet. Tension flange: angles, 36·4 ton-feet; 1st plate, 51·4 ton-feet; 2nd plate, 54·0 ton-feet; total, 141·8 ton-feet.

14. Central bars, upper boom, 36 tons push; central bars, lower boom, 32 tons pull; inclined bar nearest support, 28·28 tons push; inclined bar second from support, 16·97 tons pull; vertical bar second from support, 4 tons push.

15. 45 tons push; 40 tons pull; 42·42 tons; 23·57 tons pull; 10 tons push.

16. $\rho = 0·675$ per cent.; $A_s = 0·1004$ sq. inch; N.A. is 1·53 inches from the top; $M = 6008$ lb.-inches.

17. N.A. is 9·07 inches from the top; use stresses $c_c = 600$, $t_s = 8865$ lb. per sq. inch; $M = 489,000$ lb.-inches.

18. 6 foot-lb. **19.** 5·72 sq. inches.

20. 9·13 tons per sq. inch. **21.** 7·33 tons per sq. inch; 1·86.

22. + denotes push; − denotes pull

Panel No	1	2	3	4	5
Upper boom, tons	+14·4	+33·6	+38·4	+33·6	+14·4
Lower ,, ,,	−9·6	−26·4	−33·6	−26·4	−9·6
End posts, ,,	+14·4	—	—	—	+14·4
Diagonals, ,,	/ +13·58 \ −20·38	/ +3·4 \ −13·58	/ 3·4 \ 3·4	/ −13·58 \ +3·4	/ 20·38 \ +13·58

Chapter X. Page 248.

1. 0·114 ton. **2** 0·3305 ton.

3.

Ratio L : k	40	60	80	100	120	140	160	180	200
p tons per sq. in.	80·2	35·64	20·05	12·84	8·91	6·55	5·01	3·96	3·21

4.

Ratio L : k	40	60	80	100	120	140	160	180	200
p tons per sq. in.	321	142·6	80·2	51·36	35·64	26·2	20·04	15·84	12·84

5. 0·1325 ton. **6.** 3·576 tons. **7.** 41·3 tons.

8. 5·1 tons. **9.** 143·5 inches.

10. 0·2395 ton per sq. inch push; 0·0521 ton per sq. inch pull.

11. 0·882 inch **12.** 1000 lb.-feet.

13. H = 1·30 tons; 1·37 tons; 1·43 tons.

14. 57,700 lb.; 52,100 lb. **15.** 103·84 feet; 0·38 inch.

16. 1·98 tons per sq. inch push; 0·6[illegible]8 tons per sq. inch push.

17. (*a*) 35 tons; (*b*) 32 tons; 6·67 inches from the top edge.

18. 4·67 inches; 0·466 inch. **19.** $k_x =$ [illegible]; $k_y = 4·42$, inch units; [illegible]83 tons.

Chapter XI. Page 276.

1. 35,300 lb. feet.
2. 1·89 inches.
3. 5000 ton-inches.
4. 4320 ton-inches; 1·179.
5. 16,240 lb. inches.
6. 10·58 degrees.
7. 6·25 degrees.
8. 80·7
9. 1·42 inches.
10. 7·54 tons per sq. inch pull on a section at 40° 16′ to AB;
1·46 ,, ,, ,, 130° 16′ ,,
11. 4·905 tons per sq. inch pull on a section at 73° 9′ to AB;
5·905 ,, ,, push ,, ,, 163° 9′ ,, .
12. (*a*) 11,210 lb. inches; (*b*) 7210 lb. inches; (*c*) 2110 lb. per sq. inch push; 605 lb. per sq. inch pull; (*d*) 1·357 lb. per sq. inch.
13. 52,800 lb.-inches.
14. (*a*)

Angle with axis, degrees	0	30	45	60	90
Stress, lb. per sq. inch	8400	7570	6020	5550	4200

(*b*) 4200 lb. per sq. inch.
15 3·125 lb.
16 0·48 inch.
17 0·1 lb. inches.
18 110·4 lb.
19 0·76 inch; 0·603 inch.
20 9 plates; 0·77 inch.
21 1·745 tons per sq. inch.
22 366 lb.
23 8·534 inches.
24 6·25 tons per sq. inch pull acting on a section inclined at 18° 28′ to horizontal; 5·252 tons per sq. inch push acting on a section inclined at 108° 28′ to horizontal.

Chapter XII. Page 290.

1. 19·27 degrees.
2. 1740 lb. at 4 feet from the base, and horizontal.
3. 2160 lb. ,, ,, at 20° to the horizontal.
4. 1752 lb. ,, ,, horizontal.
5 2180 lb. ,, ,, ,,
6. 2215 lb. at 4 feet from the base and inclined at 40° to the horizontal.
7. 3340 lb. at [illegible] feet from the base, and horizontal.
8. R falls outside the middle third, at 1·97 feet from the centre of the base; maximum stress = 2543 lb. per sq. foot push; minimum stress = 1031 lb. per sq. foot pull.
9 4·48 feet.
10. 5·91 feet.
11. 3·84 feet.
12. Resultant cuts the base at 9·55 feet from the earth face, hence outside the middle third.
13. 41° 50′.

Chapter XIII. Page 322.

1 6000 tons per sq. inch.
2. 4950 tons per sq. inch; 2·39 tons per sq. inch; 0·00097 inch.
3. Ultimate tensile strength, 10·98 tons per sq. inch; elastic limit as indicated by the beam dropping, 7·2 tons per sq. inch; percentage stretch on 8 inches, 10·6; percentage stretch on 2 inches at fracture, 15; contraction of area at fracture, 19·5 per cent.
4. 12,650 tons per sq. inch; 19·6 tons per sq. inch.
5. Calculated ratio, 10·87; experimental ratio, 11·65.
6. 23·4 tons per sq. inch.

7. At elastic break-down, $c_c=680$ lb. per sq. inch; $c_s=10{,}200$ lb. per sq. inch; at rupture, $c_c=1564$ lb. per sq. inch; $c_s=23{,}420$ lb. per sq. inch; 1,385,000 lb. per sq. inch.

8. 5120 tons per sq. inch; 7·5 tons per sq. inch; 6·19 inch-lb. per cubic inch.

9. 2880 tons per sq. inch. **10.** 12,220 tons per sq. inch.

11. 4860 tons per sq. inch. **12.** 839 tons per sq. inch; $\frac{1}{3\cdot 88}$.

Chapter XIV. Page 350.

1. 600,000 foot-lb.; 0·727; 1·32. **2.** 1,656,000 foot-lb.

3. 475,200 foot-lb.; 1,584,000 foot-lb.

4. 0·00000735d; 0·0000804d; 0·0000667d; 0·000565d.

5. 15; 360 lb.; 33·33 per cent; no. **6** 142,400 foot-lb. **7.** 823 H.P.

8. 111·4 lb. **9.** 26,250 lb.-feet. **10.** 0·828 degree.

11.

Test No.	Torque, lb.-inches	Difference in vernier readings, inches.	Angle of twist, L = 51 inches, by verniers, radian	Torsion meter readings	Angle of twist, L = 25·5 inches, by torsion-meter, radian.
1	0	0	0	0	0
2	80,640	0·1725	0·001642	19·85	0·000827
3	108,864	0·2300	0·002190	26·75	0·001114
4	181,440	0·3875	0·003692	44·60	0·001857
5	254,016	0·5425	0·005165	62·65	0·002595
6	338,668	0·7200	0·006860	82·75	0·003450
7	431,424	0·9250	0·008810	105·50	0·004396

(c) 49,180 lb.-inches; (d) 48,940 lb.-inches; (e) C = 11,935,000 lb. per sq. inch; (f) shaft horse-power = 0·0647nN; (g) shaft-horse power = 4250.

12. 74·6 ton-inches. **13.** 7450 H.P. **14.** 225 foot-lb.

15. 48; 31·33 lb.; 53·1 per cent. **16.** 20 units; 33·8 per cent.

Chapter XV. Page 378.

1. 40 lb.-feet; 37,700 foot lb.; 48·5 B.T.U.

2. (a) 3 lb.-inches; 0·0143 H.P.; (b) 2·25 lb.-inches; 0·0107 H.P. **3.** 10·77 H.P.

4.

θ, degrees	0	15	30	45	60	75
P, lb.	0·25W	0·242W	0·252W	0·283W	0·35W	0·5W
Work done, foot-lb.	0·25W	0·233W	0·218W	0·2W	0·174W	0·129W

5.

θ, degrees	0	15	30	45	60	75	90
P, lb.	0·25W	0·5W	0·716W	0·884W	0·991W	1·031W	W
Work done, foot-lb.	∞	1·933W	1·433W	1·25W	1·144W	1·067W	W

6.

θ, degrees	0	15	30	45	60	75	90
P, lb.	0·25W	0·566W	0·967W	1·667W	3·5W	59·5W	∞
Work done, foot-lb.	∞	2·07W	1·675W	1·667W	2·02W	15·92W	—

P = ∞ when $\theta=76$ degrees.

7. 91 lb.; 28 per cent. **8.** 38·5 lb.

9. 232 lb.-inches. **10** 19·57 lb.

11. 730 lb. **12.** (*a*) 900 lb.; (*b*) 875 lb.

13. (*a*) 4242 lb.-inches; (*b*) 3660 lb.-inches.

14. 16,860 lb.-inches; inner angle 2°; outer angle 2·2°. **15.** 89·5.

Chapter XVI. Page 403.

1, 2 and **3**

Crank angle, degrees	Distance, feet	Average vel., feet per sec	Vel. at interval, feet per sec	Average accel., ft per sec per sec	Accel. at interval, ft per sec per sec
0	0		0		445
		5·94		413	
30	0·165		11·84		354
		15·48		249	
60	0·595		18·4		133·5
		19·15		10·18	
90	1·127		18·85		92
		16·85		−166·5	
120	1·595		14·22		222
		10·87		246·2	
150	1·897		7·37		262·2
		3·71		−265	
180	2·000		0		−267

4

Crank angle, degrees	Distance, feet	Average vel., feet per sec	Vel. at interval, feet per sec	Average accel., ft per sec per sec
0	0·0		1·8	
		8·24		454
30	0·229		14·4	
		17·85		169
60	0·725		19·1	
		19·25		10·8
90	1·26		18·8	
		16·2		208·5
120	[illegible]		13·0	
		9·36		273·5
150	1·97		5·4	
		1·08		291·5
180	2·00		2·7	
		6·12		248·1
210	1·83		9·6	
		12·6		−198
240	1·48		15·1	
		17·27		133
270	1·00		18·8	
		−18·7		+64·8
300	0·48		17·0	
		−13·95		+306
330	0·092		−8·5	
		−3·34		+335
360	0·0		9·8	

5. 217·4 seconds.
6. 3·32 feet; 0·16 feet per sec.; 0·04 feet per sec. per sec.
7. 334 feet. **8.** 2·66 seconds; 57·8 feet per sec.
9. 5·59 feet per sec. per sec.; 11·56 feet per sec. **10.** 0·667 feet per sec.
11. 1088 feet per sec.; 912 feet per sec.; 1003·8 feet per sec. at 84° 58′ to the line of motion of the train.
12. 1·047 radian per sec.; 0·0087 radian per sec. per sec.
13. 0·5236 radian per sec. per sec.; 37·5 revolutions.
14. 3·75 radians per sec. per sec.; 30 radians per sec.
15. 2·07 feet per sec. at 105° to original direction.
16. 1,778,000 feet per sec. per sec. **17.** 1·358 feet per sec. per sec.
18. 20·94 feet per sec. **19.** 49·5 radians per sec.; 19·8 radians per sec. per sec.
20. (*a*) 8·8 feet per sec. forwards; 8·8 feet per sec. backwards; (*b*) 17·6 feet per sec. forwards; 0.
21. 58·2 miles per hour at 20° to B. **22.** 0·328 second.
23. 2 h. 11 m. A.M. to 3 h. 16·9 m. A.M.

Chapter XVII. Page 428.

932 lb. weight. **2.** 976 cms. per sec. per sec.
3. 9·77 tons weight. **4.** 7·99 feet per sec. per sec.; 2·237 seconds.
5. 9·767 lb. weight. **6.** 312 foot-tons; 26 tons weight.
7. 13·38 ton-foot-second; 20·8 tons weight.
8. Truck has a velocity of 6 feet per sec. in direction opposite to that of the man.
9. 54·4 lb. weight.
10. 49·5 feet per sec.; 76·9 lb. weight; both at 135° to direction of jet.
11. 260 lb.-feet. **12.** 860 pounds and feet units.
13. 9·49 tons and feet units. **14.** 82·5 pounds and feet units.
15. 26·2 pounds and feet units. **16.** 1·414 feet; 9·7 foot-tons.
17. 1·25 foot-tons. **18.** 2·147 feet per sec. per sec.
19. 3270 lb. weight. **20.** 49° 29′.
21. (*a*) 57·8 foot tons; (*b*) 2·138 foot-tons; (*c*) 59·94 foot-tons.
22. 28·9 lb. weight, at 59° to the tangent.
23. 160·1 feet. **24.** 29·4 tons weight; 25·5 tons weight.

Chapter XVIII. Page 451.

1. 226,100 pound-foot-sec.; 312 lb.-feet.
2. 1·84 lb.-feet. **3.** 0·533 radian per sec.
4. 176·4 lb. weight; 18·85 feet per sec. **5.** 3·9 second.
6. 1·5 feet; 1·36 second. **7.** 2 feet from the top end.
8. 0·97 foot from the 4 pound mass; 44·42 pound and foot units.
9. 8·1 pounds at end of bar; 24·78 pounds at 2·694 feet from the 8·1 pound mass.
10. (*a*) 98 feet per sec. per sec. (*b*) 98 feet per sec. per sec.; 18·1 radians per sec. per sec.
11. 113,500 pound-foot-sec. **12.** 110.
13. 198 lb. weight. **14.** 1·255 pound and foot units.

15. 248 lb. weight at 4 inches from the mass centre.

16. 12·41 seconds; 323 lb. weight. 17. 0·302 second.

18. 540 pounds at axis, 60 pounds at 5 feet from axis and on same side of it as the centre of gravity; 465·8 lb.-feet.

19. (*a*) Car turning towards the right, front axle relieved by 5·725 lb. weight, back axle loaded to same extent; (*b*) car turning towards the left, the former forces are reversed.

20. 0·6085 radians per sec.

Chapter XIX. Page 491.

1. 7·7 feet per sec., upwards to right at 23° 28′ to AB produced.

2. 8·7 feet per sec., upwards to right at 35° 40′ to AB.

3.

Crank angle, degrees	0 360	30 330	60 300	90 270	120 240	150 210	180
Vel. of crosshead, feet per sec.	0	6·08	9·77	10·0	7·55	3·91	0

4. (*a*) 1·33 radians per sec.; (*b*) 1·01 feet per sec.; (*c*) 0·405 radian per sec.

5. BC produced to E, CE = 0·05 foot. 6. CP = 2·531 inches.

7.

Crank angle, degrees	0	30	60	90	120	150	180
Force in lb.	24·51	21·2	12·27	0	12·27	21·2	24·51

8.

Crank angle, degrees	0	30	60	90	120	150	180
Torque, lb. inches	0	1145	2521	2048	1300	650	0

9. 47·7 lb. weight; 81·3 lb. weight. 10. 785 revs. per min.

11. Accel. at 60°, 37·5 feet per sec. per sec.; at inner dead point, 125 feet per sec. per sec.; at outer dead point, 75 feet per sec. per sec. 581 lb.; 1940 lb.; 1102 lb.

12. (*a*) 8770 lb.-feet; (*b*) 8200 lb.-feet. 13. 50 feet per sec. per sec.

14. R = 223·5 lb., at right angles to rod and 1⅓ feet from the crank pin.

15. 1378 lb. inches [illegible]

16. Inner dead point, ω = 3·19; outer dead point, ω = 1·497; at 45°, ω = 1·341, all in radians per sec.

17. (*a*) 0·753 sec.; (*b*) 0·445 sec.; 47° 30′.

20. 2375W lb.-feet. 21. 286·2 feet per sec. per sec.

22. 48 inches; 3480 lb.; 840 lb. upwards; 3670 lb. in line of crank.

23. 9·965 feet per sec. 24. 1·36 inches; −0·1 inch; at 0·89 of the stroke.

25. 1400 lb. [illegible]

Chapter XX. Page 522.

1. 6·22 foot lb. 2. 0·0035 per cent. 3. 91·2 revs. per min.

4. 7940 revs. per min. 6. 1·16 feet per sec. 7. 0·70 pounds.

8. 128 foot lb. 9. 939 revs. per min. 10. 7·2 pounds.

11. 1712 revs. per min.; 1892 revs. per min.; 10 per cent.

12. 10·1 radians per sec.; 2·816 lb. feet.

13. 106·6 lb. weight; 17·76 lb. weight per inch compression.

14. 179 revs. per min.; 17 revs. per min.; 13·3 lb. weight. **15.** See Fig. 726.

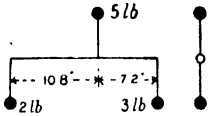


FIG. 726

16. 6·6 pounds at 1 foot radius; 90 lb. weight.

17. Angles made with the direction of the force at A: C, 166°; D, 235°; E, 30°; 14·7 lb. weight.

18. 33½ pounds making 159° with the adjacent crank.

19. 5·15 tons; angles with No. 1 crank: No. 2 crank, 144·6°, No. 3 crank, 255·6°; No. 4 crank, 55·5°.

20. Cranks A and C diametrically opposite crank B; mass of A $= \frac{c}{(a+c)}M$; mass of C $= \frac{a}{(a+c)}M$.

21. (*a*) **Revolving masses**: let $F_1 = \frac{M_1\omega^2 r}{g}$; forces balance; resultant couple $=2F_1 p\sqrt{2}$ in a plane at 45° to the vertical. **Reciprocating masses**: let $F_2 = \frac{M_2\omega^2 r}{g}$; primary forces balance; primary couple $= F_2 \times 2p$; secondary forces balance; secondary couple $= F_2 \times \frac{r}{l} \times 2p$.

(*b*) **Revolving masses**: forces balance; resultant couple $=2F_1 p\sqrt{2}$ in the horizontal plane. **Reciprocating masses**: primary forces balance; primary couples balance; no secondary forces; no secondary couple.

22. 1635 revs. per min. **23.** 822 revs. per min.

24. 1870 revs. per min. **25.** 126·2 inches.

Chapter XXI. Page 561.

1. (*a*) 240 and 1080 revs. per min. (*b*) 228 and 974·7 revs. per min.

2. One countershaft. 36 inch pulley on line shaft; [illegible] inch and 19·5 inch pulleys on countershaft; countershaft runs at 576 revs. per min.

3. T = 83·15 lb.;

Angle, degrees	0	30	60	90	120	150	180
Pull in belt, lb.	400	307·9	236·2	182	140	107·7	83·15

4. 14,700 lb. **5.** 15·77 inches. **6.** 168 lb.

7. 68·2 revs. per min.; 41·5 inches. **8.** 19 H.P. **9.** [illegible]80 lb.

10. 15 on lathe mandrel, gearing with 36 on stud; 18 on stud gearing with 45 on lead screw.

11. 1·75; 12; 8 gearing with 36, 12 gearing with 32.

12. 5·62; 9·38; 22·5. **14.** 2⅓ clockwise revolutions.

15. 1·5 clockwise revolutions; 3 clockwise revolutions. **16.** $n_D : n_H = 1 : 2$.

19.

Angle, degrees	0	30	60	90	135	180	225	270	315	360
ω of driven shaft	1·155	1·065	0·924	0·866	0·99	1·155	0·99	0·866	0·99	1·155

Chapter XXII. Page 589.

1 1200 lb; 400 lb; 300 lb **2** 15,000 lb; 15,910 lb.; 6000 lb

3. 1029 tons; 92·6 tons, R = 936·4 tons at 10·69 feet from the bottom.

4. 1885 lb; 3144 lb

5 16,500 lb.; 2·06 feet below the top edge of the opening.

6 670 tons; 7·812 feet **7** 173·8 tons.

8 Transverse BM = 9·38 feet; longitudinal BM = 104·1 feet **9.** 2·91 degrees

10 57,100 lb at 23° 30′ to the vertical and intersecting the base at a point 12·25 feet from the water face, and hence falling within the middle third; 52,500 lb intersecting the base vertically at one third its width from the water face

11.

Distance of section from top feet	Resultant force, reservoir full	Resultant force, reservoir empty
3	525 lb, vertical; 0·833 foot from water face	525 lb; vertical; 0·833 foot from water face
10	6030 lb, 14·8° to vertical, 3·4 feet from water face	5833 lb; vertical; 2·78 feet from water face.
20	25,000 lb; 21·5° to vertical; 7·75 feet from water face	23,300 lb; vertical; 5·55 feet from water face

12. 122·7 gallons per hour; 2·21 pence per hour.

13 4·72 H.P.; 3·075 H.P. **14** 21,130 foot lb **15** 17,550 foot lb

16 26,950 lb; 5550 cubic inches; 323,400 foot lb

17. 0; 1467 lb, 954 lb, 10 feet

18 8·88 cubic feet, (*a*) 5330 lb push; (*b*) 5117 lb pull.

19 14·63 inches diameter; 21·94 inches stroke; 120,000 foot lb per min; 6·07 H.P.

20 1040 lb; 707 lb. **22** 1965 lb.

23. Accumulator down, 860·3 lb per sq inch at the top; 891 lb per sq inch at the bottom. Accumulator up, 891 lb per sq. inch at the top; 91[illegible]7 lb per sq. inch at the bottom

24. 8·03 feet. **25.** 166·7 lb, 1·0 foot from the top. **26.** 0·39 foot

Chapter XXIII. Page 622.

1. 36 ft. per sec; 8·62 lb per sq inch **2** 0·352 cubic feet per sec.

3 0·047 cubic feet per sec.; 1057 gallons per hour

4. 0·072 cubic feet per sec.; 1620 gallons per hour.

5. 12·[illegible]8 feet per sec. **6.** 4·54 cubic feet per sec. **7.** 1228 lb.

8 0·267 cubic feet per sec. **9** [illegible]7,600 gallons per hour.

10. 2·233; 2·29; 2·35, all in cubic feet per sec.

11. 0·00265. **12** [illegible] foot

13. 4·42 feet per sec.; 1,880,000 cubic feet per day **14.** 2·17 feet per sec.

15. 7 85 feet per sec.; 24 65 cubic feet per sec; 5 56 feet per sec; 17 45 cubic feet per sec.

16. 3 244 miles **17.** 0 0767 foot lb. **18** 0 046 foot-lb.

20. 36 2 feet per sec. **21.** 426 feet per sec. **22.** 40 feet per sec.

23. 1 foot per sec.; 31 4 lb. per sq inch. **24** 1,440,000 gallons per hour.

25. 1 094 lb.; 25 miles per hour. **26.** 6 69 seconds

27. 59 lb. per sq. inch **28** 5 feet per sec; 9800 gallons per hour.

29. 18 3 cubic feet per sec. **30.** 0 74 foot; 0 101 H.P.

31. Frictional loss in the 6 inch pipe - - = 3 03 feet head.
,, ,, 3 ,, - - = 96 96 ,,
Loss at pipe entrance - - - - - = 0 0071 ,,
,, the contraction - - - - - = 0 0257 ,,
Kinetic energy wasted in the lower reservoir = 0 0572 ,,
100 0800

70 6 gallons per min

32 222 gallons per min.; 3 02 feet per sec.; 27 18 feet per sec.

Chapter XXIV. Page 659.

1.

Angle, degrees	0	30	45	60	90
Pressure, lb -	0	5 39	7 62	9 34	10 78

2.

Angle, degrees	0	30	60	90	120	150	180
Pressure, lb. -	21 56	20 8	18 6	15 2	10 78	5 57	0

3. 1 99 lb.; 3 33 lb.

4. (*a*) Work done by the jet = 3 98 foot-lb per sec; (*b*) work done against the jet = 6 66 foot lb. per sec

5. 4 09 H P. **6** 16 65 cubic feet per sec.; 2 27 H.P.

7. 35 25 degrees. **8** 12 feet per sec

9. 39 6 feet per sec, at 8° 41′ to the wheel circumference

10 1184 revs per min; 230 7 H P

11. (*a*) 32 78 feet per sec; (*b*) 26° 45′; (*c*) 2175 lb per sec, (*d*) 43,500 foot-lb; 59 3 H P; (*e*) 48° 49′; 33 18′; 104 revs per min, (*f*) 10 7 feet per sec.; 91 1 per cent.; (*g*) 29 2 feet per sec.; 2960 lb-feet; 32,300 foot-lb per sec.; 58 7 H P.

12 14° 29′; 72° 23′; 15° 42′; 4 72 feet per sec.; 16 62 feet per sec.; 3 27 feet.

13 22 3 lb **14.** (*a*) 87 3 lb.; (*b*) 50 per cent., (*c*) 3 57 H P.

15

v	V	$\frac{v}{V}$	E	e	$\frac{e}{E}\times 100$
81 05	67 7	1 196	395	0	0
88	35 9	2 441	505	305	60 2
85 8	30 82	2 78	469	294	62 65
85 8	28 96	2 96	469	296	63 15
86 4	18 45	4 68	477	239 8	50 2
87 3	5 65	15 45	495	83 8	16 9

2; 100 per cent.

16. 26° 26′; 45° 18′. **17.** 180 H.P. **18** 73°; 9° 52′.

20. 10 46 feet per sec.; 8 57 feet per sec; 11 43 feet per sec

21. 14,000 pound foot sec; 47 4 H.P.

22. 0 3169h; 6 15$\sqrt{h}$; 97 7 per cent **23.** 98 2 H.P.; 1 91 feet.

Chapter XXV. Page 678.

1. 0 608. **2** 0 064. **3.** [illegible]53; 0 52.

4. 2 52; 2 51 **5** 3 226, 3 17

6.

Difference in heads, feet of water	Actual discharge, Q_a, cubic feet per sec	Theoretical discharge, Q, cubic feet per sec	$\frac{Q_a}{Q}$
1 12	0 00610	0 00673	0 92
2 24	0 00890	0 0095	0 945
4 48	0 0127	0 0134	0 948
5 6	0 0142	0 015	0 947
6 73	0 0156	0 0165	0 947
8 96	0 01785	0 019	0 939
11 2	0 0203	0 02125	0 962
13 45	0 02218	0 0233	0 951
15 7	0 02325	0 02515	0 926

Average value of c 0 943

7

Velocity, feet per sec	2 51	4 43	6 81	8 85	13 73
c	94 07	97 65	100 44	102 61	105 6

$h = 1\ 22\frac{v^2}{2g}$

Expt No.	1	2	3	4	5	6	7	8	9
Pressure on plate, lb	0 07	0 082	0 09	0 0[illegible]5	0 105	0 127	0 08[illegible]	0 097	0 11[illegible]

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